Indian J. pure appl. Math., 33(6) : 933-940, June 2002

CHARACTERIZATION OF ε-CHAINABLE SETS IN METRIC SPACES

KIRAN SHRIVASTAVA* AND GEETA AGRAWAL**

*Department of Mathematics, Govt. Sarojini Naidu College, Bhopal
**Department of Mathematics, Govt. Motilal Vigyan Mahavidyalaya, Bhopal

(*Received* 20 *January* 2000; *accepted* 12 *September* 2001)

Connectedness and compactness are widely studied in Topology. In 1883 Cantor defined connectedness with the help of ε-chains. At present, however, the Riesz-Hausdorff definition using the idea of separated sets, is universally accepted. On the other hand, a lot of experiments has led to several forms of compactness. Compactness and several of its generalizations are defined in terms of open covers; e.g. compact, countably compact, paracompact, Lindeoff etc. Chainability characterizes connected sets among compact sets in the setting of metric spaces. In the same setting Beer[1] has characterized compact sets among the connected ones.

In this paper we extend the concept of ε-chain between two points to ε-chain between two sets in metric spaces. This generalization yields a simple characterization of ε-chainable sets in terms of ε-chains between their points. Some of the results of Beer[1] have been generalized. Also, results pertaining to uniformly continuous and contraction mappings in the context of ε-chainability have been obtained.

Many of the properties of connectedness and local connectedness can be extended to ε-chainable sets. In future we wish to extend some of the properties of connectedness and local connectedness to ε-chainable sets.

Key Words : Chainable; Uniformly Chainable; Connected; Uniformly Locally Compact; Contraction; Uniformly Continuous

Let A be a subset of the metric space (X, d).

For $\varepsilon > 0$, let $V_\varepsilon(A) = \{x \in X / d(x, A) < \varepsilon\}$, where $d(x, A) = \inf \{d(x, a) : a \in A\}$

Throughout this paper X will stand for metric space with metric d.

Definition 1 — For points $p, q \in X$ an ε-chain of length n from p to q is a finite sequence $a_0, a_1, a_2, ..., a_n$ in X with $a_0 = p, a_n = q$ and $d(a_{i-1}, a_i) < \varepsilon$, $1 \le i \le n$. We call X ε-chainable if each two points in X can be joined by an *ε-chain* and X is *chainable* if X is *ε-chainable* for each positive ε.

Definition 2 — Let $A, B \subset X$. An *ε-chain of length n from A to B* is a finite sequence $A_0, A_1, A_2, ..., A_n$ of subsets of X with $A = A_s, A_n = B, A_{i-1} \subset V_\varepsilon(A_i)$ and $A_i \subset V_\varepsilon(A_{i-1})$, $1 \le i \le n$. If ε-chain exists between A and B we say $\langle A, B \rangle$ is *ε-chainable* and $\langle A, B \rangle$ is *chainable* if it is *ε-chainable* for each positive ε.

Let $B_\varepsilon(A)$ be the union of all open ε-balls centered on each point of A. It may be observed that $V_\varepsilon(A) = B_\varepsilon(A)$. Also $B_\varepsilon(A)$ being the union of open sets is open. Using the notation of[1],

Mailing Address : Dr Kiran Shrivastava, F-31, Kotra Sultanabad, Bhopal (M.P.) 462 003

inductively construct the sets $V_\varepsilon^n(A)$ for each $n \in Z^+$ as follows : $V_\varepsilon^1(A) = V_\varepsilon(A)$ and for each $n \geq 2$ set $V_\varepsilon^n(A) = V_\varepsilon(V_\varepsilon^{n-1}(A))$. The following should be observed :

(1) $V_\varepsilon^n(A) \subset V_\varepsilon^{n+1}(A)$

(2) $V_\varepsilon^n(A) \subset V_{n\varepsilon}(A)$.

Using the notation in[1] of the shortest length of ε-chain between two points in a metric space we set $\Phi_\varepsilon(\langle A, B \rangle)$ to be the length of the shortest ε-chain between A and B.

