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COMPLETELY INDUCED BIFUZZY TOPOLOGICAL SPACES

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Let (X, T_1, T_2) be a bitopological space and $\tau(T_i)$ $(i = 1, 2)$ be the set of all completely lower semi continuous functions, defined from X into the closed unit interval $I = [0, 1]$. In this paper the concept of completely induced fuzzy topology due to Bhowmik and Mukherjee (Fuzzy sets and systems 47 (1992) 487-390) has been generalized to the bifuzzy topological setting and some of its basic properties are discussed. It is seen that the study of completely induced fuzzy topological space turns out to the study of a special kind induced fuzzy topological space. Finally the connections between some separation properties of the bitopological space (X, T_1, T_2) and that of its corresponding completely induced bifuzzy topological space $(X, \tau(T_1), \tau(T_2))$ are shown.

Key Words : Bifuzzy topological; Completely Lower Semi-Continuous Functions; Lower Semi Continuous Functions; Completely Induced; Strong and Weak r-cut

1. INTRODUCTION

The concept of completely induced fuzzy topological (CIFT) space was introduced by Bhowmik and Mukherjee[4]. These spaces were defined with the notions of completely lower semi-continuous (CLSC) functions introduced in [3]. The triple (X, T_1, T_2), where X is a non-empty set and T_1, T_2 are topologies on X is called a bitopological space[9]. In section 2, the concepts of completely induced bifuzzy topological spaces are introduced and some of their basic properties are studied. In section 3, we construct some weaker axioms, by using the concept of regular open sets in place of open sets in a bitopological space (X, T_1, T_2). Finally, the connections between some separation properties of a bitopological space (X, T_1, T_2) and that of its corresponding completely induced bifuzzy topological (CIBFT) space $(X, \tau(T_1), \tau(T_2))$ are shown. We use P-to denote pair wise i.e. P-regular stands for pair wise regular. If A is a subset of an ordinary topological space then we denote its characteristic function by I_A.

Let us now include some known results and definitions for ready reference :

1.1 (a) [3], [6], The function from a topological space (X, T) to the real number space (R, σ) is called a CLSC function at $x_0 \in X$ if and only if for each $\varepsilon > 0$, there exists a regular open neighbourhood $N(x_0)$ such that $x \in N(x_0)$ implies $f(x) > f(x_0) - \varepsilon$ or equivalently if and only if for all $a \in R$, $\{x \in X : f(x) > a\}$ is a union of regular open sets.

(b) [3], The characteristics functions of a regular open set is CLSC

1.2 (a) [4] The family $\tau(T)$ of all CLSC functions from a topological space (X, T) to the unit closed interval $I = [0, 1]$ forms a fuzzy topology called completely induced fuzzy topology

(CIFT) and is denoted by $\tau(T)$. The space $(X, \tau(T))$ is called a CIFT space.

(b) [4], A fuzzy subset α in a CIFT space $(X, \tau(T))$ is fuzzy open (resp. fuzzy closed) if and only if for each $r \in I$, the strong r-cut $\sigma_r(\alpha) = \{x \in X : \alpha(x) > r\}$ (resp. the weak r-cut $W_r(\alpha) = \{x \in X : \alpha(x) \geq r\}$ is a union of regular open sets (resp. is a intersection of regular closed sets) in the topological space (X, T).

1.3 [1], Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \gamma_1, \gamma_2)$ be a function from a bifuzzy topological space to another bifuzzy topological space. The function f is said to be p-fuzzy continuous if and only if the functions $f: (X, \tau_1) \rightarrow (X, \gamma_1)$ $(i = 1, 2)$ are fuzzy continuous.

1.4 [1], A fuzzy point p in a set X is a fuzzy set in X given by $p(x) = t$ for $x = x_p$ $(0 < t \leq 1)$ and $p(x) = 0$ for $x \neq x_p$, $x_p \in X$ is called the support of p and $p(x_p) = t$ the value of p. A fuzzy point p in X is said to belong to a fuzzy set λ in X i.e. $p \in \lambda$ if and only if $p(x_p) \leq \lambda(x_p)$. Finally two fuzzy points p and q are said to be distinct if and only if their supports are distinct i.e. $x_p \neq x_q$.

2. Completely Inducted Bifuzzy Topological Spaces

In this section we shall define and study completely induced bifuzzy topological (CIBFT) spaces. We start with the following definition.

