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ON SOME SPECIAL CONDITIONS OF THIRD-ORDER OVER-DAMPED NONLINEAR SYSTEM

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A simple analytical method is developed to obtain the time response of third order over-damped nonlinear systems for some special conditions. The method is an extension of the method of Krylov-Bogoliubov-Mitropolskii.

Key Words : Nonlinear System; **Krylov, Bogoliubov, Mitropolskii**

1 INTRODUCTION

In a recent paper, Shamsul *et al.*[1] have found an asymptotic solution for third-order over-damped and damped nonlinear systems based on the work of Krylov, Bogoliubov and Mitropolskii (KBM)[2, 3]. Shamsul *et al.*[4] have also studied some over-damped and damped nonlinear systems with varying coefficients. The method, developed originally by Krylov and Bogoliubov[2] for periodic solution of second-order nonlinear systems, was amplified and justified by Bogoliubov and Mitropolskii[3] and later extended by Popov[5] for damped nonlinear systems. Murty *et al.*[6, 7] developed asymptotic solutions for second-order over-damped nonlinear systems in the sense of this method. Sattar[8] and author[9] studied second order critically damped nonlinear systems. But the over-damped solution obtained in[1] is not useful for certain relations between the characteristic roots of the linear systems. The aim of the present paper is to fill that gap. The new asymptotic solution shows a good agreement with the numerical solution.

2. THE METHOD

Let us consider the third-order nonlinear differential equation

$$\dddot{x} + k_1 \ddot{x} + k_2 \dot{x} + k_3 x = -\varepsilon f(x, \dot{x}, \ddot{x}), \qquad \ldots (1)$$

where ε is a small parameter and k_1, k_2, k_3 are constants. Let us consider, $-\lambda_1, -\lambda_2, -\lambda_3$ are three characteristic roots of (1) when $\varepsilon = 0$. Therefore, the solution of the linear equation of (1) is

$$x(t, 0) = a_0 e^{-\lambda_1 t} + b_0 e^{-\lambda_2 t} + c_0 e^{-\lambda_3 t}, \qquad \ldots (2)$$

where a_0, b_0 and c_0 are arbitrary constants.

Now we seek a solution of (1) in the form of an asymptotic expansion

$$x(t, \varepsilon) = ae^{-\lambda_1 t} + b^{-\lambda_2 t} + ce^{-\lambda_3 t} + \varepsilon u_1(a, b, c, t) + \varepsilon^2 \ldots , \qquad \ldots (3)$$

where a, b and c satisfy the differential equations

$$\dot{a} = \varepsilon A_1(a, b, c, t) + \varepsilon^2 \ldots ,$$

$$\dot{b} = \varepsilon B_1(a, b, c, t) + \varepsilon^2 \ldots , \qquad \ldots (4)$$

and $$\dot{c} = \varepsilon C_1(a, b, c, t) + \varepsilon^2 \ldots .$$

Differentiating (3) three times with respect to t, using relations of (4), substituting (3) and the derivatives $\dot{x}, \ddot{x}, \dddot{x}$ in (1), and equating the coefficients of ε, we obtain

$$e^{-\lambda_1 t}\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\right)\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_3\right)A_1 + e^{-\lambda_2 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right)\left(\frac{\partial}{\partial t} - \lambda_2 + \lambda_3\right)B_1$$

$$+ e^{-\lambda_3 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_3\right)\left(\frac{\partial}{\partial t} + \lambda_2 - \lambda_3\right)C_1 \qquad \ldots (5)$$

$$+ \left(\frac{\partial}{\partial t} + \lambda_1\right)\left(\frac{\partial}{\partial t} + \lambda_2\right)\left(\frac{\partial}{\partial t} + \lambda_3\right)u_1 = -f^{(0)}(a, b, c, t),$$

where $f^{(0)} = f(x_0, \dot{x}_0, \ddot{x}_0)$ and $x_0 = ae^{-\lambda_1 t} = be^{-\lambda_2 t} + ce^{-\lambda_3 t}$.

In general, $f^{(0)}$ can be expanded in a Taylor series as :

$$f^{(0)} = \sum_{i,j,k=0}^{\infty} F_{i,j,k}(a, b, c, t)\, e^{-(i\lambda_1 + j\lambda_2 + k\lambda_3)t}. \qquad \ldots (6)$$

To solve (5) for u_1, A_1, B_1 and C_1, it is assumed that u_1 does not contain terms with $e^{-(i\lambda_1 + j\lambda_2 + k\lambda_3)t}$ where $i\lambda_1 + j\lambda_2 + k\lambda_3 \le \frac{1}{3}(i+j+k)(\lambda_1 + \lambda_2 + \lambda_3)$, so that the coefficients of terms of u_1 do not become large or u_1 does not contain secular type terms te^{-t}. Substituting the values of $f^{(0)}$ from (6) into (5) and in accordance to our assumption, we have

$$e^{-\lambda_1 t}\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\right)\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_3\right)A_1 + e^{-\lambda_2 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right)\left(\frac{\partial}{\partial t} - \lambda_2 + \lambda_3\right)B_1$$

$$+ e^{-\lambda_3 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_3\right)\left(\frac{\partial}{\partial t} + \lambda_2 - \lambda_3\right)C_1$$

$$= -\sum_{i,j,k=0}^{\infty} F_{i,j,k}\; e^{-(i\lambda_1 + j\lambda_2 + k\lambda_3)t}, \qquad \ldots (7)$$

where $$i\lambda_1 + j\lambda_2 + k\lambda_3 \leq \frac{1}{3}(i+j+k)(\lambda_1+\lambda_2+\lambda_3),$$

and $$\left(\frac{\partial}{\partial t}+\lambda_1\right)\left(\frac{\partial}{\partial t}+\lambda_2\right)\left(\frac{\partial}{\partial t}+\lambda_3\right)u_1 = -\sum_{i,j,k=0}^{\infty} F_{i,j,k}\; e^{-(i\lambda_1+j\lambda_2+k\lambda_3)t}, \quad \ldots (8)$$

where $$i\lambda_1 + j\lambda_2 + k\lambda_3 > \frac{1}{3}(i+j+k)(\lambda_1+\lambda_2+\lambda_3).$$

