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Z-SUPERCONTINUOUS FUNCTIONS

J. K. KOHLI* AND RAJESH KUMAR**

**Department of Mathematics, Hindu College, University of Delhi, Delhi* 110 007, *India*
***Department of Mathematics, University of Delhi, Delhi* 110 007, *India*

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A new class of functions, called z-supercontinuous functions, is introduced. Basic properties of z-supercontinuous functions are studied. Sufficient conditions on domain/range are given for a continuous function to be z-supercontinuous. The class of z-supercontinuous functions properly includes the class of clopen maps of Reilly and Vamanamurthy (*Indian J. Pure Appl. Math.* **14** (1983), 767-772). Moreover, the class of z-supercontinuous functions constitutes a proper subclass of each of the classes of:

(1) supercontinuous functions introduced by Munshi and Bassan (Indian J. Pure Appl. Math. **13**(2) (1982), 229-36);
(2) strongly θ-continuous functions of Noiri (J. Korean. math. Soc. **16** (1980), 161-66); and
(3) D-supercontinuous functions initiated by Kohli and Singh (*Indian J. pure appl. Math. 32(2) (2001), 227-35.*

Key Words : Supercontinuous Function; Z-Supercontinuous Function; D-Supercontinuous Function; Strongly θ-Continuous Function; Strongly Continuous Function; Perfectly Continuous Function; Clopen map, Completely Continuous Function; Z-open (closed) set.

1. INTRODUCTION

Several weak and strong variants of continuity occur in the literature. The strong variants of continuity with which we shall be dealing with in this paper include [1, 6, 8, 10, 11, 12, 14]. Certain of these strong variants of continuity coincide with continuity of domain/range space is suitably augmented. In this paper, we introduce a new strong form of continuity called "z-supercontinuous function" which coincides with continuity if domain or range is a completely regular space, or if range is a perfectly normal space. The class of z-supercontinuous functions properly includes the class of clopen maps[14] and hence contains all perfectly continuous functions of Noiri[12] which in its turn include all strongly continuous functions of Levine[8]. Moreover, the class of z-supercontinuous functions properly constitutes a subclass of each of the classes of :

(1) supercontinuous functions[10]

(2) strongly θ-continuous functions[11]; and

(3) D-supercontinuous functions[6].

Basic properties of z-supercontinuous functions are elaborated in Section 3. In Section 4, we consider the notions of z-quotient topology and a z-quotient space and compare it with the usual quotient topology and quotient space as well as with D-quotient topology[6] and δ-quotient topology[10]. Section 5 is devoted to the study of the behaviour of separation axioms under z-supercontinuous functions. In Section 6, we consider complete regularization of a space and conclude with alternative proofs of certain results of preceding sections.

