Indian J. pure appl. Math., 33(5) : 755-760, May 2002

DECOMPOSITIONS OF CONTINUITY AND COMPLETE CONTINUITY

E. HATIR

Selcuk Universitesi, Matematik Bölümü 420901, *Konya, Turkey*

AND

T. NOIRI

Department of Mathematics, Yatsushiro College of Technology, Yatsushiro, Kumamoto, 866-8501, *Japan*

(Received 9 November 2000; accepted 7 September 2001)

We introduce semipre-regular sets and the class of weak AB-sets as the intersection of an open set and a semipre-regular set. By using these sets, a new decomposition of continuity is provided. We obtain new decompositions of completely continuous functions.

Key Words : Semipre-regular; Weak AB-sets; α-gsp-closed; Continuity; Complete Continuity; Contra α-gsp-continuity, α-continuity.

1. INTRODUCTION

Recently, Hatir *et al.*[10] and Dontchev[8] have introduced C-sets and AB-sets, to obtain decompositions of continuity, respectively. In this paper we introduce the notion of semipre-regular sets and a new class of subsets, called weak AB-sets, which lies between the class of AB-sets and the class of C-sets. Using these sets we improve the following result :

Theorem A — *For a function $f : X \to Y$ the following are equivalent :*

(a) f is continuous,

(b) f is α-continuous and C-continuous (E. Hatir et al.[10]),

(c) f is pre-continuous and AB-continuous (Dontchev[8]).

We have also introduced notions of pre gsp-closed sets and α-gsp-closed sets to obtain new decompositions of completely continuous functions. We obtain some improvements of the following results using these sets:

Theorem B — *(Dontchev and Przemski[9]) — For a function $f : X \to Y$ the following are equivalent :*

(a) f is completely continuous,

(b) f is α-continuous and quasi sg-continuous.

2. Preliminaries

Throughout this paper, spaces always mean topological spaces. Let S be a subset of a topological space X. A subset S is said to be regular open (resp. regular closed) in X if $\mathrm{Int}(\mathrm{Cl}(S)) = S$ (resp. $\mathrm{Cl}(\mathrm{Int}(S)) = S$), where $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$ denote the closure and the interior of S.

First we shall recall some definitions used in the sequel.

Definition 2.1 — A subset S of a topological space X is said to be

(a) *α-open set*[15] if $S \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S)))$,

(b) *semi-open* set[12] if $S \subset \mathrm{Cl}(\mathrm{Int}(S))$,

(c) *β-open* set[1] or *semi-preopen*[2] if $S \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(S)))$,

(d) *α^*-set*[10] if $\mathrm{Int}(S) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S)))$,

(e) *C-set*[10], if $S = U \cap V$, where U is an open and V is α^*-set,

(f) *semi-regular* set[6] if it is semi-open and semi-closed,

(g) *AB-set*[8] if $S = U \cap R$, where U is an open and R is semi regular,

(h) *t-set*[17] if $\mathrm{Int}(S) = \mathrm{Int}(\mathrm{Cl}(S))$,

(i) *B-set*[17] if $S = U \cap T$, where U is an open and T is a t-set,

(j) *preopen* set[14] if $S \subset \mathrm{Int}(\mathrm{Cl}(S))$.

The *semi-preclosure*[2] of a subset S of a topological space X is defined as the intersection of all semi-preclosed sets of X containing S and is denoted by $\mathrm{sp\,Cl}(S)$. The *semi-preinterior*[2] of S, $\mathrm{spInt}(S)$, is defined as the union of all semipre-open sets of X contained in S. The semi-closure, $\mathrm{sCl}(S)$, and the semi-interior, $\mathrm{sInt}(S)$, are similarly defined.

Lemma 2.1 (Andrijevic[2]) — Let S be a subset of a topological space X. Then,

(a) $\mathrm{spCl}(S) = S \cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S)))$,

(b) $\mathrm{spInt}(S) = S \cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(S)))$,

(c) $\mathrm{sCl}(S) = S \cup \mathrm{Int}(\mathrm{Cl}(S))$,

(d) $\mathrm{sInt}(S) = S \cap \mathrm{Cl}(\mathrm{Int}(S))$.