Examples of Chainable Sets

(1) Let $\langle x_n \rangle$ be a real sequence such that $x_n \to l$. Let $A = \{x_n, n \in N, n \geq n_0\}$ and $B = \{l\}$. Then $A \subset V_\varepsilon(B)$ and $B \subset V_\varepsilon(A) \Rightarrow \langle A, B \rangle$ is ε-chainable and $\Phi_\varepsilon(\langle A, B \rangle) = 1$.

(2) Let $Y = A \cup B$, and d be the usual metric on Y where,

i. $A = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots \right\}$ and $B = \left\{ \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \ldots\ldots \right\}$, then $\langle A, B \rangle$ is ε-chainable and

$\Phi_\varepsilon(\langle A, B \rangle) = 1$ for $\varepsilon = 1$.

ii. Let $A = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots \right\}$, $A_1 = \left\{ \frac{1}{2}, \frac{1}{3} \right\}$, $A_2 = \left\{ \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2} \right\}$ and

$B = \left\{ \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \ldots\ldots \right\}$ then we have the following two chains between A and B : A, A_1, B and A, A_1, A_2, B where $\varepsilon = \frac{2}{3}$, then $\langle A, B \rangle$ is ε-chainable and

$\Phi_\varepsilon(\langle A, B \rangle) = 2$.

iii. Let A be the set of rational numbers and B the set of irrational numbers then $\langle A, B \rangle$ is chainable for any ε and $\Phi_\varepsilon(\langle A, B \rangle) = 1$.

3. Let d be the usual metric on $R \times R$. Then the following subsets of $R \times R$ are ε-chainable

i) Consider a rectangle $ABCD$ of breadth n. Divide the rectangle into m equidistant segments $AB = A_1, A_2, A_3, \ldots, A_m = CD$, then $\langle A_1, A_m \rangle$ is ε-chainable for $\varepsilon > \frac{n}{m-1}$.

ii) If we take a circle C of radius R and divide it into n concentric circles of radii, $\frac{R}{n}, \frac{2R}{n}, \frac{3R}{n}, \ldots \frac{nR}{n}$ with circumferences $C_1, C_2, C_3, \ldots, C_n = C$ respectively, then $\langle C_1, C_n \rangle$ are ε-chainable for $\varepsilon > \frac{R}{n}$.

iii) Divide a square ABCD into concentric squares such that the distance between parallel sides of any two consecutive squares is same say m, then the perimeters of the smallest and biggest square are ε-chainable for $\varepsilon > m$.

Definition 3 — Let $A, B \subset X$. $\langle A, B \rangle$ is called *uniformly ε-chainable* if there exists a positive integer n such that each point of A can be joined to a point of B by an ε-chain of length at most n and vice versa. Also $\langle A, B \rangle$ is called *uniformly chainable* if it is uniformly ε-chainable for all positive ε.

EXAMPLES

1) Let d be the usual metric on $R \times R$ and B the union of the following sets :

$$A_1 = \{ (x, y) : x^2 + y^2 = 1 \}, A_2 = \{ (x, y) : x^2 + y^2 = (1.4)^2 \}, A_3 = \{ (x, y) : x^2 + y^2 = 2^2 \} \quad ,$$

$$A_4 = \{ (x, y) : x^2 + y^2 = 3^2 \}, A_5 = \{ (x, y) : x^2 + y^2 = 4^2 \}, A_6 = \{ (x, y) : x^2 + y^2 = 5^2 \} \text{ and}$$

$A_7 = \{ (x, y) : x^2 + y^2 \geq 7^2 \}$. Consider the subspace (B, d) of $(R \times R), d)$. Then we have the following two chains between A_1 and $A_6 : A_1, A_2, A_3, A_4, A_5, A_6$ and A_1, A_3, A_4, A_5, A_6 where $\varepsilon = 1.5$ then $\langle A_1, A_6 \rangle$ is uniformly ε-chainable.