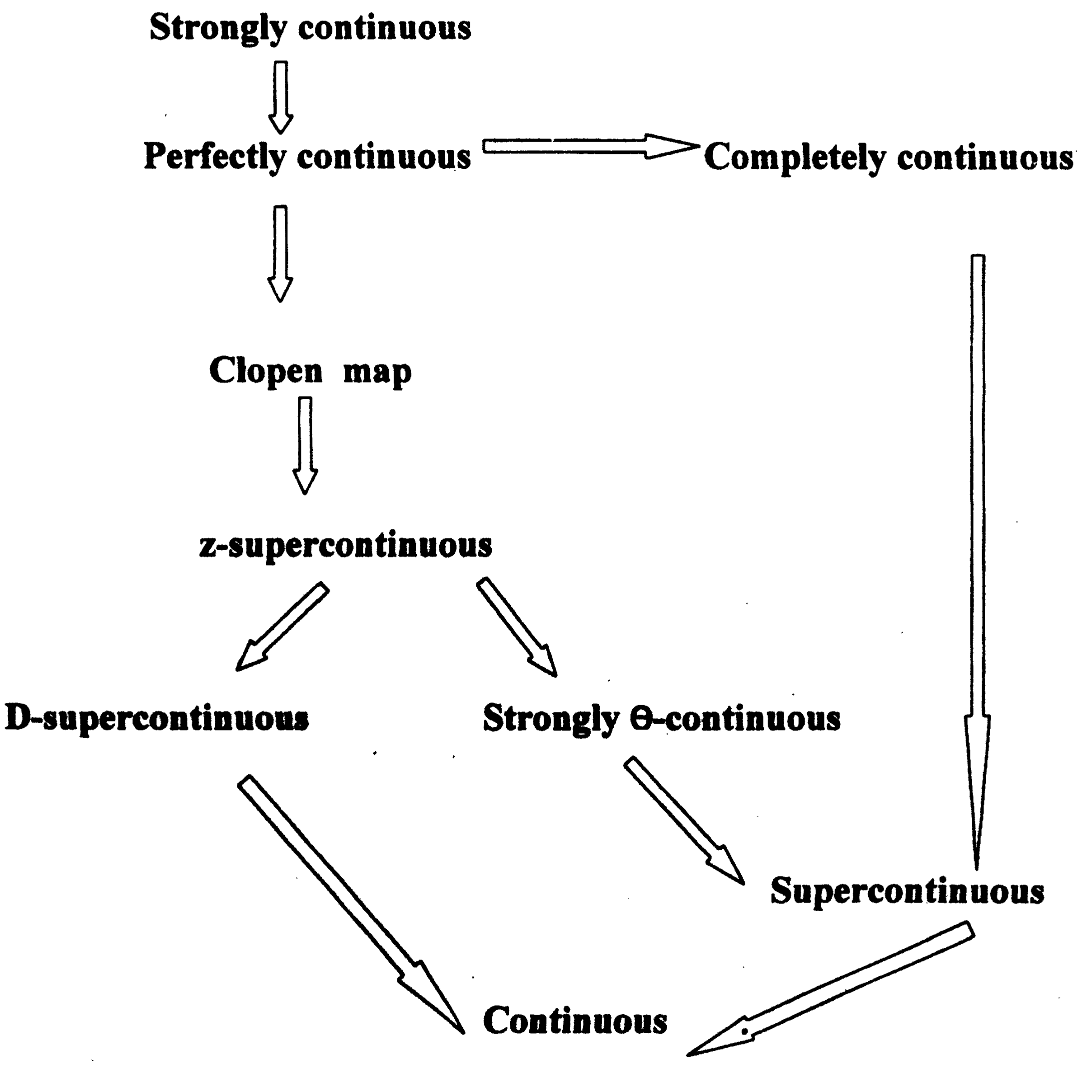
2. PRELIMINARIES AND BASIC DEFINITIONS

Definition 2.1 — A function $f: X \to Y$ from a topological space X into a topological space Y is said to be *z-supercontinuous at a point* $x \in X$ if for every open set V containing $f(x)$ there exists a cozero set U containing x such that $f(U) \subset V$. The function f is said to be *z-supercontinuous* if it is z-supercontinuous at each $x \in X$.

Definition 2.2[6] — A function $f: X \to Y$ is said to be *D-supercontinuous* at a point $x \in X$ if for every open set V containing $f(x)$ there exists an open F_σ-set U containing x such that $f(U) \subset V$. The function f is said to be *D-supercontinuous* if it is D-supercontinuous at each $x \in X$.

Definition 2.3[8] — A function $f: X \to Y$ is said to be *strongly continuous* if $f(\overline{A}) \subset f(A)$ for all $A \subset X$.

Definition 2.4[1] — A function $f: X \to Y$ is said to be *completely continuous* if for each open set $V \subset Y, f^{-1}(V)$ is regularly open.



Definition 2.5 [11] — A function $f: X \to Y$ is said to be *strongly θ-continuous* if for each $x \in X$ and each open set V containing $f(x)$ there is an open set U containing x such that $f(\overline{U}) \subset V$.

Definition 2.6[14] — A function $f: X \to Y$ is said to be *clopen* if for each $x \in X$ and each open set V containing $f(x)$ there is a clopen set U containing x such that $f(U) \subset V$.

Definition 2.7[10] — A function $f: X \to Y$ is said to be *supercontinuous* at a point $x \in X$ if for every open set U containing $f(x)$ there exists an open set N containing x such that $f((\overline{N})^0) \subset U$. The function f is *supercontinuous* if it is supercontinuous at each $x \in X$.

Definition 2.8[12] — A function $f: X \to Y$ is said to be *perfectly continuous* if for every open set $V \subset Y, f^{-1}(V)$ is clopen in X.

