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OSCILLATION THEOREMS OF SECOND-ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS UNDER IMPULSIVE PERTURBATIONS

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The present paper is devoted to the investigation of the oscillation of a kind of very extensively studied second order nonlinear neutral delay differential equations with impulses, some interesting results are gained, which extend, improve and generalize the results obtained in recent papers.

Key Words : Oscillation; Impulse; Neutral Delay Differential Equation; Nonlinearity

1. INTRODUCTION

Consider the extensively studied delay differential equation

$$(r(t)\,|\,(x(t)-x(t-\tau))'\,|^{\alpha-1}\,(x(t)-x(t-\tau))')' + f(t, x(t), x(t-\sigma)) = 0 \qquad (1)$$

under the impulsive perturbation

$$\begin{cases} x(t_k^+) - x(t_k^+ - \tau) = M_k\,(x(t_k) - x(t_k - \tau)), \\ r(t_k^+)\ |\,(x'(t_k^+) - x'(t_k^+ - \tau))\,|^{\alpha-1}\,(x'(t_k^+) - x'(t_k^+ - \tau)) \\ \quad = N_k\,(r(t_k)\,|\,(x'(t_k) - x'(t_k - \tau))\,|^{\alpha-1}\,(x'(t_k) - x'(t_k - \tau))), \end{cases} \qquad \dots (2)$$

where

$\alpha, \tau, \sigma > 0, 0 \le t_0 < t_1 < t_2 < \ldots < t_k < \ldots$, and $\lim_{t \to \infty} t_k = \infty$. Suppose that $x'(t_k) = \lim_{h \to -0} \frac{x(t_k + h) - x(t_k)}{h}$ and $x'(t_k^+) = \lim_{h \to +0} \frac{x(t_k + h) - x(t_k^+)}{h}$.

Throughout this paper, assume that the following conditions hold :

(i) $f(t, u, v)$ is continuous in $[t_0, +\infty) \times (-\infty, +\infty) \times (-\infty, +\infty)$, $uf(t, u, v) > 0$ $(u v > 0)$ and there exist functions $p(t)$ and ϕ such that

$$\frac{f(t, u, v)}{\phi(v)} \ge p(t) \; (v \ne 0),$$

where $p(t)$ is continuous in $[t_0, +\infty)$, $p(t) \ge 0$ and ϕ satisfies $x\phi(x) > 0$ $(x \ne 0)$, and $\phi'(x) \ge 0$;

(ii) $M_k(x)$ and $N_k(x)$ are continuous in $(-\infty, +\infty)$, and there exist positive numbers c_k, c_k^*, d_k, d_k^* such that $c_k^* \le \frac{M_k(x)}{x} \le c_k$, $d_k \le \frac{N_k(x)}{x} \le d_k^*$; and

(iii) $r(t)$ is a positive continuous function in $[t_0, +\infty)$ and $R(t) = \int_{t_0}^{t} \frac{ds}{r^{1/\alpha}(s)}$.

It is more than ten years since first paper on oscillation of impulsive delay differential equations[1] was published in 1989. In the recent years, there is increasing interest on the oscillation/non-oscillation of impulsive delay differential/difference equations (see Ref. [1-20] etc.), and numerous papers have been published on this class of equations and good results are obtained. But Most of them are devoted to first order delay differential equations. Only a few papers are on second-order or higher order impulsive differential equations, we refer the reader to papers[9-15].

The aim of this paper is devoted to the study of the oscillation of a type of very extensively studied second order nonlinear delay neutral differential equations with impulses and the method here, via the impulsive differential inequalities, extends and improves those employed in [12-13]. Some interesting results are gained here. In addition, some examples, which dwell upon the importance of our results, are also included.

As to some related results about the oscillation of some second-order nonlinear ODEs with impulses, we also mention two papers[21&22]. For the theory of neutral delay differential equations and impulsive differential equations, please see the recent books by Györi and Ladas[23] and Lakshmikantham, Bainov and Simeonov[2] respectively. For more detail on the existence, uniqueness and continuity of solutions of impulsive delay differential-difference equations, we refer the reader to Ref. [24 & 25] and the references therein.

For notation convenience, Let

$$y(t) = x(t) - x(t - \tau). \qquad \ldots (3)$$

A solution of (1) is said to be non-oscillatory if this solution is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.