Definition 2.1 — Let (X, T_1, T_2) be a bitopological spaces and $(\tau(T_1))$ $(i = 1, 2)$ be the set of all completely lower semi-continuous defined from X into the closed unit interval $I = [0, 1]$. Then $\tau(T_i)$ is a fuzzy topology in X^4. The triple $(X, \tau(T_1), \tau(T_2))$ is called a completely induced bifuzzy topological (CIBFT) space

In[5], Cornohan defined R-map in the following way : a function $f: (X, T) \rightarrow (Y, K)$ from a topological space (X, T) to another topological space (Y, K) is called R-map if and only if the inverse image of each regular open subset of Y is regular open in X.

Definition 2 — A function $f: (X, T_1, T_2) \rightarrow (Y, K_1, K_2)$ from a bitopological space (X, T_1, T_2) to another bitopological space (Y, K_1, K_2) is said to be a P-R-map if and only if the functions $f: (X, T_i) \rightarrow (Y, K_i)$ $(i = 1, 2))$ are R-maps.

***Theorems* 2-3** — *Let $(X T_1, T_2)$ and (Y, K_1, K_2) be two bitopological spaces. Then f : $(X, T_1, T_2) \rightarrow (Y, K_1, K_2)$ is P-R-map if only if $f: (X, \tau(T_1), \tau(T_2)) \rightarrow (Y, \tau(K_1), \tau(K_2))$ is P-fuzzy continuous.*

Proof : Let $\alpha \in \tau(K_i)$ $(i = 1, 2)$ and $r \in I$. By the result 1.2 (b), the strong r-cut of α, $\sigma_r(\alpha)$ is a union of regular open sets in (Y, K_i) $(i = 1, 2))$. Now $f: (X, T_1, T_2) \rightarrow (Y, K_1, K_2)$ is a P-R-map, hence $f^1(\sigma_r(\alpha)) = \sigma_r(f^1(\alpha))$ (by lemma 2.2 (i) of [7] is a union of a regular open sets in (X, T_i) $(i = (1, 2))$. Thus $f^1(\alpha) \in \tau(T_i)$, which implies $f : (X, \tau(T_1), \tau(T_2)) \to Y(\tau(K_1), \tau(K_2))$ is P-fuzzy continuous.

Conversely, let A be a regular open subset in (Y, K_i) $(i = 1, 2)$.

Then by result 1.1 (b), $1_A \in \tau(k_i)$ $(i = (1, 2))$

Now,

$$f^1(A) = \{x \in X : 1_A(f(x)) = 1\}$$

$$= \{x \in X : f^1(1_A(x)) > r, 0 < r \leq 1\}$$

$$= \sigma_r(f^1(1_A)).$$

Since $1_A \in \tau(K_i)$, by p-fuzzy continuity of $f, f^1(1_A) \in \tau(T_i)$. Thus $\sigma_r(f^1(1_A))$ is a union of regular open sets in (X, T_i). Since union of regilar open sets is regular open set, $f^1(A)$ is regular open in (X, T_i), $i = 1, 2$. Hence $f : (X, T_1, T_2) \to (Y, K_1, K_2)$ is a P-R-map.

The study of completely induced fuzzy (hence bifuzzy) topological space turns out to the study of a special kind of induced fuzzy topological space[2, 10]. Suppose X is a topological space then all the regular open sets in X form a base for a topology X together with this topology is denoted by X^* i.e. (X, T_n) is the semi regularization if (X, T) and denote (X, T_n) by X^*. Now, trivially a function $f : X^n \to R$ is CLSC if and only if $f : X \to R$ is lower semi continuous, hence the CIFT on X is the usual induced fuzzy topology on X.

Lemma 2.4 — Let (X, T) be a crisp space. Then $\tau(T) = \omega(T)$ is the set of all lower semi continuous function from (X, T_n) to I and (X, T_n) in the semi resularization of (X, T).

PROOF : It is trivial

Corollary 2.5 — Let (X, T) be a crisp semi-regular space. Then $\tau(T) = \omega(T)$ where $\omega(T)$ is the set of all lower semi continuous functions from (X, T) to I.

Corollary 2.6 — A function $f : (X, T_1)_n\,(T_2)_n) \Rightarrow (Y, K_1)_n, K_2)_n)$ is P-continuous if and only if $f : (X, \tau(T_1), \tau(T_2)) \to (Y, \tau(K_1), (K_2))$ is p-fuzzy continuous.

PROOF : It follows from theorem 2.3 and lemma 2.4.

