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JAYASRI SUMMABILITY

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A summability matrix of Jayasri is used to construct a sequence of positive linear operators. Convergence of these operators to continuous real functions is studied and quantitative estimates are given. Examples are discussed.

Key Words : Summability; Approximation; Positive Linear Operators

1. INTRODUCTION

Let $J = (a_{nk})$ be the matrix defined by

$$a_{00} = 1,\ a_{0k} = 0 \text{ for } k > 0, \qquad \ldots (1.1)$$

and $$\prod_{\nu=1}^{n} (f_\nu(z)\, h_\nu + 1 - h_\nu) = \sum_{k=0}^{\infty} a_{nk} z^k,$$

where $\{f_\nu\}$ is a sequence of entire functions and $\{h_\nu\}$ is a sequence of complex numbers.

This matrix was introduced by Jayasri[8] who studied many of its properties as a method of summability. The results are of considerable interest since, by means of specific choices of $\{f_\nu\}$ and $\{h_\nu\}$, the matrix reduces to any one of several well-known summability matrices. Included in these specializations are : the (f, d_n) matrix[14], the (F, d_n) matrix and the Lototsky matrix[7]. Jayasri pointed out that both (f, d_n) and (F, d_n) are generalizations of the Lototsky matrix and he referred to his new method as "generalized Lotosky summability" In the following, the matrix given by (1.1) will be denoted by $J = J(f_\nu, h_\nu)$ and will be called the *Jayasri matrix*.

Another special case of the Jayasri matrix is the Euler matrix $A = (a_{nk})$ given by

$$a_{nk} = \begin{cases} \binom{n}{k} r^k (1-r)^{n-k}, & 0 \le k \le n, \\ 0, & n < k, \end{cases} \qquad \ldots (1.2)$$

where r is a complex constant. The Euler matrix plays an important role in summability theory and

many of its properties are known[1]. Moreover, the Euler matrix appears in approximation theory as the kernel of the famous nth Bernstein polynomial $B_n(g)$, associated with a real function g defined on [0, 1].

$$B_n(g)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} g\left(\frac{k}{n}\right), \qquad \ldots (1.3)$$

where $0 \le x \le 1$. It is wellknown[13] that

$$\lim_{n \to \infty} B_n(g)(x) = g(x),$$

if g is continuous. Thus the Bernstein polynomials — and, indirectly, the Euler matrix — provide a constructive proof of the classical Weierstrass approximation theorem. The Bernstein polynomials are examples of positive linear operators. This means that $L_n(g)$ is defined on $[a, b]$ whenever the same is true for g, L_n is linear, and $L_n(g) \ge 0$ whenever $g \ge 0$. Of major interest in the theory of positive linear operators is the study of conditions which will ensure that

$$\lim_{n \to \infty} L_n(g)(x) = g(x). \qquad \ldots (1.4)$$

Since the Jayasri matrix provides a sweeping generalization of the Euler matrix, it is to be expected, that it can be used to provide generalizations of the Bernstein polynomials for which (1.4) holds. This can be easily done and theorems having (1.4) as conclusion are given in Section 2. Quantitative estimates appear in Section 3. Some examples of specific operators are given in Section 4. Regularity conditions are considered in Section 5.

The Euler matrix also appears in probability theory as the binomial distribution:

$$P(X_{n,p} = k) = \binom{n}{k} p^k (1-p)^{n-k}, \; k = 0, 1, \ldots, n,$$

where $X_{n,p}$ counts the number of heads appearing in n tosses of a coin which lands heads on each toss with probability p. In certain cases the Jayasri matrix — and the associated positive linear operators — may be given an analogous probabilitic interpretation. A brief discussion of these ideas is given in Section 6.

2. Positive Linear Operators

Two of Jayasri's basic assumptions in a study[8] of $J(f_\nu, h_\nu) = (a_{nk})$ are that the Maclaurin series for f_ν has nonnegative coefficients and that $0 \le h_\nu \le 1$ for $\mu = 1, 2, \ldots$. If, in addition, $h_\nu = h_\nu(x)$ is allowed to be a function of x with $0 \le h_\nu(x) \le 1$, $\nu = 1, 2, \ldots$ then the generating functions in (1.1) becomes

$$\prod_{\nu=1}^{n} (f_\nu(z)\, h_\nu(x) + 1 - h_\nu(x)) = \sum_{k=0}^{\infty} a_{nk}(x)\, z^k, \qquad \ldots (2.1)$$

with $a_{nk}(x) \ge 0, k = 0, 1, \ldots, n = 0, 1, \ldots$.