Obviously, the set W of all nonoscillatory solutions of eq. (1)-(2) can be divided into the following three parts ($W = W^1 \cup W^2 \cup W^3$) :

$$W^1 = \{x(t) \mid x(t)\, x'(t) \ge 0 \text{ eventually}\},$$

$$W^2 = \{x(t) \mid x(t)\, x'(t) \le 0 \text{ eventually}\},$$

and $$W^3 = \{x(t) \mid x'(t) \text{ is oscillatory}\}.$$

This paper is organized as follows: In Section 2 we shall offer three interesting Lemmas, which will be used in Section 3 to prove our main theorems. In section 4, an interesting example is also given.

2. Some Lemmas

Lemma 1 — [see 2, Theorem 1.4.1] Assume that

(A_0) $m \in PC^1[R_+, R]$ and $m(t)$ is left-continuous at t_k, $k = 1, 2...$

(A_1) for $k = 1, 2, \ldots, t \ge t_0$,

$$m'(t) \le p(t)\, m(t) + q(t), \quad t \ne t_k,$$

$$m(t_k^+) \le d_k\, m(t_k) + b_k,$$

where $q, p \in PC^1[R_+, R]$, $d_k \ge 0$ and b_k are constants. Then

$$m(t) \le m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^{t} p(s)\, ds\right)$$

$$+ \sum_{t_0 < t_k < t} \left(\sum_{t_k < t_j < t} d_j \exp\left(\int_{t_k}^{t} p(s)\, ds\right) b_k \right)$$

$$+ \int_{t_0}^{t} \prod_{s < t_k < t} d_k \exp\left(\int_{s}^{t} p(\sigma)\, d\sigma\right) q(s)\, ds \Bigg),\ ds \ge t_0.$$

Lemma 2 — Let $x(t)$ be a solution of eqn. (1)-(2). Suppose that there exists some $T \ge t_0$ such that $x(t) > 0$ for $t \ge T$. If

$$\int_{t_j}^{+\infty} \frac{1}{r^{1/\alpha}(s)} \prod_{t_j < t_i \le s} \frac{d_i^{1/\alpha}}{c_i}\, ds = +\infty \qquad \ldots (4)$$

holds for all sufficiently large $t_j \ge T\,(\ge t_1)$. Then

$$y'(t_k^+) \ge 0 \text{ and } y'(t) = (x(t) - x(t-\tau))' \ge 0, \qquad \ldots (5)$$

for $t \in (t_k, t_{k+1}]$, where $t_k \ge T$, $y(t)$ is denoted by (3).

PROOF : At first, we prove that $y'(t_k^+) \ge 0$ for any $t_k \ge T$. If not, then there exists some j such that $t_j \ge T$, $y'(t_j) < 0$ and

$$\begin{aligned} & r(t_j^+)\,|\,y'(t_j^+)\,|^{\alpha-1}\,y'(t_j^+) \\ &= r(t_j^+)\,|\,(x'(t_j^+) - x'(t_j^+ - \tau))\,|^{\alpha-1}\,(x'(t_j^+) - x'(t_j^+ - \tau)) \\ &= N_j\,(r(t_j)\,|\,y'(t_j)\,|^{\alpha-1}\,y'(t_j)) \\ &\le d_j\,r(t_j)\,|\,y'(t_j)\,|^{\alpha-1}\,y'(t_j) \\ &< 0. \end{aligned}$$

Let

$$\begin{aligned} & r(t_j^+)\,|\,(x'(t_j^+) - x'(t_j^+ - \tau))\,|^{\alpha-1}\,(x'(t_j^+) - x'(t_j^+ - \tau)) \\ &= r(t_j^+)\,|\,y'(t_j^+)\,|^{\alpha-1}\,y'(t_j^+) \\ &= -\beta\ (\beta > 0), \end{aligned}$$

and

$$\begin{aligned} S(t) &= r(t)\,|\,y'(t)\,|^{\alpha-1}\,y'(t) \\ &= r(t)\,|\,x(t) - x(t-\tau))'\,|^{\alpha-1}\,(x(t) - x(t-\tau))'. \end{aligned}$$