Definition 2.2 — A function $f: X \to Y$ is said to be *AB*-continuous[8] (resp. C-continuous[10], β-continuous[1], α-continuous[16], pre-continuous[14]) if for each open set V of Y, $f^{-1}(V)$ is an AB-set (resp. C-set, β-open set, α-open set, pre-open set) of X.

3. Semipre-Regular Sets and Weak AB-Sets

Definition 3.1 — A subset S of a space X is said to be *semipre-regular* (or simply *sp-regular*) if it is both semi-preopen and semi-preclosed (that is, the complement of a semi-preopen set), equivalently if $S = \mathrm{spInt}(\mathrm{spCl}(S))$.

Remark 3.1 : Every semi-regular set is semipre-regular but not conversely.

Example 3.1 — Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}\}$ and $S = \{c\}$. Then S is sp-regular in (X, τ). But it is not semi-regular, not even semi-open.

Definition 3.2 — A subset S of a space X is called a *weak AB-set* if $S = U \cap V$, where U is an open set and V is $\bar{s}$p-regular. The collection of all weak AB-sets in X will be denoted by $wAB(X)$.

Remark 3.1 : For some subsets defined above, we have the following implications :

$$\begin{array}{ccccc} \text{semi-regular} & \rightarrow & \text{AB-set} & \rightarrow & \text{B-set} \\ \downarrow & & \downarrow & & \downarrow \\ \text{semipre-regular} & \rightarrow & \text{weak AB-set} & \rightarrow & \text{C-set} \end{array}$$

Example 3.2 — Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $S = \{c\}$. Then S is a C-set and even a B-set. But it is not a weak AB-set.

Remark 3.2 : In the Example 3.1, the set $S = \{c\}$ is a weak AB-set but not B-set. Therefore, by Examples 3.1 and 3.2, B-sets are independent from weak AB-sets.

Every weak AB-set is semi-preopen but not conversely as shown by the following example.

Example 3.3 — Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, c\}\}$ and $S = \{a, b\}$. Then S is a semi-preopen set, but it is not a weak AB-set.

Theorem 3.1 — *For a subset S of a space X the following are equivalent :*

(a) S is sp-regular,

(b) S is semi-preclosed and a weak AB-set.

PROOF : The proof is obvious since every weak AB-set is semi-preopen.

A topological space X is said to be *submaximal* if every dense subset of X is open. Let $SPO(X)$ denote the collection of all semipre-open subsets of X. Thus we can improve Dontchev's result: [8, Corollary 2.5].

Corollary 3.1 — If X is a submaximal space, then $wAB(X) = SPO(X)$.

PROOF : Since X is submaximal, every semi-preopen set of X is an AB-set[8] and hence a weak AB-set. Conversely, since every weak AB-set is semi-preopen, this completes the proof.

Theorem 3.2 — *For a subset S of a space X the following are equivalent :*

(a) S is an open,

(b) S is a weak AB-set and an α-open set.

PROOF : It is obvious that (a) implies (b). Conversely, let S be a weak AB-set and α-open. Since every weak AB-set is a C-set, it follows from Proposition 3.5 of[10] that S is open.

A topological space X is said to be *extremally disconnected* if every open subset of X has the open closure, or equivalently if regular closed set is open.

Theorem 3.3 — *For a topological space (X, τ) the following are equivalent:*

(a) X is extremally disconnected,

(b) $\tau = wAB(X)$,

(c) every weak AB-set is open.

PROOF : The proof is straightforward and is thus omitted.

4. WEAKLY AB-CONTINUOUS FUNCTIONS

Definition 4.1 — A function $f: X \rightarrow Y$ is said to be *weakly AB-continuous* if for each open subset V of Y, $f^{1}(V)$ is a weak AB-set in X.

As a consequence of Theorem 3.2, we obtain the following decomposition of continuity.

Theorem 4.1 — *For a function $f: X \to Y$ the following are equivalent:*

(a) f is continuous,

and (b) f is weak AB-continuous and α-continuous.

The following theorem is immediate consequences of results from the beginning of this paper.

Theorem 4.2 — *For a function $f: X \to Y$ the following hold:*

(a) Every AB-continuous function is weak AB-continuous,

(b) Every weak AB-continuous function is C-continuous,

and (c) Every weak AB-continuous function is β-continuous.