The diagram on previous page well illustrates the relationships that exist among z-supercontinuous functions and various variants of continuity defined above.

However, none of the above implications in general is reversible as will be exhibited in the sequel.

Noiri[11] gave examples to show that a clopen map need not be perfectly continuous and that a perfectly continuous map need not be strongly continuous. Moreover, Noiri[12] showed that a completely continuous map need not be perfectly continuous.

Example 2.9 — Every nonconstant continuous real-valued map of a real variable is z-supercontinuoius but fails to be a clopen map.

Example 2.10 — Let X denote the Mountain chain space due to Heldermann[5]. Then X is a regular space which is not D-regular[5]. The identity function defined on X is strongly θ-continuous but not z-supercontinuous. Infact, any nonconstant continuous function defined on a regular space but not a completely regular space is strongly θ-continuous but not necessarily z-supercontinuous.

Example 2.11 — Let X denote the set of positive integers endowed with the cofinite topology. The identity function defined on X is D-supercontinuous but not supercontinuous and hence not z-supercontinuous.

Example 2.12 — Let X denote the real line endowed with usual topology. The identity function defined on X is z-supercontinuous but not completely continuous.

3. PROPERTIES OF Z-SUPERCONTINUOUS FUNCTIONS

Definition 3.1 — A set G in a topological space X is said to be *z-open* if for each $x \in G$ there exists a cozero set H such that $x \in H \subset G$, or equivalently, if G is expressible as the union of cozero sets. The complement of a z-open set will be referred to as a *z-closed* set.

Theorem 3.2 — *For a function $f: X \to Y$ form a topological space X into a topological space Y, the following statements are equivalent :*

(a) f is z-supercontinuous;

(b) inverse image of every open subset of Y is z-open in X;

(c) inverse image of every closed subset of Y is z-closed in X;

(d) For each $x \in X$ and for each open set V containing $f(x)$, there exists a z-open set U containing x such that $f(U) \subset V$.

Definition 3.3 — Let X be a topological space and let $A \subset X$. A point $x \in X$ is said to be a *z-adherent point* of A if every cozero set containing x intersects A. Let A_z denote the set of all z-adherent points of A. It is easily verified that the set A is z-closed if and only if $A = A_z$.

Theorem 3.4 — *A function f from a space X into a space Y is z-supercontinuous if and only if $f(A_z) \subset f(A)$ for every $A \subset X$.*

PROOF : Suppose that f is z-supercontinuous. Since $\overline{f(A)}$ is closed in Y, by Theorem 3.2 $f^{-1}(\overline{f(A)}))$ is z-closed in X. Now, since $A \subset f^{-1}(f(\overline{A}))$,

$A_z \subset f^{-1}(\overline{f(A)}))_z = f^{-1}(\overline{f(A)}))$ and so $f(A_z) \subset f(f^{-1}(\overline{f(A)}))) \subset \overline{f(A)}$.

Conversely, suppose that $f(A_z) \subset \overline{f(A)}$ for every $A \subset X$.

Let F be any closed set in Y. Then $f((f^{-1}(F)_z)) \subset f(f^{-1}(\overline{F})) \subset \overline{F} = F$ and hence $(f^{-1}(F))_z \subset f^{-1}(F)$. Thus $(f^{-1}(F))_z = f^{-1}(F)$ which in its turn shows that f is z-supercontinuous.

Theorem 3.5 — *A function $f: X \to Y$ from a space X into a space Y is z-supercontinuous if and only if $(f^{-1}(B))_z \subset f^{1}(\overline{B})$ for every $B \subset Y$.*

PROOF : Suppose f is z-supercontinuous. Then $f^{-1}(\overline{B})$ is a z-closed set in X for every $B \subset Y$. Now since $f^{-1}(\overline{B}) \subset f^{-1}(B))_z \subset f^{-1}(\overline{B})$.

Conversely, let F be any closed set in Y. Then $\overline{(f^{-1}(F))_z} \subset f^{-1}\overline{F} = f^{-1}(F)$. Since $f^{-1}(F) \subset \overline{f^{-1}(F)} \subset (f^{-1}(F))_z$, $f^{-1}(F) = (f^{-1}(F))_z$ which in turn implies that f is z-supercontinuous.

Definition 3.6 — A filter $\boldsymbol{F}$ is said to z-converge to a point x if every cozero set containing x contains a member of $\boldsymbol{F}$.