Solving (8), we obtain

$$u_1 = \sum_{i,j,k=0}^{\infty} \frac{F_{i,j,k}\; e^{-(i\lambda_1+j\lambda_2+k\lambda_3)t}}{(i\lambda_1+j\lambda_2+k\lambda_3-\lambda_1)(i\lambda_1+j\lambda_2+k\lambda_3-\lambda_2)(i\lambda_1+j\lambda_2+k\lambda_3-\lambda_3)}. \quad \ldots (9)$$

However, it is not easy to solve (7) for the unknown functions A_1, B_1 and C_1 if the nonlinear function, f and the characteristic roots, $-\lambda_1, -\lambda_2, -\lambda_3$ of the linear equation of (1) are not specified. We can find these functions in the forms of

$$A_1 = \sum_{i,j,k=0}^{\infty} l_{i,j,k}\; e^{-(-\lambda_1+i\lambda_1+j\lambda_2+k\lambda_3)t},$$

$$B_1 = \sum_{i,j,k=0}^{\infty} m_{i,j,k}\, e^{-(i\lambda_1-\lambda_2+j\lambda_2+k\lambda_3)t}$$

and $$C_1 = \sum_{i,j,k=0}^{\infty} n_{i,j,k}\; e^{-(i\lambda_1+j\lambda_2-\lambda_3+k\lambda_3)t}$$

such that the unknown coefficients $l_{i,j,k}, m_{i,j,k}$ and $n_{i,j,k}$ do not become large and as well as A_1, B_1 and C_1 do not become large for any time t.

3. EXAMPLE

As an example of the above procedure, we may consider the nonlinear mechanical elastic system with internal friction and relaxation[10-12]

$$m\ddot{x} + \sigma = 0,$$

$$\sigma + \gamma\dot{\sigma} = \alpha x + \beta\dot{x} + ex^3, e < 1, \quad \ldots (10)$$

where x is the deformation, m is the mass of the system, and α, β, γ and e are positive constants. The terms with α and e represent respectively the linear and nonlinear elasticity, the term with β corresponds to linear viscous damping and the term with γ reflects the linear relaxation. It is obvious that the above system can be written in the form of (1) as

$$\dddot{x} + \gamma^{-1}\ddot{x} + \beta\gamma^{-1}m^{-1}\dot{x} + \alpha\gamma^{-1}m^{-1}x = -\varepsilon x^3,\ \varepsilon = e\gamma^{-1}m^{-1}, \quad \text{... (11)}$$

where $f = x^3$. Here, the relations between roots and coefficients of the linear equation of (11) are

$$k_1 = \gamma^{-1} = \lambda_1 + \lambda_2 + \lambda_3,\ k_2 = \beta\gamma^{-1},\ m^{-1} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$

and $$k_3\,\alpha\gamma^{-1}m^{-1} = \lambda_1\lambda_2\lambda_3.$$

Now,

$$f^{(0)} = x_0^3 = a^3 e^{-3\lambda_1 t} + 3a^2 be^{-(2\lambda_1 - \lambda_2)t} + 3a^2 ce^{-(2\lambda_1 + \lambda_3)t} + 3ab^2 e^{-(\lambda_1 + 2\lambda_2)t}$$

$$+ 6abce^{-(\lambda_1 + \lambda_3 + \lambda_3)t} + 3ac^2 e^{-(\lambda_1 + 2\lambda_3)t} + b^3 e^{-3\lambda_2 t} +$$

$$+ 3b^2 ce^{-(2\lambda_2 + \lambda_3)t} + 3bc^2 e^{-(\lambda_2 + 2\lambda_3)t} + c^3 e^{-3\lambda_3 t}.$$

Therefore, (7) and (8) respectively become

$$e^{-\lambda_1 t}\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\right)\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_3\right)A_1 + e^{-\lambda_2 t}$$

$$\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right)\left(\frac{\partial}{\partial t} - \lambda_2 + \lambda_3\right)B_1$$

$$+ e^{-\lambda_3 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_3\right)\left(\frac{\partial}{\partial t} + \lambda_2 - \lambda_3\right)$$

$$C_1 = -\ (6abce^{-(\lambda_1 + \lambda_3 + \lambda_3)t} + 3ac^2 e^{-(\lambda_1 + 2\lambda_3)t}$$

$$+ b^3 e^{-3\lambda_2 t} + 3b^2 ce^{-(2\lambda_2 + \lambda_3)t} + 3bc^2 e^{-(\lambda_2 + 2\lambda_3)t} + c^3 e^{-3\lambda_3 t}), \quad \text{... (12)}$$

and

$$\left(\frac{\partial}{\partial t} + \lambda_1\right)\left(\frac{\partial}{\partial t} + \lambda_2\right)\left(\frac{\partial}{\partial t} + \lambda_3\right)u_1$$

$$= -\ (a^3 e^{-3\lambda_1 t} + 3a^2 be^{-(2\lambda_1 + \lambda_2)t} + 3a^2 ce^{-2(\lambda_1 + \lambda_3)t}$$

$$+ 3ab^2 e^{-(\lambda_1 + 2\lambda_2)t}). \quad \text{... (13)}$$

if $\lambda_1 > \lambda_2 > \lambda_3$.

Solving (13), we obtain

$$u_1 = d_1 a^3 e^{-3\lambda_1 t} + d_2 a^2 be^{-(2\lambda_1 + \lambda_2)t} + d_3 a^2 ce^{-(2\lambda_1 + \lambda_3)t}$$

$$+ d_4 ab^2 e^{-\lambda_1 + 2\lambda_2)t}, \quad \text{... (14)}$$

where $$d_1 = \frac{1}{2\lambda_1(3\lambda_1 - \lambda_2)(3\lambda_1 - \lambda_3)},$$

$$d_2 = \frac{3}{2\lambda_1(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)},$$

$$d_3 = \frac{3}{2\lambda_1(\lambda_1+\lambda_3)(3\lambda_1-\lambda_2+\lambda_3)},$$

$$d_4 = \frac{3}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)}. \quad \ldots (15)$$

It is obvious that characteristic roots, $-\lambda_1, -\lambda_2$ and $-\lambda_3$ of the linear equation of (11) are changed as m, α, β and γ are changed. For certain values of m, α, β and γ, there exit some relations between λ_1, λ_2 and λ_3, and depending on these relations of λ_1, λ_2 and λ_3, we shall obtain different functional relations of A_1, B_1 and C_1 with t.