Definition 2.7 — Let $(X, \tau(T_1), \tau(T_2))$ be a CIBFT spaces and A is a pairwise regular open subset in the bitopological space (X, T_1, T_2). Then the family $(\tau(T_1)_A = \{\{\lambda \cap 1_A : \lambda \in \tau(T_1), i = 1, 2\}$ is a fuzzy topology on A and $(A, (\tau(T_i))_A)$ is called the fuzzy sub space of the CIBFT space $(X, \tau(T_1), \tau(T_2))$.

Example 2.8 — Let $X = \{a, b, c, d\}$ and $T_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, $T_2 = \{\phi, X, \{a\}., \{b\}, \{a, b\}, \{a. c\}, \{a, b, c\}\}$ be the two topologies defined on X, then $\tau(T_1) = \{1_\phi, 1_x, 1_{(a)} 1_{(b)}\}$ and $\tau(T_2)\left\{1_\phi, 1_x, 1_{(b)}, 1_{(a,} c)\right\}$ are two CIFT spaces, also $\{b\}$ is the P-regular open subset in (X, Y_i) $i = 1, 2$. Thus considering $A = \{b\}$, we have $\tau(T_i))_A = \{1_\phi, 1_{(b)}\}$.

BIFUZZY SEPARATION AXIOMS AND THE SPACE $\tau(T)$

In this section, we construct some weaker separation axioms by using the concept of regular open sets in place of open sets in a bitopological space (X, T_1, T_2). Then we study the connections between some separation properties of the bitopological space (X, T_1, T_2) and that of its corresponding CIBFT inspace $(X, \tau(T_1), \tau(T_2))$.

Definition 3.1 — A bitopological space (X, T_1, T_2) is said to be pairwise almost T_0 (or in short p-A T_0) if and only if for any $x_p, x_q \in X$ such that $x_p \neq x_q$, there exists a *p*-regular open set A on X satisfying $x_p \in A, x_q \notin A$ or $x_q \in A, x_p \notin A$.

Definition 3.2 — A CIBFT space $(X, \tau(T_1), \tau(T_2))$ is said to be pair wise fuzzy T_0 (or in short p-F T_0 if and only if for any distinct fuzzy points p, q in X, there exists a fuzzy set $\mu \in \tau(T_1) \bigcup \tau(T_2)$ such that $p \in \mu, q \bigcap \mu = 0$ or $q \in \mu, p \bigcap \mu = 0$.

Theorems 3.3 — *Let (X, T_1, T_2) be a bitopological space, then the following statements are equivalent :*

(i) (X, T_1, T_2) is a *p-A* T_0 space.

(ii) $(X, \tau(T_1), \tau(T_2))$ is a p - F T_0 space.

Definition 3.4 — A bitopological space (X, T_1, T_2) is said to be pair wise-almost T_1 (in short p - A T_1) if and only if $x_p, x_q \in X$, where $x_p \neq x_q$ there exists two *P*-regular open subsets A and B such that $x_p \in A, x_q \notin A$ and $x_q \in B, x_p \notin B$.

Definition — A CIBFT space $(X, \tau(T_1), \tau(T_2))$ is said to be pair wise fuzzy T_1 (in short *P-F* T_1) if and only if for any two distinct fuzzy points p, q there exists a $\mu_1 \in \tau(T_1) \bigcup \tau(T_2)$ and $\mu_2 \in \tau(T_1) \bigcup \tau(T_2)$ such that $p \in \mu_1, \mu_1 \bigcap q = 0$ and $q \in \mu_2, \mu_2 \bigcap p = 0$.

Theorem 3.6 — *Let (X, T_1, T_2) be a bitopological spaces then the following are equivalent:*

(i) (X, T_1, T_2) is P-A T_1 space

(ii) $(X, \tau(T_1), \tau(T_2))$ is P-F T_1 space.

PROOF : (i) -> (ii) — Let p, q be two distinct fuzzy points in X. Since (X, T_1, T_2) is a P-A T_1 space and $x_p \neq x_q$ there exists two P-regular open subsets A and B on X such that $x_p \in A, x_q \notin A$, and

$$x_q \in B, X_p \notin B. \quad (1)$$

It is clear that $1_A : X \to [0, 1]$ is a CLSC function thus $1_A \in \tau(T_1) I = 1, 2$, also $q \notin 1_A$ because $q(x_q) > 1_A(x_p) = 0$ $(/ x_q \in A)$.

To show $q \cap 1_A = 0$, suppose $q \cap 1_A \neq 0$. This implies $(1_A(x_q)) > 0$ which gives $x_q \in A$, a contradiction, therefore $q \cap 1_A = 0$.