Theorem 3.7 — *A function $f: X \to Y$ is z-supercontinuous if and only if $f(F) \to f(x)$ for each $x \in X$ and each filter F in X that z-converge to x.*

PROOF : Suppose that f is z-supercontinuous and let $\boldsymbol{F}$ be a filter in X that z-converge to x. Let W be an open set containing $f(x)$. Then $x \in f^{-1}(W)$ and $f^{-1}(W)$ is z-open. Let H be a cozero set such that $x \in H \subset f^{-1}(W)$ and so $f(H) \subset W$. Since $\boldsymbol{F}$ z-converges to x; there exists $U \in \boldsymbol{F}$ such that $U \subset H$ and hence $f(U) \subset f(H) \subset W$. Thus $f(\boldsymbol{F}) \to f(x)$.

Conversely, let W be an open subset of Y containing $f(x)$. Now, the filter F generated by the filterbase B_x consisting of cozero sets containing x, z-converges to x. So by hypothesis $f(\boldsymbol{F}) \to f(x)$. Hence, there exists a member $f(\boldsymbol{F})$ of $f(\boldsymbol{F})$ such that $f(\boldsymbol{F}) \subset W$. Choose $B \in B_x$ such that $B \subset F$. Since B is a cozero set containing x and since $f(B) \subset f(\boldsymbol{F}) \subset W$, f is z-supercontinuous.

Theorem 3.8 — *If $f: X \to Y$ is z-supercontinuous and $f(X)$ is equipped with the subspace topology, then $f: X \to f(X)$ is z-supercontinuous.*

PROOF : Since $f: X \to Y$ is z-supercontinuous for every open set U of Y, $f^{-1}(U \cap f(X)) = f^{-1}(U) \cap f^{-1}f(X)) = f^{-1}(U) \cap X = f^{-1}(U)$ is z-open. Hence, $f: X \to f(X)$ is z-supercontinuous.

A function $f: X \to Y$ is said to be *z-open (z-closed)* if the image of every z-open (z-closed) subset of X is an open (closed) subset of Y. The function f is z-open if and only if the image of every cozero set in X is open in Y.

Theorem 3.9 — *Let function $f: X \to Y$ be a z-open, z-supercontinuous, surjection and let $g: Y \to Z$ be any function. Then gof is z-supercontinuous if and only if g is continuous.*

PROOF : Assume that gof is z-supercontinuous and let G be an open subset of Z.

Then $(g^o f)^{-1}(G) = f^{-1}(g^{-1}(G))$ is z-open in X. Since f is a z-open surjection, $f(f^{-1}(g^{-1}(G))) = g^{-1}(G))$ is z-open in Y. Hence g is continuous.

Conversely, let $V \subset Z$ be an open set. Then $g^{-1}(V)$ is open in Y and so $(gof)^{-1}(V) = f^{-1}(g^{-1}(V))$ is z-open in X, since g is continuous and f is z-supercontinuous. Hence, *gof* is z-supercontinuous.

Theorem 3.10 — *If $f: X \to Y$ is z-supercontinuous and $g: Y \to Z$ is continuous, then gof is z-supercontinuous. In particular, the composition of z-supercontinuous functions is z-supercontinuous.*

Corollary 3.11 — Let $f: X \to Y$ be z-supercontinuous. If Z is a space containing Y as a subspace, then the function $h: X \to Z$ defined by $h(x) = f(x)$ for each $x \in X$ is z-supercontinuous.

PROOF : Since h is the composition of z-supercontinuous functions $f: X \to Y$ and the inclusion mapping $i: Y \to Z$, by Theorem 3.10 it is z-supercontinuous.

Remark 3.12 : Theorem 3.10 shows that the study of z-supercontinuous functions from categorical viewpoint is useful. It seems an interesting and a rewarding exercise to study the category of topological spaces and z-supercontinuous functions.

Theorem 3.13 — *Let $\{f_a: X \to X_a \mid \alpha \in \Lambda\}$ be a family of functions and let $f: X \to \prod_{\alpha \in \Lambda} X_\alpha$ be defined by $f(x) = (f_\alpha(x))_{\alpha \in \Lambda}$. Then f is z-supercontinuous if and only if each $f_\alpha: X \to X_\alpha$ is z-supercontinuous.*

PROOF : Let $f: X \to \prod_{\alpha \in \Lambda} X_\alpha$ be z-supercontinuous. Then $f_\alpha = p_\alpha 0 f$, where p_α denotes the projection of X onto α-coordinate space X_α. Hence by Theorem 3.10 each f_α is z-supercontinuous.