First of all we may consider $\lambda_1 \approx 3\lambda_3$ or $\lambda_1 \approx \lambda_2 + 2\lambda_3$. In this case, we can choose

$$e^{-\lambda_1 t}\left(\frac{\partial}{\partial t}-\lambda_1+\lambda_2\right)\left(\frac{\partial}{\partial t}-\lambda_1+\lambda_3\right)A_1 = -(3bc^2 e^{-(\lambda_2+2\lambda_3)t} + c^3 e^{-3\lambda_3 t}), \quad \ldots (16)$$

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_2\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)B_1 = -(b^3 e^{-3\lambda_2 t} + 3b^2 ce^{(-2\lambda_2+\lambda_3)t}), \quad \ldots (17)$$

and $$e^{-\lambda_3 t}\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_3\right)$$

$$C_1 = -(6abce^{-(\lambda_1+\lambda_2+\lambda_3)t} + 3ac^2 e^{-(\lambda_1+2\lambda_3)t}). \quad \ldots (18)$$

Solving (16)-(18), we obtain

$$A_1 = l_1 bc^2 e^{(\lambda_1-\lambda_2-2\lambda_3)t} + l_2 c^3 e^{(\lambda_1-3\lambda_3)t},$$

$$B_1 = m_1 b^3 e^{-2\lambda_2 t} + m_2 b^2 ce^{-(\lambda_2+\lambda_3)t}, \quad \ldots (19)$$

$$C_1 = n_1 abce^{-(\lambda_1+\lambda_2)t} + n_2 ac^2 e^{-(\lambda_1+\lambda_3)t},$$

where $$l_1 = -3(2\lambda_3(\lambda_2+\lambda_3))^{-1},\ l_2 = -(2\lambda_3(3\lambda_3-\lambda_2))^{-1},$$

$$m_1 = -((3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3))^{-1},\ m_2 = -3(2\lambda_2(2\lambda_2+\lambda_3-\lambda_1))^{-1} \quad \ldots . (20)$$

$$n_1 = -6((\lambda_1+\lambda_3)(\lambda_2+\lambda_3))^{-1},\ n_2 = -3(2\lambda_3)(\lambda_1-\lambda_2+2\lambda_3))^{-1}.$$

Substituting the values of A_1, B_1 and C_1 from (19) into (4), we obtain

$$\dot{a} = \varepsilon(l_1 bc^2 e^{(\lambda_1-\lambda_2-2\lambda_3)t} + l_2 c^3 e^{(\lambda_1-3\lambda_3)t}),$$

$$\dot{b} = \varepsilon(m_1 b^3 e^{-2\lambda_2 t} + m_2 b^2 ce^{-(\lambda_2+\lambda_3)t}), \quad \ldots (21)$$

$$\dot{c} = \varepsilon (n_1\, abce^{-(\lambda_1+\lambda_2)t} + n_2\, ac^2\, e^{-(\lambda_1+\lambda_3)t}).$$

Now we can solve (21) by assuming that a, b and c are constants in the right hand sides of (21), since ε is a small quantity. This assumption was first made by Murty *et al.*[7] to solve similar type of nonlinear equations of (21). Thus the solution of (21) is

$$a = a_0 + \varepsilon (l_1\, b_0\, c_0^2\, (e^{(\lambda_1-\lambda_2-2\lambda_3)t} - 1)/(\lambda_1 - \lambda_2 - 2\lambda_3) + l_2\, c_0^3\, t) \quad \lambda_1 = 3\lambda_3$$

$$a = a_0 + \varepsilon (l_1\, b_0\, c_0^2\, (e^{(\lambda_1-\lambda_2-2\lambda_3)t} - 1)/(\lambda_1 - \lambda_2 - 2\lambda_3)$$

$$+ \; l_2\, c_0^3\, (e^{(\lambda_1-3\lambda_3)t} - 1)/(\lambda_1 - 3\lambda_3)), \quad \lambda_1 \neq 3\lambda_3 \qquad \ldots (22)$$

$$b = b_0 + \varepsilon (m_1\, b_0^3\, (1 - e^{-2\lambda_2 t})/(2\lambda_2) + m_2\, b_0^2\, c_0$$

$$(1 - e^{-(\lambda_2+\lambda_3)t})/(\lambda_2 + \lambda_3))$$

$$c = c_0 + \varepsilon (n_1\, a_0\, b_0\, c_0\, (1 - e^{-(\lambda_1+\lambda_2)t})/(\lambda_1 + \lambda_2)$$

$$+ n_2\, a_0\, c_0^2\, (1 - e^{-(\lambda_1+\lambda_3)t})/(\lambda_1 + \lambda_3)).$$

Secondly, we may consider the case(s) $\lambda_1 \approx 3\lambda_2$ or/and $\lambda_2 \approx 3\lambda_3$. Therefore, we choose

$$e^{-\lambda_1 t}\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\right)\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_3\right)$$

$$A_1 = -(b^3\, e^{-3\lambda_2 t} + 3b^2\, ce^{-(2\lambda_2+\lambda_3)t} + 3ac^2\, e^{-(\lambda_1+2\lambda_3)t}), \qquad \ldots (23)$$

$$e^{-\lambda_2 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right)\left(\frac{\partial}{\partial t} - \lambda_2 + \lambda_3\right)$$

$$B_1 = -(3bc^2\, e^{-(\lambda_2+2\lambda_3)t} + c^3\, e^{-3\lambda_3 t}), \qquad \ldots (24)$$

and

$$e^{-\lambda_3 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_3\right)\left(\frac{\partial}{\partial t} + \lambda_2 - \lambda_3\right)C_1 = -6abce^{-(\lambda_1+\lambda_3+\lambda_3)t}. \qquad \ldots (25)$$