2) Let $A = \{(x, y) : x^2 + y^2 = 1\}$ be a subset of $R \times R$ and consider the subspace (A, d) of $(R \times R, d)$ where d is the usual metric on $R \times R$. Then for any pair of points (x, y) in A, $d(x, y) \leq 2\pi$. Thus any two subsets of A are uniformly chainable.

3. Let (R, d) be the usual metric space and $Y = A \cup B$ where $A = [1, 2)$ and $B = (2, 3]$. Then (Y, d) is disconnected subspace of (R, d) and $\langle A, B \rangle$ is uniformly chainable in (Y, d).

4. Let $A = \left\{ \left(1, \frac{1}{n} \right) : n \geq 1, n \text{ is a natural number} \right\}$, $B = \{(x, 0) : 1 \leq x \leq 2\}$ and $Y = A \cup B$. Consider the subspace (Y, d) of $(R \times R, d)$. Then $\langle A, B \rangle$ is uniformly chainable in (Y, d).

Definition 4 — A subset A of X is called *uniformly locally compact* if there exists an $\varepsilon > 0$ for which each closed ε-ball is compact with respect to the relative metric on A.

Some Preliminary Results :

I. Let $A, B \subset X$, then

i) $A \subset V_\varepsilon(A)$

ii) $A \subset B \Rightarrow V_\varepsilon(A) \subset V_\varepsilon(B)$

iii) $V_\varepsilon(A) \cup V_\varepsilon(B) = V_\varepsilon(A \cup B)$

iv) $V_\varepsilon(A \cap B) \subseteq V_\varepsilon(A) \cap V_\varepsilon(B)$

v) $A \subseteq \bigcap_{\varepsilon > 0} V_\varepsilon(A) = \bar{A}$

vi) A is closed iff $A = \bigcap_{\varepsilon > 0} V_\varepsilon(A)$.

II. If $\langle A, B \rangle$ and $\langle C, D \rangle$ are chainable, then $\langle A \cup C, B \cup D \rangle$ is also chainable where $A, B, C, D \subset X$.

III. If $\langle A, B \rangle$ is chainable and $f: X \to X$ is a uniformly continuous function then $\langle f(A), f(B) \rangle$ is also chainable where $A, B \subset X$.

IV. Let $f: X \to X$ be a contracting mapping, that is there exists a real number $\varepsilon, 0 \le \varepsilon < 1$, such that for every $x, y \in X, d[f(x), f(y)] \le \varepsilon d(x, y)$. If $\langle A, B \rangle$ is ε-chainable then $\langle f(A), f(B) \rangle$ is also ε-chainable where $A, B \subset X$.

V. Let $A, B \subset Y$ and Y be the subspace of X. If $\langle A, B \rangle$ is chainable in Y, then $\langle A, B \rangle$ is also chainable in X but converse part is not true by the following example: Let $X = R \times R$, and d the usual metric on X and $A = \left\{ \left(1, \frac{1}{n} \right) : n \ge 1, n \text{ is a natural number} \right\}$, $B = \{(x, 0) : x \ge 0\}$, then $\langle A, B \rangle$ is chainable in X but not in Y where $Y = A \cup B$.

VI. Let $A, B \subset X$. If $d(A \cup B) \le \varepsilon$, then $\langle A, B \rangle$ is ε-chainable but converse part of this result is not true (see example 3(i)). This result also holds if $d(A) + d(B) + d(A, B) \le \varepsilon$.