By (1)-(2), for $t \in (t_{j+i-1}, t_{j+i}]$, $i = 1, 2, \ldots$, we have

$$\begin{aligned} S'(t) &= (r(t)\,|\,(x(t) - x(t-\tau))'\,|^{\alpha-1}\,(x(t) - x(t-\tau))')' \\ &= -f(t, x(t), x(t-\sigma))) \\ &\le -p(t)\,\phi\,(x(t-\sigma))) \\ &\le 0. \end{aligned}$$

Hence, $S(t)$ is monotonically decreasing in $(t_{j+i-1}, t_{j+i}]$. So

$$r(t_{j+1})\,|\,y'(t_{j+1})\,|^{\alpha-1}\,y'(t_{j+1}) \le r(t_j^+)\,|\,y'(t_j^+)\,|^{\alpha-1}\,y'(t_j^+) = -\beta^{\alpha} < 0.$$

and

$$r(t_{j+2})\,|\,y'(t_{j+2})\,|^{\alpha-1}\,y'(t_{j+2}) \le r(t_{j+1}^+)\,|\,y'(t_{j+1}^+)\,|^{\alpha-1}\,y'(t_{j+1}^+)$$

$$= N_{j+1}(r(t_{j+1})\,|y'(t_{j+1})|^{\alpha-1}y'(t_{j+1}))$$

$$\le d_{j+1}\,r(t_{j+1})\,|y'(t_{j+1})|^{\alpha-1}y'(t_{j+1})$$

$$= -d_{j+1}\,\beta^{\alpha}$$

$$< 0.$$

By induction, we obtain

$$r(t)\,|y'(t)|^{\alpha-1}y'(t)$$

$$= r(t)\,|(x(t)-x(t-\tau))'|^{\alpha-1}(x(t)-x(t-\tau))'$$

$$\le -(d_{j+1}d_{j+2}\cdots d_{j+n})\,\beta$$

$$< 0,$$

for $t \in (t_{j+n}, t_{j+n+1}]$. Therefore,

$$y'(t) = (x(t)-x(t-\tau))' \le -\frac{\beta \prod_{t_j<t_k<t} d_k^{1/\alpha}}{r^{1/\alpha}(t)}.$$

In view of (3) and condition (ii), we have $y(t_k^+) \le c_k\,y(t_k)$. Applying Lemma 1, we obtain

$$y(t) \le y(t_j^+)\prod_{t_j<t_k<t} c_k - \beta \int \prod_{s<t_k<t} c_k \prod_{t_j<t_i<s} d_i^{1/\alpha}\frac{1}{r^{1/\alpha}(s)}\,ds,\ t \ge t_j.$$

In view of the fact that $\prod_{t_j<t_k<t} c_k = \prod_{t_j<t_i\le s} c_i \prod_{s<t_l<t} c_l$, we have

$$y(t) \le \prod_{t_j<t_k<t} c_k\left\{y(t_j^+) - \beta\int_{t_j}^{t}\frac{1}{r^{1/\alpha}(s)}\prod_{t_j<t_i\le s}\frac{d_i^{1/\alpha}}{c_i}\,ds\right\},\ t > t_j. \qquad \text{... (7)}$$

If $y(t) > 0$ eventually, then, in view of (4), one can find that the left side of (7) is eventually positive whereas the right dise is eventually negative, which is a contradiction. If not, then there exists $T_0 \ge T$ and a positive constant M such that $y(t) = x(t) - x(t-\tau) \le -M$ for $t \ge T_0$, which leads to

$$x(t) \le x\left(t - \left[\frac{t-T_0}{\tau}\right]\tau\right) - \left[\frac{t-T_0}{\tau}\right]M \to -\infty\ (t \to \infty).$$

which is a contradiction. Therefore,

$$y'(t_k) \geq 0 \ (t_k \geq T).$$

Because $S(t)$ is decreasing in $(t_{j+i-1}, t_{j+i}]$, we get, for $t \in (t_{j+i-1}, t_{j+i}]$, $S(t) \geq 0$, which implies

$$y'(t) = (x(t) - x(t-\tau))' \geq 0.$$

The proof of this Lemma is complete.

Remark 1 : In the case that $x(t)$ is eventually negative, if (4) holds true, then $y'(t_k^+) = (x(t_k^+) - x(t_k^+ - \tau))' \leq 0$ and $(x(t) - x(t-\tau))' \leq 0$ for $t \in (t_{j+i-1}, t_{j+i}]$, where $t_k \geq T$.