These observations lead to an easy construction of a sequence of positive linear operators which will be called *Jayasri operators* in the sequel.

Theorem 2.1 — *Let* $(a_{nk}(x))$ *be given by* (2.1). *Let*

(i) f_ν *be an entire function for* $\nu = 1, 2, \ldots,$

(ii) $f_\nu(1) = \nu = 1, 2, \ldots,$

(iii) $f_\nu^{(k)}(0) \geq 0, \nu = 1, 2, \ldots, k = 0, 1, 2, \ldots,$

(iv) $h_\nu = h_\nu(x)$ *be defined on* $[0, 1]$, $\nu = 1, 2, \ldots,$ *and*

(v) $0 \leq h_\nu(x) \leq 1, \nu = 1, 2, \ldots$ *and* $0 \leq x \leq 1.$

Let g *be a real-valued function which is bounded on* $[0, \infty)$ *and continuous on* $[0, 1]$. *Let*

$$J_n(g)(x) = \sum_{k=0}^{\infty} a_{nk}(x)\, g\left(\frac{k}{n}\right), n = 0, 1, \ldots \qquad \ldots (2.2)$$

Then $\lim_{n \to \infty} J_n(g)(x) = g(x)$

for $0 \leq x \leq 1$ *if*

$$(a) \lim_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n} f'_\nu(1)\, h_\nu(x) = x,$$

$$(b) \lim_{n \to \infty} \frac{1}{n^2} \sum_{\nu=1}^{n} f''_\nu(1)\, h_\nu(x) = 0, \text{ and}$$

$$(c) \lim_{n \to \infty} \frac{1}{n^2} \sum_{\nu=1}^{n} (f'_\nu(1)\, h_\nu(x))^2 = 0.$$

PROOF : The operators J_n, given by (2.2), are linear and, because of (iii) and (iv) $J_n(g) \geq 0$ whenver $g \geq 0$. Korovkin's Theorem [12, p. 14] ensures that,

$$\lim_{n \to \infty} J_n(g)(x) = g(x)$$

for each $g \in C[0, 1], x \in [0, 1]$ provided

$$\lim_{n \to \infty} J_n(e_k)(x) = e_k(x),$$

for the three functions $e_k(x) = x^k$, $k = 0, 1, 2.$

Fix $x \subset [0, 1]$ and let

$$P_n(z) = \prod_{\nu=1}^{n} [f_\nu(z)\, h_\nu(x) + 1 - h_\nu(x)]. \qquad \ldots (2.3)$$

Then $$L_n(e_0)(x) = \sum_{k=0}^{\infty} a_{nk}(x) = P_n(1) = 1, \qquad \ldots (2.4)$$

since (ii) holds. Also

$$\log P_n(z) = \sum_{\nu=1}^{n} \log (f_\nu(z)\, h_\nu(x) + 1 - h_\nu(x)),$$

so that $$P_n'(z) = P_n(z) \sum_{\nu=1}^{n} \frac{f_\nu'(z)\, h_\nu(x)}{f_\nu(z)\, h_\nu(x) + 1 - h_\nu(x)}$$

when the differentiation is with respect to z.

Eq. (2.1) gives

$$P_n'(z) = \sum_{k=0}^{\infty} k a_{nk}(x)\, z^{k-1}.$$

Therefore, $$J_n(e_1)(x) = \frac{1}{n} P_n'(1)$$

or $$J_n(e_1)(x) = \frac{1}{n} \sum_{\nu=1}^{n} f_\nu'(1)\, h_\nu(x). \qquad \ldots (2.5)$$

A second differentiation gives, analogously,

$$J_n(e_2)(x) = \frac{P_n''(1)}{n^2} + \frac{P_n'(1)}{n^2}$$

or $$J_n(e_2)(x) = \frac{1}{n^2} \left(\sum_{\nu=1}^{n} f_\nu'(1)\, h_\nu(x) \right)^2 - \frac{1}{n^2} \sum_{\nu=1}^{n} (f_\nu'(1)\, h_\nu(x))^2$$

$$+ \frac{1}{n^2} \sum_{\nu=1}^{n} f_\nu''(1)\, h_\nu(x) + \frac{1}{n^2} \sum_{\nu=1}^{n} f_\nu'(1)\, h_\nu(x). \qquad \ldots (2.6)$$

Eqs. (2.4), (2.5) and (2.6) and hypotheses (a), (b) and (c) give

$$\lim_{n \to \infty} J_n(e_k)(x) = e_k(x), \quad k = 0, 1, 2$$

and Theorem 2.1 is proved.