Solving (23)-(25), we obtain

$$A_1 = l_1\, b^3\, e^{(\lambda_1-3\lambda_2)t} + l_2 b^2\, ce^{(\lambda_1-2\lambda_2-\lambda_3)t} + l_3\, ac^2\, e^{-2\lambda_3 t},$$

$$B_1 = m_1 bc^2\, e^{-2\lambda_3 t} + m_2\, c^3\, e^{(\lambda_2-3\lambda_3)t}, \qquad \ldots (26)$$

$$C_1 = n_1\, abce^{-(\lambda_1+\lambda_2)t},$$

where

$$l_1 = -(2\lambda_2\,(3\lambda_2 - \lambda_3))^{-1}, \quad l_2 = -3\,(2\lambda_2\,(\lambda_2 + \lambda_3))^{-1},$$

$$l_3 = -3\,((\lambda_1 - \lambda_2 + 2\lambda_3)(\lambda_1 + \lambda_3))^{-1},\ m_1 = -3\,((-\lambda_1 + \lambda_2 + 2\lambda_3)(\lambda_2 + \lambda_3))^{-1}, \ldots \quad (27)$$

$$m_2 = -(2\lambda_3(-\lambda_1 + 3\lambda_3))^{-1},\ n_1 = -6\,((\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3))^{-1}.$$

Substituting the values of A_1, B_1 and C_1 from (26) into (4), we obtain

$$\dot{a} = \varepsilon(l_1 b^3 e^{(\lambda_1 - 3\lambda_2)t} + l_2 b^2 c e^{(\lambda_1 - 2\lambda_2 - \lambda_3)t} + l_3 a c^2 e^{-2\lambda_3 t}),$$

$$\dot{b} = \varepsilon(m_1 b c^2 e^{-2\lambda_3 t} + m_2 c^3 e^{(\lambda_2 - 3\lambda_3)t}), \quad \ldots (28)$$

$$\dot{c} = \varepsilon n_1 abc e^{-(\lambda_1 + \lambda_2)t},$$

Again we can integrate (28) with respect to t, by assuming that a, b and c are constant in the right hand sides and obtain

$$a = a_0 + \varepsilon(l_1 b_0^3 t + l_2 b_0^2 c_0 (e^{(\lambda_1 - 2\lambda_2 - \lambda_3)t} - 1)/(\lambda_1 - 2\lambda_2 - \lambda_3)$$

$$+ l_3 a_0 c_0^2 (1 - e^{-2\lambda_3 t})/(2\lambda_3)),\ \lambda_1 = 3\lambda_2$$

$$a = a_0 + \varepsilon(l_1 b_0^3 (e^{(\lambda_1 - 3\lambda_2)t} - 1)/(\lambda_1 - 3\lambda_2) + l_2 b_0^2 c_0$$

$$(e^{(\lambda_1 - 2\lambda_2 - \lambda_3)t} - 1)/(\lambda_1 - 2\lambda_2 - \lambda_3) + l_3 a_0 c_0^2 (1 - e^{-2\lambda_3 t})/(2\lambda_3),\ \lambda_1 \neq 3\lambda_2 \quad \ldots (29)$$

$$b = b_0 + \varepsilon(m_1 b_0 c_0^2 (1 - e^{-(2\lambda_3 t)}/(2\lambda_3)) + m_2 c_0^3 t)),\ \lambda_1 = 3\lambda_2$$

$$b = b_0 + \varepsilon(m_1 b_0 c_0^2 (1 - e^{-2\lambda_3 t})/(2\lambda_3) + m_2 c_0^3 (e^{(\lambda_2 - 3\lambda_3)t} - 1)/(\lambda_2 - 3\lambda_3))$$

$$c = c_0 + \varepsilon n_1 a_0 b_0 c_0 (1 - e^{(-(\lambda_1 + \lambda_2)t})/(\lambda_1 + \lambda_2),\ \lambda_2 \neq 3\lambda_3.$$

Thus the first order solution of (11) is

$$x = ae^{-\lambda_1 t} + be^{-\lambda_2 t} + ce^{-\lambda_3 t} + \varepsilon u_1, \quad \ldots (30)$$

where a, b and c are given by (22) when $\lambda_1 \approx 3\lambda_3$ or $\lambda_1 \approx \lambda_2 + 2\lambda_3$ and u_1 is given by (14). But when $\lambda_1 \approx 3\lambda_2$ or/and $\lambda_2 \approx 3\lambda_3$, a, b and c should be computed by (29) instead of (22). In the second case(s) u_1 is also given by (14). The procedure can be carried out in second or higher order in the same way.

4. Discussion of Unified Method of Shamsul *et al.*[1]

Shamsul *et al.*[1] found an asymptotic solution for third-order over-damped and damped nonlinear systems in the form

$$x(t, \varepsilon) = e^{\xi} + e^{\eta} \cosh \varphi + \varepsilon \tilde{u}_1 (\xi, \eta, \varphi) + \varepsilon^2 .... \quad \text{... (31)}$$

where ξ, η and φ satisfy the differential equations

$$\dot{\xi} = -\lambda + \varepsilon \tilde{A}_1 (\xi, \eta) + \varepsilon^2 ...,$$

$$\dot{\eta} = -\mu + \varepsilon \tilde{B}_1 (\xi, \eta) + \varepsilon^2 ..., \quad \text{... (32)}$$

$$\dot{\varphi} = \omega + \varepsilon \tilde{C}_1 (\xi, \eta) + \varepsilon^2 ...,$$

and $-\lambda, -\mu - \omega$ and $-\mu + \omega$ are the characteristic roots of the linear equation of (1). It is noted that when ω is real, (31) represents an over-damped nonlinear system and when ω is a pure imaginary, it represents a damped nonlinear system. We may choose $\lambda_1 = \lambda, \lambda_2 = \mu + \omega$ and $\lambda_3 = \mu - \omega, \omega > 0$, so that we keep $\lambda_1 > \lambda_2 > \lambda_2$. It is interesting to note that unified solution (31) can be found from (5). Substituting $\lambda_1 = \lambda, \lambda_2 = \mu + \omega$ and $\lambda_3 = \mu - \omega$ into (5), we obtain

$$\begin{aligned} & e^{-\lambda t}\left(\frac{\partial}{\partial t} - \lambda + \mu + \omega\right)\left(\frac{\partial}{\partial t} - \lambda + \mu - \omega\right)\tilde{A}_1 \\ & + e^{-\mu t}\left(e^{-\omega t}\left(\frac{\partial}{\partial t} + \lambda - \mu - \omega\right)\left(\frac{\partial}{\partial t} - 2\omega\right)\tilde{B}_1 + e^{\omega t}\right. \\ & \left.\left(\frac{\partial}{\partial t} + \lambda - \mu + \omega\right)\left(\frac{\partial}{\partial t} + 2\omega\right)\tilde{C}_1\right) \\ & + \left(\frac{\partial}{\partial t} + \lambda\right)\left(\frac{\partial}{\partial t} + \mu + \omega\right)\left(\frac{\partial}{\partial t} + \mu - \omega\right)\tilde{u}_1 = -f^{(0)}(a, b, c, t), \end{aligned} \quad \text{... (33)}$$

where $f^{(0)} = f(x_0, \dot{x}_0, \ddot{x}_0)$ and $x_0 = ae^{-\lambda t} + e^{-\mu t}(be^{-\omega t} + ce^{\omega t})$.