Conversely, let $f_\alpha: X \to X_\alpha$ be z-supercontinuous for each $\alpha \in \Lambda$. To show that the function f is z-supercontinuous in view of Theorem 3.2, it is sufficient to show that $f^{-1}(U)$ is z-open for each open set U in the product space $\prod_{\alpha \in \Lambda} X_\alpha$. Since the finite intersections and arbitrary unions of z-open sets are z-open, it suffices to prove that $f^{-1}(S)$ is z-open for every subbasic open set S

in the product space $\prod_{\alpha \in \Lambda} X_\alpha$.

Let $U_\beta \times \prod_{\alpha \neq \beta} X_\alpha$ be a subbasic open set in $\prod_{\alpha \in \Lambda} X_\alpha$.

Then $f^{-1}\left(U_\beta x \prod_{\alpha \neq \beta} X_\alpha \right) = f^{-1}(p\,\beta^{-1}(U_\beta)) = f_\beta^{-1}(U_\beta)$ is z-open. Hence f is z-supercontinuous.

Theorem 3.14 — *Let $f: X \to Y$ be a function and let $g: X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, be the graph function. Then g is z-supercontinuous if and only if f is z-supercontinuous and X is completely regular.*

PROOF : Suppose that g is z-supercontinuous. By Theorem 3.10 $f = p_y 0 g$ is z-supercontinuous, where p_y is the projection form $X \times Y$ onto Y. Let U be any open set in X and let $x \in U$. Then $U \times Y$ is an open set containing $g(x)$. Since g is z-supercontinuous there exists a cozero set W containing x such that $g(W) \subset U \times Y$. Thus $x \in W \subset U$, which shows that U is z-open and so X is completely regular.

Conversely, let $x \in X$ and let W be an open set containing $g(x)$. There exist open sets $U \subset X$ and $V \subset Y$ such that $(x, f(x)) \in U \times V \subset W$. Since X is completely regular, there exists a cozero set G_1 in X such that $x \in G_1 \subset U$. Since f is z-supercontinuous, there exists a cozero set G_2 in X containing x such that $f(G_2) \subset V$. Let $G = G_1 \cap G_2$, then G is a cozero set containing x and $g(G) \subset U \times V \subset W$, which shows that g is z-supercontinuous.

Definition 3.15[16] — A function $f: X \to Y$ is said to be *z-continuous* if for each $x \in X$ and each cozero set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset V$.

Lemma 3.16 — For a function $f: X \to Y$, the following statements are equivalent :-

(a) f is z-continuous.

(b) $f(\overline{A}) \subseteq (f(A))_z$ for all $A \subseteq X$.

(c) $\overline{f^{-1}(B)} \subseteq f^{1}(B_z)$ for all $B \subseteq Y$.

(d) Inverse image of every z-closed set is closed, i.e., $f^{-1}(F)$ is closed in X for every z-closed set $F \subseteq Y$.

(e) Inverse image of every z-open set is open, i.e., $f^{-1}(V)$ is open in for every z-open set $V \subseteq Y$.

PROOF : (a) $\Rightarrow$ (b) : Let $y \in f(\overline{A})$. Choose $x \in \overline{A}$ such that $f(x) = y$. Let V be a cozero set containing y. Since f is z-continuous, $f^{-1}(V)$ is an open set containing x. This gives $f^{-1}(V) \cap A \neq \phi$ which in turn implies that $V \cap f(A) \neq \phi$ and consequently $y \in (f(A))_z$. Hence,

$f(\overline{A}) \subseteq (f(A))_z$.

(b) $\Rightarrow$ (c) : Let B be any subset of Y, then $f(\overline{f^{-1}(B)}) \subseteq f(f^{-1}(B))_z$ and consequently $\overline{f^{-1}(B)} \subseteq f^{1}(B_z)$.

(c) $\Rightarrow$ (d) : Since a set A is z-closed if and only if $A = A_z$, therefore the implication (c) $\Rightarrow$ (d) is obvious.