Theorem 1 — *Let $A, B \subset X$ and $\langle A, B \rangle$ be ε-chainable, then there exists an ε-chain from every point of A to some point of B and vice-versa. Also the converse holds.*

PROOF : We prove the necessary part first. As $\langle A, B \rangle$ is ε-chainable there exists a sequence $A_0, A_1, \ldots, A_n$ of subsets of X with $A = A_0, A_n = B, A_i \subset V_\varepsilon(A_{i+1})$ and $A_{i-1} \subset V_\varepsilon(A_i), 1 \le i \le n$. Let $x \in A$ be arbitrary. The $x \in A \Rightarrow x \in V_\varepsilon(A_1) \Rightarrow d(x, x_0) < \varepsilon$ for some $x_1 \in A_1$. Again $x_1 \in A_1 \Rightarrow d(x_1, x_2) < \varepsilon$ for some $x_2 \in A_2$. Repeating the above process n times we obtain a sequence of points $x_1, x_2, \ldots\ldots x_n = y \in B$ such that $d(x_i, x_{i-1}) < \varepsilon, 1 \le i \le n$ and $x_i \in A_i$, showing that there exist an ε-chain from x to y. Likewise we can obtain an ε-chain from every point of B to a point of A.

We next prove the sufficient part. Let there exist an ε-chain from every point of A to some point of B and vice-versa. Let $A_1 = \{y \in X / d(y, x) < \varepsilon$ for some $x \in A$ and $y \ne x\}$. Clearly $A_1 \ne \phi$

and $A_i \subset V_\varepsilon(A)$. Next, we show that $A \subset V_\varepsilon(A_1)$. If $x \in A$, then there exists a sequence $x = x_0, x_1, x_2, \ldots.. x_n = y \in B$ such that $d(x, x_1) < \varepsilon \Rightarrow x_1 \in A_1$ and $d(x, A_1) < \varepsilon \Rightarrow x \in V_\varepsilon(A_1)$.

Again let $A_2 = \{y \in X / d(y, x) < \varepsilon$ for some $x \in A_1$ and $y \neq x\}$. Clearly $A_2 \neq \phi, A_2 \subset V_\varepsilon(A_1)$ and it can be shown as above that $A_1 \subset V_\varepsilon(A_2)$. Repeating the above process n times we obtain a sequence $A = A_0, A_1, \ldots., A_n = B$ of subsets of X, such that $\langle A, B \rangle$ is ε-chainable.

***Theorem* 2** — *Sequential Characterization of ε-chainable Sets in Metric Spaces — Let $\langle \varepsilon_n \rangle$ be a monotonically increasing sequence of positive real numbers converging to ε (arbitrary). Then $\langle A, B \rangle$ is ε-chainable iff here exists a subsequence $\langle \varepsilon_{n_k} \rangle$ of $\langle \varepsilon_n \rangle$ such that $\langle A, B \rangle$ is ε_{n_k} chainable for each $k \in N$.*

PROOF : Let $\langle \varepsilon_n \rangle$ be a monotonically increasing sequence such that $\varepsilon_n \to \varepsilon$. Then for every $\delta > 0$, there exists a positive integer n_0 such that $n \geq n_0 \Rightarrow |\varepsilon_n - \varepsilon| < \delta \Rightarrow \varepsilon_n - \delta < \varepsilon < \varepsilon_n + \delta$. $\langle A, B \rangle$ is ε-chainable $\Rightarrow$ there exists a sequence of sets $A = A_0, A_1, A_2, \ldots\ldots, A_l = B$ such that $A_i \subset V_\varepsilon(A_{i-1})$ and $A_{i-l} \subset V_\varepsilon(A_i)$ for $i = 1, 2, \ldots., l$, $\Rightarrow A_i \subset V_{\varepsilon_n + \delta}(A_{i-1})$ and $A_{i-1} \subset V_{\varepsilon_n + \delta}(A_i)$ $\forall\ n \geq n_0$ or $A_i \subset V_{\varepsilon_{n_0} + \delta}(A_{i-1})$ and $A_{i-1} \subset V_{\varepsilon_{n_0} + \delta}(A_i)$. Let $m > n_0$ be the least positive integer such that $\varepsilon_{n_0} + \delta \leq \varepsilon_m$ for if $\varepsilon_m \leq \varepsilon_{n_0} + \delta\ \forall\ m \in N$, then the sequence $\langle \varepsilon_n \rangle$ is not monotonically increasing. $\Rightarrow A_i \subset V_{\varepsilon_m}(A_{i-1})$ and $A_{i-1} \subset V_{\varepsilon_m}(A_i) \Rightarrow A_i \subset V_{\varepsilon_n}(A_{i-1})$ and $A_{i-1} \subset V_{\varepsilon_n}(A_i)$ $\forall\ n \geq m$. Let $n_1 = m, n_2 = m + 1, \ldots\ldots$ Then $A_i \subset V_{\varepsilon_{n_k}}(A_{i-1})$ and $A_{i-1} \subset V_{\varepsilon_{n_k}}(A_i)$ $\forall\ k \in N$ and $i = 1, 2, \ldots.. l$. Hence $\langle A, B \rangle$ is ε_{n_k} chainable for each $k \in N$.