Let the function $f^{(0)}$ be expanded in a Taylor series

$$\begin{aligned} f^{(0)} = {} & F_0 (a, b, c, t) + F_1 (a, b, c, t) e^{\omega t} + G_1 (a, b, c, t) e^{-\omega t} \\ & + F_2 (a, b, c, t) e^{2\omega t} + G_2 (a, b, c, t) e^{-2\omega t} + ... \end{aligned} \quad \text{... (34)}$$

To solve (33) for $\tilde{u}_1, \tilde{A}_1, \tilde{B}_1$ and $\tilde{C}_1$, it was assumed[1] that the function $\tilde{u}_1$ does not contain the fundamental terms, $e^{0\omega t}, e^{\omega t}$ and $e^{-\omega t}$. Substituting $f^{(0)}$ from (34) into (33) and equating the coefficients of $e^{0\omega t}, e^{\omega t}$ and $e^{-\omega t}$, we obtain

$$e^{-\lambda t}\left(\frac{\partial}{\partial t} - \lambda + \mu + \omega\right)\left(\frac{\partial}{\partial t} - \lambda + \mu - \omega\right)\tilde{A}_1 = -F_0, \quad \text{... (35)}$$

$$e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu-\omega\right)\left(\frac{\partial}{\partial t}-2\,\omega\right)\tilde{B}_1=-G_1, \qquad \ldots (36)$$

$$e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu+\omega\right)\left(\frac{\partial}{\partial t}+2\,\omega\right)\tilde{C}_1=-F_1, \qquad \ldots (37)$$

and

$$\left(\frac{\partial}{\partial t}+\lambda\right)\left(\frac{\partial}{\partial t}+\mu+\omega\right)\left(\frac{\partial}{\partial t}+\mu-\omega\right)\tilde{u}_1$$

$$=-(F_2\,e^{2\,\omega t}+G_2\,e^{-2\,\omega t}+\ldots). \qquad \ldots (38)$$

Here,

$$f^{(0)}=a^3\,e^{-3\,\lambda t}+6abce^{-(\lambda+2\,\mu)\,t}+3\,(a^2\,ce^{-(2\,\lambda+\mu)\,t}$$

$$+bc^2\,e^{-3\,\mu t})\,e^{\omega t}+3\,(a^2\,be^{-(2\,\lambda+\mu)\,t}+b^2\,ce^{-3\,\mu t})\,e^{-\omega t}$$

$$+\,3\,ae^{-(\lambda+2\,\mu)t}\,(c^2\,e^{2\,\omega t}+b^2\,e^{-2\,\omega t})+e^{-3\,\mu t}\,(c^3\,e^{3\,\omega t}+b^3\,e^{-3\,\omega t}).$$

Therefore, we have

$$F_0=a^3\,e^{-3\,\lambda t}+6abce^{-(\lambda+2\,\mu)\,t},$$

$$F_1=3\,(a^2\,ce^{-(2\,\lambda+\mu)\,t}+bc^2\,e^{-3\,\mu t}),\ G_1=3\,(a^2\,be^{-(2\,\lambda+\mu)\,t}+b^2\,ce^{-3\,\mu t}), \qquad \ldots (39)$$

$$F_2=3ac^2\,e^{-(\lambda+2\,\mu)\,t},\ G_2=3ab^2\,e^{-(\lambda+2\,\mu)\,t},$$

$$F_3=c^3\,e^{-3\,\mu t},\ G_3=b^3\,e^{-3\,\mu t}.$$

Substituting the values of F_0 from (39) into (35), F_1 into (37) and G_1 into (36) and then solving them, we obtain

$$\tilde{A}_1=\frac{-a^3\,e^{-2\,\lambda t}}{(3\,\lambda-\mu)^2-\omega^2}+\frac{-6\,abce^{-2\,\mu t}}{(\lambda+\mu)^2-\omega^2},$$

$$\tilde{B}_1=-\frac{3}{2}\times\left(\frac{a^2\,be^{-2\,\lambda t}}{(\lambda+\omega)\,(\lambda+\mu+\omega)}+\frac{b^2\,ce^{-2\,\mu t}}{(\mu+\omega)\,(-\lambda+3\,\mu+\omega)}\right) \qquad \ldots (40)$$

and

$$\tilde{C}_1=-\frac{3}{2}\times\left(\frac{a^2\,ce^{-2\,\lambda t}}{(\lambda-\omega)\,(\lambda+\mu-\omega)}+\frac{bc^2\,e^{-2\,\mu t}}{(\mu-\omega)\,(-\lambda+3\,\mu-\omega)}\right),$$

where $\lambda\neq 3\,\mu\pm\omega$, i.e., $\lambda_1\neq 2\,\lambda_2+\lambda_3$ and $\lambda_1\neq\lambda_2+2\,\lambda_3$. In these cases the unified method has been failed to give the desired results.