5. Decompositions of Complete Continuity

We recall the definitions of some modifications of generalized closed sets.

Definition 5.1 — A subset A of a topological space X is said to be

(a) *gs-closed*[4] (resp. *gsp*-closed)[7] if $\mathrm{sCl}(A) \subset U$ (resp. $\mathrm{spCl}\ (A) \subset U$ whenever $A \subset U$ and U is open,

(b) *sg-closed*[5] (resp. *spg-closed*[11]) if $\mathrm{sCl}(A) \subset U$ (resp. $\mathrm{spCl}(A) \subset U$) whenever $A \subset U$ and U is semi-open.

In 1995, Dontchev[7] obtained decompositions of regular open sets as follows :

Proposition 5.1 (Dontchev[7]) — For a subset A of a topological space X, the following are equivalent :

(a) A is regular open;

(b) A is open and *gs*-closed;

(c) A is open and gsp-closed.

In 1981, Maheshwari *et al.*[13] obtained a decomposition of regular open sets as follows: a set is regular open if and only if it is open and semi-regular. Dontchev and Przemski[9] improved their result as follows :

Proposition 5.2 (Dontchev and Przemski[9]) — For a subset A of a topological space X the following are equivalent :

(a) A is regular open;

(b) A is open and semi-regular;

(c) A is open and semi-closed;

(d) A is open and *sg*-closed;

(e) A is α-open and sg-closed.

Moreover, in order to improve the above results we define some new sets.

Definition 5.2 — A subset A of a topological space X is said to be *pre gsp-closed* (resp. *α-gsp-closed*) if $\mathrm{spCl}A \subset U$ whenever $A \subset U$ and U is pre-open (resp. α-open).

Remark 5.1 : For several subsets defined above, we have the following implications.

$$\begin{array}{ccccccc} \text{semi-closed} & \rightarrow & \text{sg-closed} & \rightarrow & \text{spg-closed} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{semi-preclosed} & \rightarrow & \text{pre-gsp-closed} & \rightarrow & \alpha\text{-gsp-closed} & \rightarrow & \text{gsp-closed} \end{array}$$

Theorem 5.1 — 2 *For a subset A of a topological space X the following conditions are equivalent :*

(a) A is a regular open;

(b) A is open and sp-regular;

(c) A is open and semi-pre-closed;

(d) A is open and pre-gsp-closed;

(e) A is α-open and spg-closed;

(f) A is α-open and pre-gsp-closed;

(g) A is α-open and α-gsp-closed.

PROOF : We prove only that (*g*) implies (*a*). Since every regular open set is semi-regular, it is obvious that (*a*) implies (*e*) and (*e*) implies (*g*). The other implications are obvious by Remark 5.1.

(*g*) $\Rightarrow$ (*a*) — Let A be an α-open and α-gsp-closed set. Then, we have $\text{spCl}(A) \subset A$ and hence A is semi-preclosed. Therefore, we obtain $\text{Int}(\text{Cl}(\text{Int}(A))) \subset A$. Since every α-open set is semi-open, $\text{Cl}(\text{Int}(A)) = \text{Cl}(A)$. Therefore, we obtain

$\text{Int}(\text{Cl}(A)) = \text{Int}(\text{Cl}(\text{Int}(A))) \subset A \subset \text{Int}(\text{Cl}(\text{Int}(A))) \subset \text{Int}(\text{Cl}(A))$.

This shows that A is regular open.

The following example shows that "α-gsp-closed" in Theorem 5.1 (g) can not be replaced by "gsp-closed".

Example 5.1 — Let $X = \{a, b, c, d\}$ $\tau = \{\phi, X, \{a\}\}$ and $A = \{a, b\}$. Then A is an α-open and gsp-closed set which is not regular open. Moreover A is not α-gsp-closed since $\text{spCl}(A) = X$.

The following example shows that "α-open" in Theorem 5.1 (f) can not be replaced by "preopen".

Example 5.2 — Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, d\}, \{a, b, d\}\}$ and $A = \{b\}$. Then A is a preopen and pre-gsp-closed set which is not regular open (not even open). For, $\text{Int}(\text{Cl}(A)) = \text{Int}(\{b, c, d\}) = \{b, d\} \supset A$ and A is preopen. Moreover $\text{spCl}(A)$ $= A \cup \text{Int}(\text{Cl}(\text{Int}(A))) = A$ and A is pre-gsp-closed. But A is not α-open.