(e) $\Rightarrow$ (e) : Let V be a z-open set in Y. Then $Y - V$ is z-closed in Y and so $f^{-1}(Y-V) = X - f^{-1}(V)$ is closed in X implying that $f^{-1}(V)$ is open in X.

(f) $\Rightarrow$ (a) : Since every cozero set is z-open and since a function is z-continuous if and only if the inverse image of every cozero set is open. Hence, (e) $\Rightarrow$ (a).

Theorem 3.17 — *Let X, Y and Z be topological spaces and let the function $f: X \rightarrow Y$ be z-continuous and $g: Y \rightarrow Z$ be z-supercontinuous. Then $gof : X \rightarrow Z$ is continuous.*

PROOF : Let U be an open set in Z. By z-supercontinuity of g, $g^{-1}(U)$ is z-open set in X. Now, $(gof)^{-1}(U) = f^{-1}(g^{-1}(U))$. Therefore, by Lemma 3.16 $f^{-1}(g^{-1}(U))$ is open in X. Hence, $gof: X \rightarrow Y$ is continuous.

Theorem 3.18 — *Let $f: X \rightarrow Y$ be a continuous function defined on a completely regular space. Then f is z-supercontinuous.*

PROOF : In a completely regular space, every open set is z-open.

Theorem 3.19 — *Let $f: X \rightarrow Y$ be a continuous function. If Y is perfectly normal, then f is z-supercontinuous.*

PROOF : In a perfectly normal space, every open set is a cozero set and a continuous function lifts cozero sets to cozero set.

Theorem 3.20 — *Let $f: X \rightarrow Y$ be a continuous function and let Y be a completely regular space. Then f is z-supercontinuous.*

PROOF : In a completely regular space every open set is z-open and it is easily verified that a continuous function lifts z-open sets to z-open sets.

We may recall that a space X is quasicompact[4] if every cover of X by cozero sets admits a finite subcover.

Theorem 3.21 — *Let $f: X \rightarrow Y$ be a z-supercontinuous function from a quasicompact space X onto Y. Then Y is compact.*

PROOF : Let $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. Then each $f^{-1}(V_\alpha)$ is a z-open set in X and so it is a union of cozero sets. This in turn yields a cover ζ of X consisting of cozero sets. Since X is quasicompact there is a finite subcollection $\{C_1, \ldots, C_n\}$ of ζ which covers X. Suppose $C_i \subset f^{-1}(V_{\alpha i})$ for some $\alpha_i \in \Lambda$ (i = 1, ..., n). Then $\{V_{\alpha 1}, \ldots, V_{\alpha n}\}$ is a finite subcover of $\mathcal{V}$. Thus X is compact.

Definition 3.22[2] — A space X is said to be *almost compact* if every open covering of X has a finite subcollection the closures of whose members covers X.

Almost compact spaces have been referred to as $H(i)$ spaces by Scorborough and Stone[15] and are called absolutely closed spaces by Liu[9], while Porter and Thomas[13] call them quasi-H-closed spaces.

Definition 3.23[18] — Let X be a topological space and let $A \subset X$. A point $x \in X$ is called a *θ-limit point* of A if every closed neighbourhood of x intersects A. Let $cl_\theta A$ denote the set of all θ-limit points of A. The set A is called θ-closed if $A = cl_\theta A$.

The complement of a *θ-closed set* is called a *θ-open set.*

Definition 3.24[18] — A space X is called θ-compact if every θ-open cover of X has a finite subcover.

It is observed in [7] that every almost compact is θ-compact and every θ-compact space is quasicompact. However, none of the reverse implications hold.

The following corollaries are immediate from Theorem 3.21.

Corollary 3.25 — If $f: X \to Y$ is a z-supercontinuous function from a θ-compact space X onto Y. Then Y is compact.

Corollary 3.26 — If $f: X \to Y$ is a z-supercontinuous function from an almost compact space X onto Y, then Y is compact.

Theorem 3.27 — *Let $f: X \to Y$ be a z-continuous function from quasicompact space X onto a space Y. Then Y is quasicompact.*

We omit simple proof of Theorem 3.27.