Now substituting the values of A_1, B_1 and C_1 from (40) into (4), we obtain

$$\dot{a} = -\varepsilon\left(\frac{a^3 e^{-2\lambda t}}{(3\lambda-\mu)^2-\omega^2}+\frac{6abce^{-2\mu t}}{(\lambda+\mu)^2-\omega^2}\right),$$

$$\dot{b} = -\frac{3\varepsilon}{2}\times\left(\frac{a^2 be^{-2\lambda t}}{(\lambda+\omega)(\lambda+\mu+\omega)}+\frac{b^2 ce^{-2\mu t}}{(\mu+\omega)(-\lambda+3\mu+\omega)}\right) \quad \ldots (41)$$

and eq. (4) becomes

$$\dot{c} = -\frac{3\varepsilon}{2}\times\left(\frac{a^2 ce^{-2\lambda t}}{(\lambda-\omega)(\lambda+\mu-\omega)}+\frac{bc^2 e^{-2\mu t}}{(\mu-\omega)(-\lambda+3\mu-\omega)}\right)$$

Under the transformations

$$a = a,$$

$$b = \frac{\beta e^{-\psi}}{2}, \quad \ldots (42)$$

$$c = \frac{\beta e^{\psi}}{2}, \text{ (41) becomes}$$

$$\dot{a} = \varepsilon(l_1' a^3 e^{-2\lambda t} + l_2' a\beta^2 e^{-2\mu t}),$$

$$\dot{\beta} = \varepsilon(m_1' a^2\beta e^{-2\lambda t} + m_2'\beta^3 e^{-2\mu t}), \quad \ldots (43)$$

$$\dot{\psi} = \varepsilon(n_1' a^2 e^{-2\lambda t} + n_2'\beta^2 e^{-2\mu t}),$$

where $$l_1' = \frac{-1}{(3\lambda-\mu)^2-\omega^2},\ l_2' = \frac{-3}{2((\lambda+\mu)^2-\omega^2)},$$

$$m_1' = \frac{-3(\lambda^2+\lambda\mu+\omega^2)}{2(\lambda^2-\omega^2)((\lambda+\mu)^2-\omega^2))},$$

$$m_2' = \frac{-3(-\lambda+3\mu^2+\omega^2)}{8(\mu^2-\omega^2)((-\lambda+3\mu)^2-\omega^2)}, \quad \ldots (44)$$

$$n_1' = \frac{-3(2\lambda+\mu)\omega}{2(\lambda^2-\omega^2)((\lambda+\mu)^2-\omega^2)},$$

$$n_2' = \frac{-3(-\lambda+4\mu)\omega}{8(\mu^2-\omega^2)((-\lambda+3\mu)^2-\omega^2)}$$

Lastly under the transformations

$$ae^{-\lambda t} = e^{\xi},$$
$$\beta e^{-\mu t} = e^{\eta}, \quad \ldots (45)$$
$$\psi = \varphi - \omega t,$$

(43) becomes

$$\dot{\xi} = -\lambda \xi + \varepsilon (l_1' e^{2\xi} + l_2' e^{2\eta}),$$
$$\dot{\eta} = -\mu \eta + \varepsilon (m_1' e^{2\xi} + m_2' e^{2\eta}), \quad \ldots (46)$$
$$\dot{\varphi} = \omega + \varepsilon (n_1' e^{2\xi} + n_2' 2^{2\eta}).$$

In general, (46) has been computed by a numerical method[1,11]. Now substituting the values of F_2, G_2, F_3 and G_3 from (39) into (38) and solving it, we obtain

$$\tilde{u}_1 = \frac{3ae^{-(\lambda+2\mu)t}}{2} \times \left(\frac{b^2 e^{-2\omega t}}{(\lambda = \mu - \omega)(\lambda + \mu - 3\omega)(\mu - \omega)} \right.$$
$$\left. + \frac{c^2 e^{2\omega t}}{(\lambda + \mu + \omega)(\lambda + \mu + 3\omega)(\mu + \omega)} \right) \quad \ldots (47)$$
$$+ \frac{e^{-3\mu t}}{2} \times \left(\frac{b^3 e^{-3\omega t}}{(-\lambda + 3\mu - 3\omega)(\mu - \omega)(\mu - 2\omega)} \right.$$
$$\left. + \frac{c^3 e^{3\omega t}}{(-\lambda + 3\mu + 3\omega)(\mu + \omega)(\mu + 2\omega))} \right),$$

where $\lambda \neq 3\mu \pm 3\omega$ and $\mu \neq 2\omega$ i.e., $\lambda_1 \neq 3\lambda_2$, $\lambda_1 \neq 3\lambda_3$, and $\lambda_2 \neq 3\lambda_3$. In these cases, the unified method has also been failed to give the correct results. Since, in these cases $\tilde{u}_1$ contains *secular* type terms te^{-t}.

Under the transformations (42) and (45), (47) becomes

$$\tilde{u}_1 = a\beta^2 e^{-(\lambda+2\mu)t} (c_2' \cosh 2(\omega t + \psi) + d_2' \sinh 2(\omega t + \psi))$$
$$+ \beta^3 e^{-32\mu t}) (c_3' \cosh 3(\omega t + \psi) + d_3' \sinh 3(\omega t + \psi)),$$

or,
$$\tilde{u}_1 = e^{\xi + 2\eta} (c_2' \cosh 2\varphi + d_2' \sinh 2\varphi) + e^{3\eta}$$
$$(c_3' \cosh 3\varphi + d_3' \sinh 3\varphi), \quad \ldots (48)$$

where
$$c_2' = \frac{3(\mu(\lambda+\mu)^2 + (4\lambda+7\mu)\omega^2)}{4((\lambda+\mu)^2-\omega^2)((\lambda+\mu)^2-9\omega^2)(\mu^2-\omega^2)},$$

$$d_2' = \frac{3((\lambda+\mu)(\lambda+5\mu)-3\omega^2)\omega}{4((\lambda+\mu)^2-\omega^2)((\lambda+\mu)^2-9\omega^2)(\mu^2-\omega^2)}, \quad \text{... (49)}$$

$$c_3' = \frac{\mu^2(-\lambda+3\mu)+(-2\lambda+15\mu)\omega^2}{16((-\lambda+3\mu)^2-9\omega^2)(\mu^2-\omega^2)(\mu^2-4\omega^2)},$$

$$d_3' = \frac{3(\mu(-\lambda+4\mu)+2\omega^2)\omega}{16((-\lambda+3\mu)^2-9\omega^2)(\mu^2-\omega^2)(\mu^2-4\omega^2)}.$$

It is obvious that under the transformations (42) and (45), the first order solution of (11) becomes

$$x = ae^{-\lambda t} + \beta e^{-\mu t}\cosh\varphi + \varepsilon\tilde{u}_1,$$

or
$$x = e^{\xi} + e^{\eta}\cosh\varphi + \varepsilon\tilde{u}_1, \quad \text{... (50)}$$

where ξ, η and φ are given by (46) and $\tilde{u}_1$ is given by (48). From (49) it is clear that when $\lambda+\mu = 3\omega$ or $\mu = 2\omega$, i.e., $\lambda_1 = 3\lambda_2$, $\lambda_1 = 3\lambda_3$, and $\lambda_2 = 3\lambda_3$, $\tilde{u}_1$ in (48), contains *secular* type terms $\varphi\cos 2\varphi$, $\varphi\sin 2\varphi$ or $\varphi\cos 3\varphi$, $\varphi\sin 3\varphi$ (which are equivalent to te^{-t}) and for this why (50) will not give desired results.