Lemma 3 — Let $x(t)$ is a nonoscillatory solution of eqn. (1)-(2) and y_n is defined by (3). If (5) holds and there exist T_1 such that $x(t) > 0$ for $t \geq T_1$, then either

$$y(t) > 0, \ W^+ \neq \phi$$

and it is an unbounded set,

or $$y(t) < 0, \ \lim_{t \to \infty} y(t) = 0, \ W^- \neq \phi$$

and it is an unbounded set,

where W^+, W^- are defined respectively as follows :

$$W^+ = \{t : x'(t) \geq 0, t \geq T_1\},$$

and $$W^- = \{t : x'(t) < 0, t \geq T_1\}.$$

Furthermore, if $W^- \neq \phi$ and it is an unbounded set, then

$$\lim_{t \to \infty,\ t \in W^-} x'(t) = 0.$$

The proofs can be followed from (5) and the monotone convergence theorem and mathematical induction, thet are omitted.

3. Main Results

Theorem 1 — *Assume that (4) holds and* $x(t_k^+) = x(t_k), x(t_k^+ - \tau) = x(t_k - \tau)$. *If*

$$\int_T^{+\infty} p(s) \prod_{T < t_k \leq s} \frac{1}{d_k^*} ds = +\infty \qquad \text{... (8)}$$

holds for all sufficiently large T. Then $W^1 = \phi$.

PROOF : Without loss of generality, we can assume $k_0 = 1$. If (1)-(2) has a non-oscillatory solution $x(t) \in W^1$, we might assume that $x(t) > 0$ for $t \geq t_0$ (the proof is similar for the case $x(t) < 0$), then $x'(t) \geq 0$ for $t \geq T_1 \geq t_0$. It follows from Lemma 2 that $y'(t) = (x(t) - x(t-\tau))' \geq 0$ for $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots$. Let

$$w(t) = \frac{r(t) \,|\, y'(t) \,|^{\alpha - 1} y'(t)}{\phi(x(t-\sigma))}$$

$$= \frac{r(t) \,|\, (x(t) - x(t) - \tau))' \,|^{\alpha-1} (x(t) - x(t-\tau))'}{\phi(x(t-\sigma))}. \quad \ldots (9)$$

Then $w(t_k^+) \geq 0$ $(k = 1, 2, \ldots)$, $w(t) \geq 0$ for $t \geq t_0$. In view of $x'(t) \geq 0$ and condition (i), Lemma 3 and eq. (1)-(2), we get

$$w'(t) = -\frac{f(t, x(t), x(t-\sigma))}{\phi(x(t-\sigma))}$$

$$-\frac{r(t) \,|\, (x(t) - x(t-\tau))' \,|^{\alpha-1} (x(t)\ x(t-\tau))' \,\phi'(x(t-\sigma))\, x'(t-\sigma)}{\phi^2(x(t-\sigma))}$$

$$\leq -p(t). \quad \ldots (10)$$

It follows from condition (ii), $x'(t) \geq 0$, $\phi'(x) \geq 0$ that

$$w(t_k^+) = \frac{r(t_k^+) \,|\, y'(t_k^+) \,|^{\alpha-1} y'(t_k^+)}{\phi\, x(t_k^+ - \sigma))}$$

$$\leq \frac{d_k^* \, r(t_k) \,|\, y'(t_k) \,|^{\alpha-1} y'(t_k)}{\phi(x(t_k - \sigma))} = d_k^* \, w(t_k).$$

Therefore, $w(t)$ satisfies the following differential inequalities :

$$w'(t) \leq -p(t), \quad t \neq t_k,$$

$$w(t_k^+) \leq d_k^* \, w(t_k).$$

Then, applying Lemma 1, we obtain

$$w(t) \leq \prod_{t_j < t_k < t} d_k^* \left\{ w(t_j^+) - \int_{t_j}^{t} p(s) \prod_{t_j < t_k \leq s} \frac{1}{d_k^*} \, ds \right\}, \; t \geq t_j,$$

which, in view of (8) and $w(t) \geq 0$, leads to a contradiction as $t \to \infty$.

The proof of Theorem 1 is complete.

