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ON CONFORMAL TRANSFORMATION OF FINSLER SPACES WITH *m*th ROOT METRIC

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In this paper, we have considered the conformal changes of Finsler space with *m*th root metric. We have obtained the conformal change of Cartan's and Berwald's connections in terms of *m*th root metrics. The necessary and sufficient condition, under which a Berwald or a Landsberg space with *m*th root metric is conformally transformed to the spaces of the same character, has been discussed.

Key Words : Finsler Space; Conformal Change, *m*th Root Metric

1. INTRODUCTION

The theory of *m*th root metrics has been first developed by Shimada[7] as an interesting example of Finsler metrics, immediately following Matsumoto and Numata's theory of cubic metrics[4]. By introducing the regularity of the metric, various fundamental quantities as a Finsler metric have been found. In 1995, Matsumoto and Okubo[5] obtained the Berwald and Cartan connections of Finsler space with *m*th root metric and hence they obtained the equation of geodesics. They have introduced the higher order Christoffel symbols which have an important role in his theory.

On the other hand the conformal theory of Finsler metric based on the theory of Finsler spaces by Matsumoto[3] has been developed by Hashiguchi[1]. To treat the conformal theory of Finsler metrics, Knebelman[2] defined two metric functions L and $\bar{L}$ as conformal if the length of an arbitrary vector in the one is proportional to the length in the other i.e., $\bar{g}_{ij} = \phi\, g_{ij}$. The length of a vector ξ means $\{g_{ij}(x, y)\, \xi^i \xi^j\}^{1/2}$, where g_{ij} is the Finsler metric tensor. Hashiguchi gave a some what different definition for conformal transformation of Finsler metrics which coincides with Knebelman's one. He has obtained the conformal change of all the torsion and curvature tensors of the Cartan's, Rund's, Berwald's and Hashiguchi's connections.

2. Preliminaries

The mth root metric $L(x, y)$ of an n-dimensional differentiable manifold M^n is defined by *H.* Shimada[7] as

$$L(x, y)^m = a_{i_1, i_2 \dots i_m}(x)\, y^{i_1} y^{i_2} \dots y^{i_m}, \qquad \dots (2.1)$$

where the coefficients $a_{i_1 i_2 \dots i_m}(x)$ are components of a symmetric tensor field covariant of order m. Consequently, the second root metric is a Riemannian metric. Therefore, we shall restrict $m > 2$ throughout the paper. The manifold M^n equipped with metric (2.1) will be called Finsler space F^n with mth metric.

Let us first define the tensors $a_i(x, y)$, $a_{ij}(x, y)$ and $a_{ijk}(x, y)$ as follows :

$$\text{(a)}\ L^{m-1} a_i = a_{ij_1 j_2 \dots j_{m-1}} y^{j_1} y^{j_2} \dots y^{j_{m-1}},$$

$$\text{(b)}\ L^{m-2} a_{ij} = a_{ijk_1 k_2 \dots k_{m-2}} y^{k_1} y^{k_2} \dots y^{k_{m-2}}, \qquad \dots (2.2)$$

$$\text{(c)}\ L^{m-3} a_{ijk} = a_{ijkl_1 l_2 \dots l_{m-3}} y^{l_1} y^{l_2} \dots y^{1_{m-3}},$$

The normalized supporting element $l_i = \dot{\partial}_i L$, $(\dot{\partial}_i = \partial/\partial y^i)$, the angular metric tensor $h_{ij} = L(\dot{\partial}_i, \dot{\partial}_j L)$, the fundamental tensor $g_{ij} = \dot{\partial}_i \dot{\partial}_j (L^2/2)$ and the C-tensor $C_{ijk} = \dot{\partial}_i, \dot{\partial}_j \dot{\partial}_k (L^2/4)$ of F^n are written as

$$\text{(a)}\ l_i = a_i,\ \text{(b)}\ h_{ij} = (m-1)(a_{ij} - a_i a_j)$$

$$\text{(c)}\ g_{ij} = (m-1)\, a_{ij} - (m-2)\, a_i a_j \qquad \dots (2.3)$$

$$\text{(d)}\ 2LC_{.ijk} = (m-1)(m-2)(a_{ijk} - a_{ij} a_k - a_{jk} a_i - a_{ki} a_j + 2a_i a_j a_k).$$