Corollary 2.2 — If $0 \leq f''_v(1) \leq f'_v(1) \leq 1$, $v = 1, 2, \ldots$, in addition to (i), (ii), (iii) and (iv), then

$$\lim_{n \to \infty} J_n(g)(x) = g(x)$$

for $0 \leq x \leq 1$ provided only

$$\text{(a)} \lim_{n \to \infty} \frac{1}{n} \sum_{v=1}^{n} f'_v(1)\, h_v(x) = x.$$

PROOF : Suppose (a) holds and $0 \leq f''_v(1) \leq f'_v(1) \leq 1$, $v = 1,2, \ldots$ Then

$$0 \leq \frac{1}{n^2} \sum_{v=1}^{n} f''_v(1)\, h_v(x) \leq \frac{1}{n^2} \sum_{v=1}^{n} f'_v(1)\, h_v(x)$$

and the last expression tends to zero. So hypotheses (b) of Theorem 2.1 is valid.

Also

$$0 \leq \frac{1}{n^2} \sum_{v=1}^{n} (f'_v(1)\, h_v(x))^2 \leq \frac{1}{n^2} \sum_{v=1}^{n} f'_v(1)\, h_v(x).$$

Thus hypothesis (c) of Theorem 2.1 also holds and the corollary is proved.

Hypothesis (a) can be expressed in terms of Cesàro summability, namely, that the sequence $\{f'_v(1)\, h_v(x)\}$ is $(C, 1)$-summable to x. Since the $(C, 1)$ method is regular [6, p. 109], i.e., transforms convergent sequences into convergent sequences and preserves limits, hypothesis (a) holds if

$$\lim_{v \to \infty} f'_v(1)\, h_v(x) = x. \qquad \ldots (2.7)$$

Corollary 2.3 — If $f'_v(1) = 1$, $v = 1, 2, \ldots$, $\{f''_v(1)\}$ is a bounded sequence and if (i), (ii), (iii) and (iv) hold then

$$\lim_{n \to \infty} J_n(g)(x) = g(x)$$

for $0 \leq x \leq 1$ provided the sequence $\{h_v(x)\}$ is $(C, 1)$-summable to x.

PROOF : The $(C, 1)$-summability hypothesis says

$$\lim_{n \to \infty} \frac{1}{n} \sum_{v=1}^{n} h_v(x) = x.$$

Thus (a) holds since $f'_v(1) = 1$, $v = 1, 2, \ldots$ Then (b) and (c) hold because $|f''_v(1)| \leq M$ for each $v \geq 1$ and some M and $0 \leq h_v(x) \leq 1$, for $v \geq 1$. So the hypothesis of Theorem 2.1 hold and the corollary is proved.

Theorem 2.1 and the corollaries deal with pointwise convergence of $\{J_n(g)(x)\}$ to $g(x)$. However, Korovkin's Theorem also gives uniform convergence provided the convergence hypotheses hold uniformly.

3. ESTIMATES

Let $\{L_n\}$ be a sequence of positive linear operators on $C[a, b]$, $g \in C[a, b]$, and $x \in [a, b]$. It is known [4, p. 30] that

$$|L_n(g)(x) - g(x)| \leq \omega(g, \delta)\left[L_n(e_0)(x) + \frac{1}{\delta}(L_n(e_0)(x))^{1/2}\alpha_n(x)\right] + g(x)|L_n(e_0)(x) - e_0(x)|, \quad \ldots (3.1)$$

where $\omega(g, \delta)$ is the modulus of continuity of g and $\alpha_n^2(x) = L_n(\phi_t)(x)$, with $\phi_t(x) = (t-x)^2$.

Applying (3.1) to the Jayasri operators gives

Theorem 3.1 — *Let $\{J_n\}$ be the sequence of Jayasri operators satisfying the hypotheses of Theorem* 2.1. *Let g be a real-valued function which is bounded on $[0, \infty)$ and continuous on $[0, 1]$. Then for $x \in [0, 1]$,*

$$|J_n(g)(x) - g(x)| \leq \omega(g, \delta)\left[1 + \frac{\alpha_n(x)}{\delta}\right]$$

where $\omega(g, \delta)$ is the modulus of continuity of g and

$$\alpha_n^2(x) = \frac{1}{n^2}\left(\sum_{\nu=1}^{n} f'_\nu(1)\, h_\nu(x)\right)^2 - \frac{1}{n^2}\sum_{\nu=1}^{n}(f'_\nu(1)\, h_\nu(x))^2 + \frac{1}{n^2}\sum_{\nu=1}^{n} f''_\nu(1)\, h_\nu(x) + \frac{1}{n^2}\sum_{\nu=1}^{n} f'_\nu(1)\, h_\nu(x) - \frac{2x}{n}\sum_{\nu=1}^{n} f'_\nu(1)\, h_\nu(x) + x^2.$$