Theorem 2 — *Assume that (4) and (8) hold. If eq. (1)-(2) has a non-oscillatory solution* $x(t) \in W^2$, *then*

$$\lim_{t \to \infty} x(t) = 0.$$

PROOF : We might assume that $x(t) > 0$ $(t \geq t_0)$ (the proof is similar for the case $x(t) < 0$). Then $x'(t) \leq 0$, which implies there exist a nonnegative constant M such that

$$\lim_{t \to \infty} x(t) = M.$$

If $M = 0$, then the proof is complete. If not, then $M > 0$ and $x(t-\sigma) \geq M$ and $\phi(x(t-\sigma)) \geq \phi(M)$ for $t \geq T_1 \geq T$. In view of condition (i) and eq. (1), we have

$$(r(t) \,|\, (x(t) - x(t-\tau))' \,|^{\alpha-1} (x(t) - x(t-\tau))')' = -f(t, x(t), x(t-\sigma))$$

$$\leq -p(t)\,\phi(x(t-\sigma))$$

$$\leq -p(t)\,\phi(M).$$

Let $$w(t) = r(t) \,|\, (x(t) - x(t-\tau))' \,|^{\alpha-1} (x(t) - x(t-\tau))'.$$

It follows from Lemma 1 that $w(t) \geq 0$. In view of (2), we find that $w(t)$ satisfies the following difference inequalities :

$$w'(t) \leq -\phi(M)\,p(t), \quad t \neq t_k,$$

$$w(t_k^+) \leq d_k^* \, w(t_k).$$

Then, applying Lemma 1, we obtain

$$w(t) \leq \prod_{t_j < t_{0,n} < t} d_k^*$$

$$\left\{ w(t_j^+) - \phi(M) \int_{t_j}^{t} p(s) \prod_{t_j < t_k \leq s} \frac{1}{d_k^*} \, ds \right\}, \; t \geq t_0. \qquad \text{... (11)}$$

In view of (8), the right side of inequality (11) is eventually negative whereas the left side is nonnegative, which is a contradiction.

The proof of Theorem 2 is complete.

The following result is an immediate consequence of Theorem 1 and Theorem 2.

Theorem 3 — *If (4) and (8) hold, and* $x(t_k^+) = x(t_k)$, $x(t_k^+ - \tau) = x(t_k - \tau)$. *Then every solution* $x(t)$ *of eq. (1)-(2) has one of the following properties :*

(1) $x(t)$ *is oscillatory;*

(2) $x'(t)$ *is oscillatory; and*

(3) $x(t)$ monotonically converges to zero as $t \to \infty$.

Using Theorem 3, we can obtain some corollaries as follows :

Corollary 1 — Assume that (4) holds, $x(t_k^+) = x(t_k), x(t_k^+ - \tau) = x(t_k - \tau)$, and there exists a positive integer k_0 such that $d_k^* \leq 1$ for $k \geq k_0$. If

$$\int^{+\infty} p(s)\, ds = +\infty. \qquad \text{... (12)}$$

Then the conclusions of Theorem 3 hold true.

Corollary 2 — Assume that (4) holds, $x(t_k^+) = x(t_k), x(t_k^+ - \tau) = x(t_k - \tau)$ and there exist a positive integer k_0 and a constant $\gamma > 0$ such that

$$\frac{1}{d_k^*} \geq \left(\frac{t_{k+1}}{t_k} \right)^{\gamma}, \text{ for } k \geq k_0 \qquad \text{... (13)}$$

and
$$\int^{+\infty} t^{\gamma} p(t)\, dt = +\infty. \qquad \text{... (14)}$$

Then the conclusions of Theorem 3 hold true.

The proofs of the above corollaries are similar to those for Corollary 1-2 in [12] or [13] and they are omitted.

Now we consider eq. (1) under some special impulsive perturbations, i.e.,

$$\begin{cases} t_{k+1} - t_k \equiv \tau, \\ x(t_k^+) = cx(t_k), \quad x'(t_k^+) = dx'(t_k), \end{cases} \qquad \text{... (15)}$$

where c, d are two positive constants, and $r(t)$ is eventually positive and continuous. Obviously, $M_k(x) \equiv cx, N_k(x) \equiv dx, c_k \equiv c_k^* \equiv c, d_k \equiv d_k^* \equiv d$.