Conversely, let $\langle A, B \rangle$ is ε_{n_k} chainable for each $k \in N$, where $\langle \varepsilon_{n_k} \rangle$ is a subsequence of $\langle \varepsilon_n \rangle$ converging to ε and $\langle \varepsilon_n \rangle$ is monotonically increasing. Now $n \geq n_0 \Rightarrow |\varepsilon_n - \varepsilon| < \delta$. Without loss of generality we may assume $|\varepsilon_{n_k} - \varepsilon| < \delta$ for $k \geq n_0 \Rightarrow \varepsilon_{n_k} - \delta < \varepsilon < \varepsilon_{n_k} + \delta$ $\forall k \geq n_0$ or $\varepsilon_{n_0} - \delta < \varepsilon < \varepsilon_{n_0} + \delta$ ... (1). Let $\varepsilon_{n_m} < \varepsilon_{n_0} - \delta$. If no such n_m exists then in (1) we may take instead of n_0 some n_k for much larger k so that such a n_m exists. Now $\langle A, B \rangle$ is ε_{n_m} chainable and hence ε-chainable.

***Theorem* 3** — *Let $A, B \subset X$ and $\langle A, B \rangle$ be chainable then $\langle \overline{A}, \overline{B} \rangle$ is chainable.*

PROOF : Let $\langle A, B \rangle$ be chainable and $\varepsilon > 0$. Choose $\varepsilon' > 0$ such that $2\varepsilon' < \varepsilon$. Now $\langle A, B \rangle$ is ε'-chainable $\Rightarrow \exists$ a sequence $A = A_0, A_1, \ldots\ldots A_n = B$ of subsets of X such that $A_{i-1} \subset V_{\varepsilon'}(A_i)$ and $A_i \subset V_{\varepsilon'}(A_{i-1})$. This implies $\overline{A_{i-1}} \subset \overline{V_{\varepsilon'}(A_i)} \subset V_{2\varepsilon'}(\overline{A}_i) \subset V_\varepsilon(\overline{A}_i)$ and $\overline{A}_i \subset \overline{V_{\varepsilon'}(A_{i-1})} \subset V_{2\varepsilon'}(\overline{A_{i-1}}) \subset V_\varepsilon(\overline{A}_{i-1}) \Rightarrow \langle \overline{A}, \overline{B} \rangle$ is chainable.

Preposition — Let $A, B \subset X$ and $\varepsilon \bullet > \max\{d(A), d(B), d(A, B)\}$, then $\langle A, B \rangle$ is ε-chainable and $\Phi_\varepsilon(\langle A, B \rangle) = 2$.

PROOF : Let $x \in A$. Now $d(A, B) < \varepsilon \Rightarrow \exists\ a \in A, b \in B$ such that $d(a, b) < \varepsilon$. Also $d(A) < \varepsilon \Rightarrow d(x, a) < \varepsilon$. Thus $d(x, b) \leq d(x, a) + d(a, b) < 2\varepsilon \Rightarrow x$ is chainable to b. Similarly if $y \in B$, then y is chainable to a. Thus, $\langle A, B \rangle$ is ε-chainable and $\Phi_\varepsilon(\langle A, B \rangle) = 2$.