We have the following relations among a_i, a_{ij} and a_{ijk}

$$\text{(a)}\ a_i y^i = L,\ a_{ij} y^j = La_i,\ a_{ijk} y^k = La_{ij}$$

$$\text{(b)}\ (a_{ij} - a_i a_j) y^j = 0,\ (a_{ijk} - a_{ij} a_k) y^k = 0 \qquad \dots (2.4)$$

$$\text{(c)}\ (a_{ijk} - a_{ij} a_k) y^i = L(a_{ij} - a_j a_k)$$

$$\text{(d)}\ L(\dot{\partial}_k a_{ij}) = (m-2)(a_{ijk} - a_{ij} a_k).$$

The tensor $a_{ij}(x, y)$ is called the basic tensor, because it plays an important role in the paper. The metric L is called regular if the basic tensor has the non-vanishing determinant.

The reciprocal $g^{ij}(x, y)$ of $g_{ij}(x, y)$ is given by

$$(m-1)\, g^{ij} = a^{ij} + (m-2)\, a^i a^j, \qquad \text{... (2.5)}$$

where $a^{ij}(x, y)$ is reciprocal of $a_{ij}(x, y)$ and $a^i = a^{ij} a_j = l^i$. Throughout this paper, we raise and lower the indices by the tensor a^{ij} and a_{ij}, for instance $a^k_{ij} = a^{kh} a_{hij}$.

We quote the following results which have been obtained in [5].

The coefficients $(F^i_{jk}, G^i_j, C^i_{jk})$ of Cartan's connection of a Finsler space F^n with **m**th root metric are given by

$$F^i_{jk} = a^{ih} \left\{ f_{jhk} - \frac{(m-2)}{2L} (a_{jhr} - a_{jh} a_r)\, G^r_k - \frac{(m-2)}{2L} (a_{hkr} - a_{hk} a_r)\, G^r_j \right.$$

$$\left. + \frac{(m-2)}{2L} (a_{jkr} - a_{jk} a_r)\, G^r_h \right\}, \qquad \text{... (2.6)}$$

$$G^i_j = \frac{(m-2)}{2(m-1)} a^{ih} \left\{ f_{jh0} + \frac{m}{m-2} f_{jhi} + \frac{1}{L^2} f_{hji}\, a_{jh} - \frac{1}{L} (f_{h00}\, a_j - f_{j00}\, a_h + f_{0r0}\, a^r_{jh}) \right\} \text{... (2.7)}$$

and

$$C^i_{jk} = \frac{m-2}{2L} \{ a^i_{jk} - (\delta^i_j a_k + \delta^i_k a_j) + a^i (2a_j a_k - a_{jk}) \}, \qquad \text{... (2.8)}$$

where

$$2f_{ijk} = \partial_k a_{ij} + \partial_i a_{jk} - \partial_j a_{ki}, \quad (\partial_k = \partial / \partial x^k) \qquad \text{... (2.9)}$$

and 0 denotes the contraction with y^k, for instance $f_{ij0} = f_{ijk} y^k$.

3. Conformal Change of F^n

Let $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L})$ be two Finsler space on the same underlying manifold M^n. If we have a function $\alpha(x)$ in each coordinate neighbourhood of F^n, such that $\overline{L}(x, y) = e^{\alpha} L(x, y)$, then F^n is called conformal to $\overline{F}^n$ and change $L \to \overline{L}$ of metric is called conformal.

As to the mth root metric $L(x, y)$ given by (2.1) we have

$$\overline{L}(x, y)^m = e^{m\alpha} a_{i_1 i_2 \dots i_m} y^{i_1} y^{i_2} \dots y^{i_m}.$$

This gives the following

Proposition 3.1 — By a conformal transformation, the Finsler space F^n with mth root metric is transformed to a Finsler spaces with **m**th root metric and we have

$$\overline{a}_{i_1 i_2 \dots i_m} = e^{m\alpha} a_{i_1 i_2 \dots i_m} \qquad \text{... (3.1)}$$

Throughout this paper, the quantities corresponding to the Finsler space $\overline{F}^n$ will be denoted by putting bar on the top of that quantity. From (2.2) and (3.1) we have

$$\overline{a}_i = e^{\alpha} a_i, \; \overline{a}_{ij} = e^{2\alpha} a_{ij}, \; \overline{a}_{ijk} = e^{3\alpha} a_{ijk}. \qquad \text{... (3.2)}$$