Similarly, we can show the other case, i.e. when $x_q \in, x_p \notin B$ and $p \cap 1_B = 0$. (ii) -> (i) Let $(X, \tau(T_1), \tau(T_2))$ be a P-F T_1 space and x_p, x_q be two distinct points in X, take p, q the two fuzzy points in X for which there exists $\mu_1 \in \tau(T_1) \cup \tau(T_2)$ and $\mu_2 \in \tau(T_1) \cup \tau(T_2)$ such that $p \in \mu_1, \mu_1 \cap q = 0$ and $q \in \mu_2, \mu_2 \cap p = 0$. Consider the 1st case, since $\mu_1 \in \tau(T_1) \cup \tau(T_2)$ for $r \in I$, $\sigma_r(\mu_1)$ is a union of regular open sets in (X, T_i), $i = 1, 2$, hence regular open in (X, T_i) which contains x_p but not x_q. Let $A = \sigma_r(\mu_1)$ if $x_p \notin A$, then $p(x_p) > 1_A(x_p) = 0$ which contradicts $p \in \mu_1$. To show $\mu_1 \cap q = 0$ suppose $\mu_1 \cap q \neq 0$, this implies $(\mu_1(x_q)) > 0$ which gives $x_q \in A$, a contradiction, therefore, $\mu_1 \cap q = 0$. Similarly, we can show the other cases.

Definition 3.7 — A bitopological space (X, T_1, T_2) is called a P-almost regular space if and only if for each T_i-regular open subsets A of X is a union of T_i-regular open subsets A_j's of X such that C1 $Aj \subseteq A$ for each j.

According to Hutton and Reilly[8], a fuzzy space (X, τ) is called a fuzzy regular space if and only if each fuzzy open subset δ of X is a union of fuzzy open subsets δ_n's of X such that $C1\ \delta_a \subseteq \delta$ for each a. Taking similar definition for a CIBFT space we have the following.

Definition 3.8 — A CIBFT space $(X, \tau(T_1), \tau(T_2))$ is called a *P*-fuzzy regular space if and only if for each $\tau(T_i)$-open subset δ of X is a union of $\tau(T_i)$-open subset δ_n's of X suchthat C_l $\delta_a \subseteq \delta$ for each a.

Theorem 3.9 — *A CIBFT space $(X, \tau(T_1), \tau(T_2))$ is a P-fuzzy regular space if and only if the bitopological space (X, T_1, T_2) is a P-almost regular space.*

PROOF : Let $(X, \tau(T_1), \tau(T_2))$ be a P-fuzzy regular space and A be a *P*-regular open subset in (X, T_i) $(i = 1, 2)$.

Then $1_A \in \tau(T_1)$ (By result 1.1 (b)). Thus $1_A = \bigcup \delta_a$ where δ_a's are $\tau(T_i)$-open subsets with $Cl\ \delta_a \subset 1_A$. Now for each $r \in I$.

$A = \sigma_r(\bigcup_a \delta_a) = \bigcup \sigma_r(\delta_a)$ by lemma 1.1 (i) [7] where $\sigma_i(\delta_a)$ is a union of regular open set in (X, T_i), hence regular open in (X, T_i), for each a. Also, $Cl\ (\sigma_r(\delta_a) \subseteq W_r(Cl\ \delta_a)\ W_r(1_A) = A^7$. Thus (X, T_1, T_2) is a p-almost regular space. Conversely, let (X, T_1, T_2) be a p-almost regular space and $\alpha \in \tau(T_i)$ (i = 1, 2). Then for each $r \in I, \sigma_r(\alpha)$ is a union of regular open sets in (X, T_i),. Now, $\sigma_r(\alpha) = \bigcup V_j$ where v_j's are regular open subsets in (X, T_i) (i = 1, 2) with Cl $(V_j) \subseteq \sigma_r(\alpha)$. By decomposition theorem[7].

$$\alpha = \bigcup_{r \in I} r \cdot 1_{\sigma_r(\alpha)}$$

$$= \bigcup_{r \in I} r \cdot 1_{\bigcup_j v_j} = \bigcup_{r \in I} \bigcup_j r \cdot 1_{v_j}$$

where $r \cdot 1_{v_j} \in \tau(T_i)$ with $r \cdot Cl\ 1\ v_j = r \cdot 1_{cl\ V_j} \subseteq \alpha$

Hence $(X, \tau(T_i), \tau(T_2))$ is a p-fuzzy regular space.

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