PROOF : The proof follows easily from (2.4), (2.5), (2.6) and (3.1) since, here,

$$\alpha_n^2 = J_n(\phi_t)(x)$$

$$= J_n(e_2)(x) - 2xJ_n(e)(x) + x^2.$$

4. EXAMPLES

In the following $J(f_\nu, h_\nu) = (a_{nk})$ is the Jayasri matrix given by (2.1) and $\{J_n\}$ is the associated sequence of positive linear operators given by (2.2).

Example 4.1 — Let $f_\nu(z) = z$ and $h_\nu(x) = x$, $\nu = 1,2, \ldots$ Then $J(f_\nu, h_\nu)$ is the Euler matrix, $E(x)$, and $J_n(g)(x) = B_n(g)(x)$, the nth Bernstein polynomial.

In this case Theorem 3.1 gives

$$|B_n(g)(x) - g(x)| \le \omega(g, \delta)\left[1 + \frac{1}{\delta}\sqrt{\frac{x(1-x)}{n}}\right].$$

Then $\delta = \frac{1}{\sqrt{n}}$ gives $|B_n(g)(x) - g(x)| \le \frac{3}{2}\omega\left(g, \frac{1}{\sqrt{n}}\right)$,

a known result of Popoviciu [2, p. 297].

Example 4.2 — Let $f_\nu(z) = z$ and $h_\nu = h_\nu(x)$ with $0 \le h_\nu(x) \le 1$ for $0 \le x \le 1$ and $\nu = 1, 2, \ldots$ Then, under the transformation $d_\nu = h_\nu(1 - h_\nu)^{-1}$, $J(f_\nu, h_\nu)$ becomes the generalized Lototsky matrix (F, d_n)[7]. The associated operators $J_n(g)(x) = F_n(g)(x)$ were introduced by King[8] who showed

$$\lim_{n \to \infty} F_n(g)(x) = g(x)$$

for $g \in C[0, 1]$ provided $\{h_\nu(x)\}$ in $(C, 1)$-summable to x, $0 \le x \le 1$.

The operators, F_n, are called Lototsky-Bernstein operators by Eisenberg and Wood[3] who extended them to the complex plane. Atomare and Campetti[2] refer to F_n as Bernstein-King operators.

5. REGULARITY OF THE JAYASRI MATRIX

Jayasri proved that, when $\{h_\nu\}$ is a sequence of real numbers, then $J(f_\nu, h_\nu) = (a_{nk})$ is a regular summability matrix provided hypotheses (i), (ii), (iii) and (v) of Theorem 2.1 hold provided

(vi) $\lim_{\nu \to \infty} \sup f_\nu(0) < 1$,

(vii) $\sum_{\nu=1}^{\infty} h_\nu = \infty$.

(Jayasri also proved that, in the presence of (i), (ii), (iii), (v) and (vi), hypothesis (vii) is also a necessary condition for regularity of $J(f_\nu, h_\nu)$.

Jayasri's proof involves rather sophisticated analysis i.e. of the use of (the somewhat unnatural) hypothesis (vi) to show

$$\lim_{n \to \infty} a_{nk} = 0, \quad k = 0, 1, 2, \ldots,$$

which is one of the three well-known Silvaman-Toeplitz conditions for regularity of (a_{nk}).

Moreover, Jayasri's hypothesis (vii) may require proper interpretation. To see this consider :

Example 5.1 — For each $n = 1, 2, \ldots$, take $f_\nu(z) = z$ and $h_\nu = r_n$ for $\nu = 1, 2, \ldots, n$. Then the Jayasri matrix $J(f_\nu, h_\nu) = (a_{nk})$ becomes

$$a_{nk} = \begin{cases} \binom{n}{k} r_n^k (1 - r_n)^{n-k}, & 0 \le k \le n \\ 0, & n \le k. \end{cases}$$

This is the $E(r_n)$-matrix introduced by Wuyts-Torfs[15]. In this case the infinite series in hypothesis (vii) has no meaning.