From this equation it follows that, if F^n is regular then $\overline{F}^n$ is also regular and we have

$$\overline{a}^{ij} = e^{-2\alpha} a^{ij}, \ \overline{a}^{i} = e^{-\alpha} a^{i}. \qquad \text{... (3.3)}$$

For a symmetric tensor field $a_{i_1 i_2 \dots i_m}(x)$ covariant of order m, the Christoffel symbols of mth order are defined by[5]

$$\{i_1 i_2 \dots i_m, j\} = \frac{1}{2(m-1)} (\partial_{i_1} a_{i_2 i_3 \dots i_m j} + \partial_{i_2} a_{i_3 i_4 \dots i_1 j}$$
$$+ \dots + \partial_{i_m} a_{i_1 i_2 \dots i_{m-1} j} - \partial_j a_{i_1 i_2 \dots i_m}), \qquad \text{... (3.4)}$$

where the cyclic permutation is applied to $(i_1 i_2 \dots i_m)$ in the first m terms of the right hand side.

From (3.1) and (3.4) we get the following law of transformation for the Christoffel symbols of mth order

$$\overline{\{i_1 i_2 \dots i_m, j\}} = e^{m\alpha}[\{i_1 i_2 \dots i_m, j\} + \frac{m}{2(m-1)} (\alpha_{i_1} a_{i_2 i_3 \dots i_m j}$$
$$+ \alpha_{i_2} a_{i_3 i_4 \dots i_1 j} + \dots + \alpha_{i_m} a_{i_1 i_2 \dots i_{m-1} j} - \alpha_j a_{i_1 i_2 \dots i_m})], \qquad \text{... (3.5)}$$

where $\alpha_i = \partial \alpha / \partial x^i$.

The differential equation of the geodesic curves is given by[5]

$$a_{ij} \frac{d^2 x^j}{ds^2} + \frac{2}{m} \{j_1 j_2 \dots j_m, i\} \frac{dx^{j_1}}{ds} \frac{dx^{j_2}}{ds} \dots \frac{dx^{j_m}}{ds} = 0. \qquad \text{... (3.6)}$$

If $\overline{s}$ denotes the arc length of a curve with respect to the metric function $\overline{L}(x, y)$, then from (3.1) we have

$$\frac{d\overline{s}}{ds} = e^{\alpha}, \qquad \text{... (3.7)}$$

which gives

$$\frac{d^2 x^j}{d\overline{s}^2} = e^{-2\alpha} \left(\frac{d^2 x^j}{ds^2} - \alpha_0 \frac{dx^j}{ds} \right), \ \alpha_0 = \alpha_i \frac{dx^i}{ds}. \qquad \text{... (3.8)}$$

From (3.3), (3.5), (3.7) and (3.8), we get

$$\overline{a}_{ij} \frac{d^2 x^j}{d\overline{s}^2} + \frac{2}{m} \overline{\{j_1 j_2 \dots j_m, i\}} \frac{dx^{j_1}}{d\overline{s}} \frac{dx^{j_2}}{d\overline{s}} \dots \frac{dx^{j_m}}{d\overline{s}} \qquad \text{... (3.9)}$$

$$= a_{ij} \frac{d^2 x^j}{ds^2} + \frac{2}{m} \{j_1 j_2 \dots j_m, i\} \frac{dx^{j_1}}{ds} \frac{dx^{j_2}}{ds} \dots\dots \frac{dx^{j_m}}{ds} - \frac{1}{m-1} \alpha_0 \alpha_i,$$

where we have used the fact that

$$a_{i_1 i_2 \dots i_m} \frac{dx^{i_1}}{ds} \frac{dx^{i_2}}{ds} \dots \frac{dx^{i_m}}{ds} = 1, \; a_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 1.$$

From (3.6) and (3.9) we get the following well known[6] theorem for a Finsler space.