Theorem 4 — *Let $A, B \subset X$. If $(A \cup B)$ is connected and $\varepsilon >$ max $\{d(A)\, d(B)\}$ then $\langle A, B \rangle$ is ε-chainable.*

PROOF : $A \cup B$ is connected $\Rightarrow A \cap \overline{B} \neq \phi$ or $\overline{A} \cap B \neq \phi$. Suppose $A \cap \overline{B} \neq \phi$, then there exists an $a \in A$ and $a \in \overline{B}$. Now $a \in \overline{B} \Rightarrow a \in \bigcap_{\varepsilon > 0} V_\varepsilon(B) \Rightarrow$ for each $\varepsilon > 0$ there exists $b_1 \in B$ with $d(a, b_1) < \varepsilon$. Let $x \in A, y \in B$ then as $\varepsilon > \max\{d(A), d(B)\}$ and $x, a \in A \Rightarrow d(x, a) < \varepsilon$, hence x is ε-chainable to some point b_1 in B. Again y of B is ε-chainable to $a \in A$. Hence by Theorem 1, $\langle A, B \rangle$ is ε-chainable. A similar result is obtained by taking $\overline{A} \cap B \neq \phi$.

The following example shows that the converse part of the above theorem is not true, for if $(R,\ d)$ is the usual metric space, $A = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ and $B = \{1\}$ subsets of R, then $\overline{A} = \left\{0, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\}$ and $\overline{B} = B$. Also, $\overline{A} \cap B = \phi$ and $A \cap \overline{B} = \phi \Rightarrow A \cup B$ is not connected. But $A \subset V_\varepsilon(B)$ and $B \subset V_\varepsilon(A) \Rightarrow \langle A, B \rangle$ is ε-chainable where $\varepsilon = 1$.

Theorem 5 — *X is ε-chainable if and only if $\langle A, B \rangle$ is ε-chainable for every pair of subsets A, B of X.*

PROOF : Obvious

Theorem 6 — *Let A and B be totally bounded subsets of X and further let $\langle A, B \rangle$ be chainable, then $\langle A, B \rangle$ is uniformly chainable. If $A \cup B$ is uniformly locally compact then X is complete.*

PROOF : For the above theorem refer to Lemma given in[1].

***Theorem* 7** — *Let $A, B \subset X$. Then $\overline{A} = \overline{B}$ if and only if $\langle A, B \rangle$ is chainable and $\Phi_\varepsilon(\langle A, B \rangle) = 1$.*

PROOF : $\overline{A} = \bigcap_{\varepsilon > 0} V_\varepsilon(A), \overline{B} = \bigcap_{\varepsilon > 0} V_\varepsilon(B)$. Then $B \subset \overline{B} = \overline{A} \subset V_\varepsilon(A)$ and $A \subset \overline{A} = \overline{B} \subset V_\varepsilon(B)$ $\forall\ \varepsilon > 0$, that is $\langle A, B \rangle$ is chainable and $\Phi_\varepsilon(\langle A, B \rangle) = 1$. For converse $A \subset V_\varepsilon(B)$ and $B \subset V_\varepsilon(A)$ $\forall\ \varepsilon > 0$ implies $A \subset \bigcap_{\varepsilon > 0} V_\varepsilon(B) = \overline{B}$ and $B \subset \bigcap_{\varepsilon > 0} V_\varepsilon(A) = \overline{A}$, that is $\overline{A} = \overline{B}$.