It has been already noted that when ω is a pure imaginary quantity, (50) represents a damped nonlinear system[1]. In this case, denominator(s) of the terms of (44) and (49) do not vanish and $\tilde{u}_1$ does not contain *secular* terms te^{-it} (*i.e. t cos t* and *t* sin *t*). So, (50) can be used as a damped solution for all positive values of λ and μ.

5. RESULTS AND DISCUSSION

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we sometimes compare the approximate solution to numerical solution (considered to be exact). With regard to such a comparison concerning the presented asymptotic solution obtained by KBM method of this paper, we refer to the works of Murty *et al*[6] and Mendelson[13].

First of all, we may consider the case in which $\lambda_1 \approx 3\lambda_3$. For $m = 1$, $\alpha = \gamma = \frac{\sqrt{3}}{4+\sqrt{3}}$, $\beta = 1$ and $e = \frac{1}{10}\times\frac{\sqrt{3}}{4+\sqrt{3}}$, we have been obtained $\lambda_1 = \sqrt{3}$, $\lambda_2 = 1$, $\lambda_3 = 1/\sqrt{3}$ and ε = 0.1. Therefore, $\lambda_1 = 3\lambda_3$. So, we have computed $x(t)$ by (30) in which a, b and c have been evaluated by (22) with initial conditions $[x(0) = 1, \dot{x}(0) = 0, \ddot{x}(0) = -0.75]$, or a_0 = – 0.868936, b_0 = 0.406290 and c_0 = 1.450118, and the results for various time, t have been given in second column of **Table I**. Corresponding numerical solution (denoted by $x^{\#}$) has been computed by *Runge-Kutta* (fourth-order) method and given in third column of the **Table I**. Then percentage errors have calculated and given in fourth column. From **Table I**, it is clear that percentage errors are not greater than 1%. Thus the new asymptotic solution, (30) agrees with numerical solution, since errors have occurred in an order of ε^2, (noted that ε = 0.1).

TABLE I

t	X	$x^{\#}$	$E(\%)$
0.0	1.000000	1.000000	0.0000
0.5	0.931646	0.930349	0.1394
1.0	0.789337	0.789436	- 0.0125
1.5	0.635627	0.636697	- 0.1681
2.0	0.496331	0.497956	- 0.3263
2.5	0.380265	0.382052	- 0.4675
3.0	0.287988	0.289659	- 0.5765
3.5	0.216618	0.218036	- 0.6504
4.0	0.162310	0.163443	- 0.6926
4.5	0.121376	0.122246	- 0.7117
5.0	0.090685	0.091338	- 0.7138
5.5	0.067738	0.068221	- 0.7065
6.0	0.050602	0.050987	- 0.6967

Now for m = 1, $\alpha = \gamma = 40/129$, β = 1 and e = 4/129, we have been obtained $\lambda_1 = 8/5, \lambda_2 = 1, \lambda_3 = 5/8$ and $\varepsilon = 0.1$. Therefore, $\lambda_1 < 3\,\lambda_3$. So, we have again computed $x(t)$ by (30) in which a, b and c have been evaluated by (22) with initial conditions $[x(0) = 1, \dot{x}(0) = 0, \ddot{x}(0) = -0.75]$, or $a_0 = -0.975325, b_0$ = 0.353649 and $c_0 = 1.599778$, and the results have been given in second column of **Table II**. Corresponding numerical solution has been computed and given in third column of the **Table II**. Then percentage errors have been calculated and given in fourth column. In this case, the new asymptotic solution shows a good agreement with numerical solution, since percentage errors are much smaller than 1%. In a similar way, we can show that this solution is useful when $\lambda_1 > 3\,\lambda_3$ or $\lambda_1 \approx \lambda_2 + 2\,\lambda_3$.

TABLE II

t	x	x^*	$E(\%)$
0.0	1.000000	1.000000	0.0000
0.5	0.931731	0.929551	0.2345
1.0	0.786059	0.785544	0.0656
1.5	0.628428	0.628632	- 0.0325
2.0	0.485725	0.486150	- 0.0874
2.5	0.367336	0.367726	- 0.1061
3.0	0.273937	0.274180	- 0.0883
3.5	0.202480	0.202561	- 0.0400
4.0	0.148843	0.148797	0.0316
4.5	0.109055	0.108929	0.1157
5.0	0.079754	0.079590	0.2061
5.5	0.058267	0.058096	0.2943
6.0	0.042550	0.042390	0.3774

Then we consider the case(s) $\lambda_1 \approx 3\lambda_2$ or/and $\lambda_2 \approx 3\lambda_3$. For $m = 1$, $\alpha = \gamma = 3/13$, $\beta = 1$ and $e = 3/130$, we have been obtained $\lambda_1 + 3, \lambda_2 = 1$, $\lambda_3 = 1/3$ and $\varepsilon = 0.1$. Therefore, we have got the equalities, $\lambda_1 = 3\lambda_2$ and $\lambda_2 = 3\lambda_3$. In this case, we have computed $x(t)$ by (30) in which a, b and c have been evaluated by (29) with initial conditions $[x(0) = 1, \dot{x}(0) = 0, \ddot{x} = -0.75]$, or $a_0 = -0.066960, b_0 = -0.087016$ and $c_0 = 1.153966$, and the results have been given in second column of **Table III**. Corresponding numerical solution has been computed and given in third column of the **Table**. Then percentage errors have been calculated and given in fourth column. Here, the ratio of the largest and the smallest roots is 9, *i.e.*, an order of ε^{-1}, yet percentage errors of the new solution do not exceed 1%. It is noted that in this case, errors have increased monotonically.