Theorem 3.1 — *The conformal transformation leaves the geodesic of mth root metric is invariant, if and only if the transformation is homothetic.*

4. Conformal Change of Finsler Connections

To find the conformal change of Cartan's connection $C\Gamma = \left\{ F^i_{jk}, G^i_j, C^i_{jk} \right\}$, we first find the conformal change of f_{ijk} which is obtained from (2.9) and (3.2)

$$\bar{f}_{ijk} = e^{2\alpha} (f_{ijk} + a_{ij}\, \alpha_k + a_{jk}\, \alpha_i - a_{ki}\, \alpha_j). \quad \dots (4.1)$$

Contracting (4.1) with y^k, y^j and y^i successively and using (2.4), we get

$$\text{(a)}\ \bar{f}_{ij0} = e^{2\alpha} (f_{ij0} + \alpha_0\, a_{ij} + L\, \alpha_i\, a_j - L\, \alpha_j\, a_i),$$

$$\text{(b)}\ \bar{f}_{i00} = e^{2\alpha} (f_{i00} + L^2\, \alpha_i), \quad \dots (4.2)$$

$$\text{(c)}\ \bar{f}_{000} = e^{2\alpha} (f_{000} + L^2\, \alpha_0).$$

Again contracting (4.1) with $y^i y^k$, we get

$$\bar{f}_{0j0} = e^{2\alpha} (f_{0j0} + 2\, \alpha_0\, a_j - L^2\, \alpha_j). \quad \dots (4.3)$$

From (3.2), (3.3), (4.1), (4.2), (4.3) and (2.7) we obtain the conformal change of G^i_j as follows :

$$\bar{G}^i_j = G^i_j + B^i_j, \quad \dots (4.4)$$

where

$$B^i_j = \frac{L}{2(m-1)} \left\{ (m-2)\, a^i_{\alpha j} + m\, (L^{-1}\, \alpha_0\, \delta^i_j + a^i\, \alpha_j - \alpha^i\, a_j) \right\} \quad \dots (4.5)$$

and index α stands for contraction with α^r, i.e., $a^i_{\alpha j} = a^i_{rj}\, \alpha^r$.

Furthermore from (2.6), (3.2), (3.1), (4.1) and (4.4) we get the conformal change of F^i_{jk} given by

$$\bar{F}^i_{jk} = F^i_{jk} + U^i_{jk}, \quad \dots (4.6)$$

where $$U^i_{jk} = (\delta^i_j \alpha_k + \delta^i_k \alpha_j - a_{jk} \alpha^i) + [(m-2)^2 (a^t_{\alpha j} a^i_{kt} + a^t_{\alpha k} a^i_{jt} - a^i_{\alpha t} a^t_{jk})$$

$$- m (m-2) \{L^{-1} \alpha_0 a^i_{jk} - (a^i_{\alpha j} a_k + a^i_{\alpha k} a_j - a_{\alpha jk} a^i)\}$$

$$+ (m-2) (\delta^i_j a_\alpha a_k + \delta^i_k a_{\alpha j} - a_{jk} \alpha^i)]/4(m-1).$$

Also from (2.8), it follows that

$$\overline{C}^i_{jk} = C^i_{jk}. \quad \text{... (4.8)}$$

Summarising these results, we get

Proposition 4.1 — The conformal change of Cartan's connection $C\Gamma$ of Finsler space with mth root metric are given by (4.6), (4.4) and (4.8).

To find the conformal change of Berwald's connection $B\Gamma = \left\{ G^i_{jk}, G^i_j, 0 \right\}$, we notice that $G^i_{jk} = \partial_k G^i_j$, the non linear connection G^i_j is same as that of $C\Gamma$. From (4.4) we get

$$\overline{G}^i_{jk} = G^i_{jk} + B^i_{jk}, \quad \text{.. (4.9)}$$

where $$B^i_{jk} = \dot\partial_k B^i_j. \quad \text{... (4.10)}$$

To find the value of B^i_{jk} explicitly in terms of metric, we obtained that

$$\dot\partial_k \alpha^i = (m-2) L^{-1} (\alpha^i a_k - a^i_{rk} \alpha^r),$$

$$\dot\partial_k a_i = (m-1) L^{-1} (a_{ik} - a_i a_k), \quad \text{... (4.11)}$$

$$\dot\partial_k a^{ij} = (m-2) L^{-1} (a^{ij} a_k - a^{ij}_k)$$

$$\dot\partial_k a^i_{rj} = (m-3) L^{-1} (a^i_{rjk} - a^i_{rj} a_k) - (m-2) L^{-1} (a^i_{hk} a^h_{rj} - \delta^i_j a_k)$$

where $$L^{m-4} a_{ijkl} = a_{ijklh_1 h_2 \ldots h_{m-4}} y^{h_1} y^{h_2} \ldots y^{h_{m-4}}.$$