***Theorem* 8** — *If $V^n_\varepsilon(A) \subset B \subseteq V^{n+1}_\varepsilon(A)$, then $\langle A, B \rangle$ is ε-chainable and $\Phi_\varepsilon(\langle A, B \rangle) = n + 1$.*

PROOF : The sets $\{A, V_\varepsilon(A), V^2_\varepsilon(A), \ldots\ V^n_\varepsilon(A), B\}$ form an ε-chain of length $n + 1$. Now, to prove $\Phi_\varepsilon(\langle A, B \rangle) = n + 1$. Let $A = A_0, A_1, A_2, \ldots, A_m = B$ be some other ε-chain between A and B. Then $A \subset V_\varepsilon(A_1)$ and $A_1 \subset V_\varepsilon(A)$. Also $A_2 \subset V_\varepsilon(A_1) \subset V^2_\varepsilon(A_1)$. Continuing the above process $A_{m-1} \subset V^{m-1}_\varepsilon(A_1)$. If possible let $m \leq n$. (i) Say $m < n$. Then $B = A_m \subset V_\varepsilon(A_{m-1}) \subset V^m_\varepsilon(A) \subset V^n_\varepsilon(A)$ which is a contradiction. (ii) say $m = n$, then $B = A_m \subset V^m_\varepsilon(A) = V^n_\varepsilon(A)$ which is again a contradiction, hence $m > n$, that is $\Phi_\varepsilon(\langle A, B \rangle) = n + 1$.

***Theorem* 9** — *Let A be any subset of X which is not necessarily opn but contained in a finite number of proper open subsets (say n) of X. Then A is uniformly chainable to the largest open set containing A.*

PROOF : We have $A \subset \overline{A} \subset V_\varepsilon(A)$ for all $\varepsilon > 0$. Then by theorem 10.6, page 141 [4] there exists an open set A_1 such that $A \subset \overline{A} \subset A_1 \subset \overline{A}_1 \subset V_\varepsilon(A)$, thus we get $A \subset V_\varepsilon(A_1)$ and $A_1 \subset V_\varepsilon(A)$ for all $\varepsilon > 0$.

Again $A_1 \subset \overline{A}_1 \subset V_\varepsilon(A_1)$ for all $\varepsilon > 0$ implies $A_2 \subset V_\varepsilon(A_1)$ and $A_1 \subset V_\varepsilon(A_2)$, for some open set A_2. Continuing the above process atmost n times we have A is chainable to the largest open set A_n containing A.

Let $A, B_1, \ldots\ B_m, A_n$ be some other chain between A and A_n. Then $A \subset B_1 \subset B_2 \subset \ldots \subset B_m \subset A_n$. Hence $A \subset V_\varepsilon(B_1) \subset V_\varepsilon(B_2) \subset \ldots \subset V_\varepsilon(B_m) \subset A_n$. Each $V_\varepsilon(B_m)$ being open $m \leq (n-1)$. So any chain between A and A_n is of length atmost n. Thus $\langle A, A_n \rangle$ is uniformly chainable.

Theorem 10 — *Let X be ε-chainable. Define a relation ~ on X as follows :*

$\langle A, B \rangle \sim \langle C, D \rangle$ iff $\Phi_\varepsilon(\langle A, B \rangle) = \Phi_\varepsilon(\langle C, D \rangle)$. Then ~ is an equivalence relation on X, which partitions X into disjoint equivalence classess denoted by $\overline{\langle A, B \rangle}, \overline{\langle C, D \rangle}$. Define a metric d on the quotient set $X \sim$ as follows : $d : X/\sim x\ X / \sim \rightarrow R$ by $d\ \overline{\langle A, B \rangle}, \overline{\langle C, D \rangle} = | \Phi_3(\langle A, B \rangle) - \Phi_\varepsilon(\langle C, D \rangle)|$. Then d is a metric on the quotient set $X / \sim$.

PROOF : Obvious.

REFERENCES

1. B. Gerald, *Proc. Amer. Nath. Soc.* **83**, (1981).
2. G. Simmons, *Introduction to topology and modern analysis*, Mc-Graw Hill, New York, 1963.
3. Kelly, *John General topology*, Van Nostrand Reinhold Company, New York, 1969.
4. Lipschutz, Seymour, *Schaum's outline of theory and problems of General Topology*, 1965.
5. M. H. A. Newmann, Elements of topology of plane sets of points, Cambridge Univ. Press, New York, 1961.