***Theorem* 4** — *Assume that $\phi(ab) \geq \phi(a)\phi(b)$ for any $ab > 0$. If (4) holds and*

$$\int_T^{+\infty} p(s) \prod_{T < t_{0,n} \leq s} \frac{1}{\mu_{0,n}}\, ds = +\infty \qquad \text{... (16)}$$

holds for all sufficiently large $T\,(> t_0)$, where

$$\mu_{0,n} = \begin{cases} \dfrac{d}{\phi(c)}, & t_{0,n} = t_k + \sigma = t_m \ (m > k), \\ d, & t_{0,n} = t_k \text{ and } t_k - \sigma \neq t_m, \\ \dfrac{1}{\phi(c)}, & t_{0,n} = t_k + \sigma \neq t_m \ (m > k), \\ \dfrac{d}{\phi(c)}, & t_{0,n} = t_k \text{ and } t_k - \sigma = t_m \ (0 < m < k). \end{cases} \qquad \text{... (17)}$$

Then $W^1 = \phi$.

PROOF : If (1) has a non-oscillatory solution $x(t) \in W^1$, without loss of generality, we can assume $x(t) > 0$ $(t \geq t_0)$. Then $x'(t) \geq 0$ for $t \geq T$. Let $w(t)$ be defined by (9). Then

$$w(t_k^+) \geq 0, \; w(t) \geq 0 \; (t \geq T).$$

It is easy to see that

$$w(t_k^+) = \frac{r(t_k^+) \,|\, y'(t_k^+) \,|^{\alpha-1}\, y'(t_k^+)}{\phi(x(t_k^+ - \sigma))} \leq \frac{dr(t_k \,|\, y'(t_k) \,|^{\alpha-1}\, y'(t_k)}{\phi(x(t_k - \sigma))} = dw(t_k), \qquad \dots (18)$$

for $t_k - \sigma \neq t_m$ $(0 < m < k)$ or

$$w(t_k^+) \leq \frac{dr(t_k) \,|\, y'(t_k) \,|^{\alpha-1}\, y'(t_k)}{\phi(cx(t_m))} \leq \frac{dr(t_k) \,|\, y'(t_k) \,|^{\alpha-1}\, y'(t_k)}{\phi(c)\, \phi(x(t_k - \sigma))} = \frac{d}{\phi(c)} w(t_k),$$

for $t_k - \sigma = t_m$ $(0 < m < k)$. Similarly, we can obtain

$$w(t_k^+ + \sigma) \leq \begin{cases} \dfrac{1}{\phi(c)} w(t_k + \sigma), & t_k + \sigma \neq t_m \; (m > k), \\[2ex] \dfrac{d}{\phi(c)} w(t_k + \sigma), & t_k + \sigma = t_m \; (m > k). \end{cases} \qquad \dots (19)$$

It follows from inequalities (10), (18) and (19) that

$$w'(t) \leq -p(t), \; t \neq t_{0,n}$$

$$w(t_{0,n}^+) \leq \mu_{0,n}\, w(t_{0,n}),$$

where $t_{0,n} = t_k$ or $t_k + \sigma$ $(t_1 = t_{0,1} < t_{0,2} < \dots < t_{0,n} < t_{0,n+1} < \dots)$ and $\mu_{0,n}$ is defined by (17). Then, applying Lemma 1, we obtain

$$w(t) \leq \prod_{t_j < t_{0,n} < t} \mu_{0,n} \left\{ w(t_j^+) - \int_{t_j}^{t} p(s) \prod_{t_j < t_{0,n} \leq s} \frac{1}{\mu_{0,n}} ds \right\}, \; t \geq t_j \geq T \qquad \dots (20)$$

In view of (16), (20) and $w(t) \geq 0$, we get a contradiction as $t \to \infty$.

The proof of Theorem 4 is complete.

Corollary 3 — Assume that (4) hold and $t_{k+1} - t_k \equiv \tau > \sigma$, $\phi(ab) \geq \phi(a)\,\phi(b)$ for any $ab > 0$. Furthermore, suppose that there exist a positive integer k_0 and a constant $\gamma > 0$ such that

$$\frac{\phi(c)}{d} \geq \left(\frac{t_{k+1}}{t_k} \right)^{\gamma}, \text{ for } k \geq k_0, \qquad \dots (21)$$

and (14) hold. Then the conclusions of theorem 3 for eq. (1)-(14) hold true.

Corollary 3 can be deduced from Theorem 2 and Theorem 4. Its proof is similar to tha of Corollary 2 and it is omitted.