TABLE III

t	x	$x^{\#}$	$E(\%)$
0.0	1.000000	1.000000	0.0000
0.5	0.938992	0.938953	0.0042
1.0	0.828536	0.828213	0.0390
1.5	0.714059	0.713204	0.1199
2.0	0.608677	0.607245	0.2358
2.5	0.515930	0.514045	0.3667
3.0	0.436047	0.433899	0.4950
3.5	0.368029	0.365796	0.6104
4.0	0.310472	0.308290	0.7078
4.5	0.261919	0.259875	0.7865
5.0	0.221019	0.219161	0.8478
5.5	0.186578	0.184924	0.8944
6.0	0.157569	0.156170	0.8958
6.5	0.133123	0.131908	0.9211
7.0	0.112510	0.111462	0.9402
7.5	0.095118	0.094220	0.9531
8.0	0.080436	0.079668	0.9640

In our previous paper[1], the following nonlinear equation

$$\dddot{x} + 8\ddot{x} + 19\dot{x} + 12x = 0.1x^3, \quad \ldots (51)$$

was solved with initial conditions $[x(0) = 1, \dot{x}(0) = -2, \ddot{x}(0) = 5.5]$. The characteristic roots of the linear equation of (51) are -4, -3 and -1. In accordance to unified theory, it was considered that $\lambda = 4$, $\mu = 2$ and $\omega = 1$. It is obvious that $\mu = 2\omega$. So, $\tilde{u}_1$ was not taken the simple form

$$\tilde{u}_1 = g_2(\xi, \eta)\cosh 2\varphi + h_2(\xi, \eta)\sinh 2\varphi + gh_3(\xi, \eta)\cosh 3\varphi + h_3(\xi, \eta)\sinh 3\varphi. \quad \ldots (52)$$

In[1], $\tilde{u}_1$ actually became

$$\tilde{u}_1 = -\frac{e^{\xi + 2\eta}}{1260} \times (34\cosh 2\varphi + 29\sinh 2\varphi) + \frac{e^{3\eta}}{1920}$$

$$\times((6\varphi-1)\cosh 3\varphi+(6\varphi-1)\sinh 3\varphi). \quad \ldots (53)$$

However one can verify that the equations for ξ, η and φ *i.e.*, (46) is exact form of (32) and they are similar to those obtained in[1] when $\lambda=4, \mu=2$ and $\omega=1$. In (53), $\tilde{\mu}_1$ contained the secular type terms $\varphi\cosh 3\varphi$ and $\varphi\sinh 3\varphi$. In according to our new asymptotic solution, we can choose $\lambda_1=4, \lambda_2=3$ and $\lambda_3=1$. Here, we have got the equality, $\lambda_2=3\lambda_3$ and we should compute $x(t)$ by (30) in which a, b and c are given by (29). We have solved (51) again with same initial conditions by the new method by choosing $\varepsilon = -0.1$, and the results are given in second column of **Table IV**. Corresponding numerical solution has been computed and given in third column of the **Table IV**. Then percentage errors have been calculated and given in fourth column. In this case, the new asymptotic solution shows a good coincidence with numerical solution. To compare the results with solution obtained in[1], the previous results have been given in fifth column (denoted by x^*) and their percentage errors have been given in sixth column. Though percentage errors of previous solution are less than 1%, but they are much greater than those obtained for the new solution, (30). It is noted that the ratio of the coefficients of x and x^3 in (51) is $\frac{\varepsilon}{12}$. So, errors should occur in an order of $\frac{\varepsilon^2}{144}$. Our new solution follows this order of accuracy, while the previous solution follows the order ε^2.

TABLE IV

t	x	x^*	$E(\%)$	x^*	$E^*(\%)$
0.0	1.00000000	1.00000000	0.000000	1.000000	0.0000
0.1	0.82475811	0.82475823	- 0.000015	0.824746	- 0.0015
0.2	0.68975127	0.68975186	- 0.000086	0.689684	- 0.0098
0.3	0.58414936	0.58415037	- 0.000173	0.583985	- 0.0283
0.4	0.50025105	0.50025254	- 0.000298	0.499968	- 0.0569
0.5	0.43255368	0.43255529	- 0.000372	0.432450	- 0.0243
0.6	0.37710252	0.37710437	- 0.000491	0.376961	- 0.0380
0.7	0.33103582	0.33103779	- 0.000595	0.330859	- 0.0540
0.8	0.29226553	0.29226759	- 0.000705	0.282059	- 0.0714
0.9	0.25925419	0.25925624	- 0.000791	0.259024	- 0.0896
1.0	0.23085757	0.23085940	- 0.000793	0.230610	- 0.1080
1.1	0.20621413	0.20621583	- 0.000824	0.205956	- 0.1260
1.2	0.18466690	0.18466842	- 0.000823	0.184404	- 0.1432
1.3	0.16570802	0.16570938	- 0.000821	0.165440	- 0.1626
1.4	0.14893912	0.14894034	- 0.000819	0.148679	- 0.1754
1.5	0.13404328	0.13404427	- 0.000739	0.133790	- 0.1897
1.6	0.12076456	0.12076544	- 0.000729	0.120521	- 0.2024
1.7	0.10889342	0.10889421	- 0.000725	0.108661	- 0.2142
1.8	0.09825592	0.09825664	- 0.000733	0.098036	- 0.2245
1.9	0.08870590	0.08870656	- 0.000744	0.088499	- 0.2340

The unified method is a generalization of KBM method and was covered over-damped and damped nonlinear systems. Sometimes, the solution brings secular type term, te^{-t} in correction terms, $\tilde{u}_1, \tilde{u}_2$ ... and then it gives incorrect results. But our new asymptotic solution, obtained again by KBM method, gives desired results in the case(s) where the previous method has been failed. One can check that we choose the values of m, α, β and γ such a way that the coefficients of x in (11) is 1. Therefore, ratio the coefficients of x and x^3 in (11) is ε (in each case), and errors of our new solution (30) in Tables 1-3 have been occurred in an order ε^2.

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