In order to obtain decompositions of complete continuity, we introduce some new functions.

Definition 5.3 — A function $f: X \rightarrow Y$ is said to be *completely continuous*[3] (resp. *sp-perfectly continuous*) if for each open subset V in Y, $f^{-1}(V)$ is regular-open (resp. sp-regular) in X.

Definition 5.4 — A function $f: X \rightarrow Y$ is said to be *contra α-gsp-continuous* (resp. *contra pre-gsp continuous, contra spg-continuous*) if for each open set V in Y, $f^{-1}(V)$ is α-gsp-closed (resp. pre-gsp-closed, spg-closed) in X.

By Theorem 5.1, we obtain the following decompositions of complete continuity.

Theorem 5.2 — *For a function $f: X \rightarrow Y$, the following conditions are equivalent.*

(a) f is completely continuous;

(b) f is continuous and sp-perfect continuous;

(c) f is continuous and contra pre-gsp-continuous;

(d) f is α-continuous and contra spg-continuous;

(e) f is α-continuous and contra pre-gsp-continuous; and

(f) f is α-continuous and contra α-gsp-continuous.

Remark 5.2 : By the following example, α-continuity and contra α-gsp-continuity are independent concepts.

Example 5.3 — Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a, \{a, b\}\}$, $\sigma = \{\phi, X, \{a\}\}$ and $v = \{\phi, X\}$. Then (1) the identity function $f: (X, \tau) \to (X, \sigma)$ is α-continuous (even continuous) [9, Example 5.2]. But it is not contra α-gsp-continuous since $f^{-1}(\{a\}) = \{a\}$ is not α-gsp-closed. For $\{a\} \in \tau$ and $\{a\}$ is α-open however $\text{spCl}(\{a \to \{a\}\}) = \{a\} \cup \text{Int}(\text{Cl}(\text{Int}(\{a\}))) = \{a\} \cup \text{Int}(X) = X$ and hence $\text{spCl}(\{a\})$ is not contained in $\{a\}$.

(2) The identity function $g: (X, v) \to (X, \sigma)$ is sp-perfect and contra α-gsp-continuous. But g is not α-continuous [9, Example 5.2]. For $\{a\}$ is sp-regular, but it is not α-open in (X, V).

REFERENCES

1. M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, *Bull. Fac. Sci. Assiut Univ.* **12** (1983) 77-90.
2. D. Andrijevic, *Mat. Vesnik* **38** (1986) 24-32.
3. S. P. Arya and R. Gupta, *Kyungpook Math. J.* **14** (1974) 131-143.
4. S. P. Arya and T. Nour, *Indian J. pure appl. Math.* **21** (1990) 717-719.
5. P. Bhattacharya and B. K. Lahiri, *Indian J. Math.* **29** (1987) 375-382.
6. G. Di Maio and T. Noiri, *Indian J. pure appl. Math.* **18** (1987) 226-233.
7. J. Dontchev, *Mem. Fac. Sci. Kochi Univ. Ser. A, Math.* **16** (1995) 35-48.
8. J. Dotchev, Between *A and B-sets*, (archiv math. From internet?) 1998.
9. J. Dontchev and M. Przemski, *Acta Math. Hungar.* **71** (1996) 109-120.
10. E. Hatir, T. Noiri and S. Yuksel, *Acta Math. Hungar.* **70** (1996) 145-50.
11. M. K. R. S. V. Kumar, *Mem. Fac. Sci. Kochi Univ. Ser. A, Math.* **20** (1999) 33-46.
12. N. Levine, *Amer. Math. Monthly* **70** (1963) 36-41.
13. S. N. Maheshwari, G. I. Chae and S. S. Thakur, *Ulsan Inst. Tech. Rep.* **12** (1981) 141-45.
14. A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, *Proc. math. phys. Soc. Egypt* **53** (1982) 47-53.
15. O. Njastad, *Pacific J. Math.* **15** (1965) 961-970.
16. T. Noiri, *Casopis Pest Mat.* **109** (1984) 118-126.
17. J. Tong, *Acta Matha Hungar.* **54** (1989) 51-55.