Theorem 5 — *Assume that (4) holds and there exists a positive integer* k_0 *such that* $c \geq 1$ *for* $k \geq k_0$. *If*

$$\int_T^{+\infty} p(s) \prod_{T < t_{0,n} \leq s} \frac{1}{\theta_{0,n}} ds = +\infty \qquad \text{... (22)}$$

holds for all sufficiently large T, where

$$\theta_{0,n} = \begin{cases} 1, & t_{0,n} = t_k + \sigma \neq t_m \ (m > k) \\ d, & t_{0,n} = t_k . \end{cases}$$

Then $W^1 = \phi$.

The proof is similar to that of Theorem 4 and it is omitted.

Using Theorem 5, we can obtain some corollaries as follows :

Corollary 4 — Assume that (4) and (12) hold, and there exists a positive integer k_0 such that $c \geq 1, d \leq 1$ for $k \geq k_0$. Then the conclusions of theorem 3 for eq. (1)-(15) hold true.

Corollary 5 — If (4), (13) and (14) hold, and there exist a positive integer k_0 such that $c \geq 1$ for $k \geq k_0$, then the conclusions of theorem 3 hold true.

The proofs of the above corollaries are similar to those for Corollary 1-2 and they are omitted.

Remark 2 : Using the same technique and the same argument as above, one also can obtain new criteria about the oscillation of the advanced differential equation

$$(r(t) \mid (x(t) - x(t-\tau))' \mid^{\alpha-1} (x(t) - x(t-\tau))')' + f(t, x(t), x(t+\sigma)) = 0$$

with impulses

$$\begin{cases} x(t_k^+ - \tau) = M_k (x(t_k) - x(t_k - \tau)), \\ r(t_k^+) \mid (x'(t_k^+) - x'(t_k^+ - \tau)) \mid^{\alpha-1} (x'(t_k^+) - x'(t_k^+ - \tau)) = \\ N_k (r(t_k) \mid (x'(t_k) - x'(t_k - \tau)) \mid^{\alpha-1} (x'(t_k) - x'(t_k - \tau))). \end{cases}$$

4. Examples

Example 1 — Consider the impulsive delay differential equation

$$\begin{cases} (x(t) - x(t-\tau))'' + \dfrac{\lambda}{t^2} x(t-\sigma)) = 0, & t \neq k, k = 1, 2, \ldots \\ x(k^+) = x(k), & \\ x'(k^+) - x'(k^+ - \tau) = \left(\dfrac{k}{k+1}\right)(x'(k) - x'(k-\tau)), & \end{cases} \quad \ldots (23)$$

where $c_k = c_k^* \equiv 1$, $d_k = d_k^* = \dfrac{k}{k+1}$, $p(t) = \dfrac{\lambda}{t^2}$, $t_k = k$, and $\phi(x) = x$. Obviously, $\alpha = 1$; the conditions (i) and (ii) are satisfied and

$$\int_{t_j}^{+\infty} \frac{1}{r^{1/\alpha}(s)} \prod_{t_j < t_i < s} \frac{d_i^{1/\alpha}}{c_i} ds = R(t_{j+1}) - R(t_j)$$

$$+ \frac{d_{j+1}^{1/\alpha}}{c_{j+1}} (R(t_{j+2} - R(t_{j+1}))$$

$$+ \ldots + \frac{d_{j+1}^{1/\alpha} d_{j+2}^{1/\alpha} \cdots d_{j+n}^{1/\alpha}}{c_{j+1} c_{j+2} \cdots c_{j+n}} (R(t_{j+n+1}) - R(t_{j+n})) + \ldots$$

$$= 1 + \frac{j+1}{j+2} + \ldots + \frac{j+1}{j+n} + \ldots$$

$$= \infty.$$

Let $k_0 = 1$, $\gamma = 1$. Then

$$\frac{1}{d_k^*} = \frac{k+1}{k} = \frac{t_{k+1}}{t_k}$$

and $$\int^{+\infty} t^\gamma p(t)\, dt = \int^{+\infty} tp(t)\, dt = = \infty .$$

By corollary 2, we know that every solution $x(t)$ of eq. (23) has one of the following properties :

(1) $x(t)$ is oscillatory;

(2) $x'(t)$ is oscillatory;

(3) $x(t)$ monotonically converges to zero as $t \to \infty$.

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