The approach to the Jayasri matrix via the associated positive linear operators (in Section 2) gives sufficient conditions for regularity which avoid both (vi) and (vii). A theorem of King and Swetits[11] gives an immediate consequence to Theorem 2.1.

Corollary 5.2 — When hypotheses (i), (ii), (iii), (iv), (v), (a), (b), and (c) of Theorem 2.1 hold, $J(f_\nu, h_\nu(x))$ is a regular summability matrix for $0 < x \le 1$.

It follows, of course, that the hypotheses of Corollary 2.2 or of Corollary 2.3 also imply that $J(f_\nu, h_\nu(x))$ is regular for $0 < x \le 1$.

Example 5.3 — Let $f_\nu(z) = z$, $h_\nu = h_\nu(x)$, with $0 \le h_\nu(x) \le 1$ for $\nu = 1, 2, \ldots$ and $x \in [0, 1]$. It follows from Corollary 2.3 and the above-mentioned theorem of King and Swetits that $J(f_\nu, h_\nu(x))$ is regular for $0 < x \le 1$ provided

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n} h_\nu(x) = x.$$

In the special case, $J(f_\nu, h_\nu(x))$ becomes the matrix studied by King in[8]. For $h_\nu(x) = x$, $\nu = 1, 2, \ldots$ the matrix specializes further to the Euler matrix $E(x)$ which is known to be regular for $0 < x \le 1$.

6. PROBABILITY

Let $J_n(g)(x)$ be the nth Jayasri operator given by (2.2). Hypotheses (i), (ii), (iii), (iv) and (v) of Theorem 2.1 give

$$a_{nk}(x) \ge 0, \quad k \ge 0, n \ge 0 \qquad \ldots (6.1)$$

and

$$\sum_{k=0}^{\infty} a_{nk}(x) = 1, \quad n \ge 0. \qquad \ldots (6.2)$$

Thus, for each $n = 0, 1, 2, \ldots$ the number $a_{nk}, k = 0, 1, 2, \ldots$ may be considered as probabilities. It follows that for each $x \in [a, b]$ and each $n = 0, 1, 2, \ldots$ there exists a random variable $X_{n,x}$ such that (see, in particular, [10] or [2, Sec. 5.2]

$$J_n(g)(x) = E(g(X_{n\,x})), \qquad \ldots (6.3)$$

where E denotes the expectation operator.

It is now possible to express inequalities such as (3.1) in probabilistic terms. For example [9]:

$$| J_n(g)(x) - g(x) | \leq \left[1 + \frac{1}{\delta} \sqrt{V(X_{n,x}) + (E(X_{n,x}) - x)^2} \right] \omega(g, \delta), \qquad \ldots (6.4)$$

where V denotes the variance operator.

In certain special cases the random variables have illuminating physical interpretations. Consider, for example, a biased coin which lands heads with probability $x \in [0, 1]$. Let p denote the probability of heads on a single toss. Toss the coin n times and let Y_n denote the number of heads which appear.

Example 6.1 — Let $p = x$ and $X_{n,x} = \frac{Y_n}{n}$. Then

$$J_n(g)(x) = E(g(X_{n,k}))$$

is the nth Bernstein polynomial.

Example 6.2. — Let the probability p vary from trial-to-trial; say $p = h_i(x)$, $i = 1, 2, \ldots, n$. Let $X_{n,k} = \frac{Y_n}{n}$. Then $J_n(g)(x)$ becomes the generalized Bernstein polynomial studied in [8] and [3].

Example 6.3. — Let p depend only on the number of tosses; say $p = r_n(x)$. Let $X_{n,x} = \frac{Y_n}{n}$. In this case the Jayasri summability matrix becomes the $E(r_n(x))$ matrix of Wuyts-Torfs [15] and the associated positive linear operators are [5, p. 77]

$$J_n(g)(x) = \sum_{k=0}^{n} \binom{n}{k} (r_n(x))^k (1 - r_n(x))^{n-k} g\left(\frac{k}{n}\right).$$

Theorem 2.1 gives in this special case,

$$\lim_{n \to \infty} J_n(g)(x) = g(x)$$

for $g \in C[0, 1], x \in [0, 1]$ provided

$$\lim_{n \to \infty} r_n(x) = x. \qquad \ldots (6.5)$$

It follows from the theorem of King and Swetits[11] that $E(r_n(x))$ is a regular summability matrix for $0 < x \leq 1$ provided (6.5) holds. This result agrees with the more general regularity condition produced by Wuyts-Torfs[15].

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