Thus from (4.5) and (4.10), we get

$$B^i_{jk} = [(m-2) \alpha^r \{(m-3) a^i_{rjk} + m (a^i_{rj} a_k + a^i_{rk} a_j) - (m-2) (a^i_{hk} a^h_{rj} + a^i_{hj} a^h_{rk})\}$$

$$+ m \{\alpha_j \delta^i_k + \alpha_k \delta^i_j - (m-1) a_{jk} \alpha^i - (m-2) a_j a_k \alpha^i\}]/2(m-1). \quad \text{... (4.12)}$$

Proposition 4.2 — The conformal change of Berwalds connection $B\Gamma$ of Finsler space with ***m***th root metric are given by (4.9) and (4.4), where B^i_j and B^i_{jk} are given by (4.5) and (4.12).

5. Conformal Change of Some Special Finsler Spaces with *m*th Root Metric

A Finsler space of dimension $n \geq 4$ is called S3-like[3] if there exists a scalar S such that the v-curvature tensor S_{hijk} of Cartan's connection $C\Gamma$ is written in the form

$$L^2 S_{hijk} = S\,(h_{hj}\, h_{ik} - h_{hk}\, h_{ij}).$$

As for the Finsler space with *m*th root metric, we have :

Proposition 5.1[1] — Let $F^n\,(n \geq 4)$ be a Finsler space with the *m*th root metric (2.1). If the tensor $U_{hijk} = (a^r_{hk}\, a_{rij} - a^r_{hj}\, a_{rik})$ is written in form

$$U_{hijk} = \lambda\,(h_{hj}\, h_{ik} - h_{hk}\, h_{ij}) \qquad \text{... (5.1)}$$

then F^n is an S3-like Finsler space.

From (2.3) and (3.2), it follows that $\bar{h}_{ij} = e^{2\alpha} h_{ij}$, $\bar{a}^i_{jk} = e^{\alpha} a^i_{jk}$ and

$$\bar{U}_{hijk} = e^{4\alpha}\, U_{hijk}. \qquad \text{... (5.2)}$$

Thus condition (5.1) in proposition (5.1) is equivalent to the condition

$$\bar{U}_{hijk} = \lambda\,(\bar{h}_{hj}\, \bar{h}_{ik} - \bar{h}_{hk}\, \bar{h}_{ij}).$$

In view of proposition (3.1) this condition gives the following :

Theorem 5.1 — *Let $F^n\,(n \geq 4)$ be a Finsler space with mth root metric (2.1), then the condition (5.1) is invariant under any conformal transformation and under the condition (5.1) and S3-like Finsler space is conformally transformed to an S3-like Finsler space.*

Next we discuss the conformal change of Berwald and Landsberg spaces. The Finsler space F^n with the ***m***th root metric (2.1) is a Berwald space (resp. Landsberg space) if and only if $a_{ijklh} = 0$ (resp. $a_{ijkl0} = 0$)[7]. Here short solidus stands for the h-covariant differentiation with respect to Cartan's connection $C\Gamma$. From (3.2), (4.4) and (4.6) it follows that

$$\bar{a}_{ijklh} = e^{3\alpha}\,[a_{ijklh} + 3a_{ijk}\, \alpha_h - (\dot{\partial}_r\, a_{ijk})\, B^r_h - a_{rjk}\, U^r_{ih} - a_{irk}\, U^r_{jh} - a_{ijr}\, U^r_{kh}]. \qquad \text{... (5.3)}$$

Since $\qquad \dot{\partial}_r\, a_{ijk} = (m-3)\, L^{-1}\,(a_{ijkr} - a_{ijk}\, a_r)$, where

we have the following theorems :

Theorem 5.2 — *A Berwald space with mth root metric is conformally transformed to a Berwald space, if and only if*

$$3a_{ijk}\,\alpha_h + (m-3)\,L^{-1}\,(a_{ijkr} - a_{ijk}\,a_r) - a_{rjk}\,U^r_{ih} - a_{irk}\,U^r_{jh} - a_{ijr}\,U^r_{kh} = 0$$

Theorem 5.3 — *A Landsberg space with mth root metric is conformally transformed to a Landsberg space, if and only if*

$$3\,\alpha_0\,a_{ijk} + (m-3)\,L^{-1}\,(a_{ijkr} - a_{ijk}\,a_r)\,B^r_0 - a_{rjk}\,U^r_{i0} - a_{irk}\,U^r_{j0} - a_{ijr}\,U^r_{k0} = 0.$$

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