4. Z-QUOTIENT TOPOLOGIES AND Z-QUOTIENT SPACES

Definition 4.1 — Let $f: X \to Y$ be a surjection from a topological space X onto a set Y. The topology on Y for which a subset $A \subset Y$ is open if and only if $f^{-1}(A)$ is z-open in X is called the *z-quotient topology* and the map f is called the *z-quotient map.*

A subset A in a space X is said to be *d-open*[6] if it is expressible as a union of open F_σ-sets.

Definition 4.2[6] — Let $f: X \to Y$ be a function from a topological space X onto a set Y. The topology on Y for which a subset $A \subset Y$ is open if and only if $f^{-1}(A)$ is d-open in X is called the *D-quotient topology* and the map f is called the *D-quotient map.*

Definition 4.3[10] — Let $f: X \to Y$ be a function from a topological space X onto a set Y. The topology on Y for which a subset $A \subset Y$ is open if and only if $f^{-1}(A)$ is δ-open in X is called the *δ-quotient topology* and the map f is called the *δ-quotient map.*

It is clear from the definitions that in general z-quotient topology on Y is coarser than D-quotient topology which in its turn is coarser than the quotient toplogy on Y and the three coincide if X is completely regular. The last two coincide in case X is D-regular. Moreover, z-quotient topology is coarser than δ-quotient topology, while D-quotient topology and δ-quotient topology are independent of each other.

Example 4.4 — Let (X, τ) denote the Tychonoff Corkskrew [17, p. 109]. Let $Y = X$ and f denote the identity map. Then z-quotient topology on Y is indiscrete, while δ-quotient topology = quotient topology = τ.

Example. 4.5 — Let (X, τ_c) denote the set of positive integers endowed with the cofinite topology. Let $Y = X$ and let f denote the identity map on X. Then z-quotient topology on Y is indiscrete while D-quotient topology = τ_c.

Theorem 4.6 — *Let f be a function from a topological space (X, τ_1) onto a topological the space (Y, τ_2), where τ_2 is z-quotient topology on Y. Then f is z-supercontinuous. Moreover, τ_2 is the finest topology on Y which makes $f: X \to Y$ z-supercontinuous.*

PROOF : The z-supercontinuity of f follows from the definition of z-quotient topology. Let τ_3 be a topology on Y such that $f: (X, \tau_1) \to (X, \tau_3)$ is z-supercontinuous. Let G be a τ_3 open set in Y. By z-supercontinuity of f, $f^{-1}(G)$ is z-open in X. Now, by the definition of the z-quotient topology, G is τ_2-open and hence $\tau_3 \subset \tau_2$.

Theorem 4.7 — *Let $f: X \to Y$ be a z-quotient map. Then a function $g: Y \to Z$ is continuous if and only if gof is z-supercontinuous.*

PROOF : If U is an open set in Z and gof is z-supercontinuous, then $(gof)^{-1}(U) = f^{-1}(g^{-1}(U))$, which is z-open in X. Since f is z-quotient map, $g^{-1}(U)$ is open in Y. Thus g is continuous. Conversely, let $g: Y \to Z$ be continuous. Let U be an open set in Z. By z-supercontinuity of f, $(gof)^{-1}(U) = f^{-1}(g^{-1}(U))$ is z-open in X.

5. SEPARATION AXIOMS AND Z-SUPERCONTINUOUS FUNCTIONS

Theorem 5.1 — *Let $f: X \to Y$ be a z-supercontinuous open bijection. Then X and Y are regular spaces.*

PROOF : Let B be any closed subset of Y and let $y \notin B$. Then $f^{-1}(B) \cap f^{-1}(y) = \phi$. Since f is z-supercontinuous by Theorem 3.1, $f^{-1}(B)$ is z-closed and so $f^{-1}(B) = \bigcap_{\alpha \in \Lambda} Z_\alpha$, where each Z_α is a zero set. Since f is one-one, $f^{-1}(y)$ is a singleton and so there exists $\alpha_0 \in \Lambda$ such that $f^{-1}(y) \notin Z_{\alpha 0}$. Thus $f^{-1}(y)$ and $Z_{\alpha 0}$ are functionally separated and so in particular there exist disjoint open sets U and V containing $Z_{\alpha 0}$ and $f^{-1}(y)$, respectively. Since f is an open bijection, $f(U)$ and $f(V)$ are disjoint open sets containing B and y, respectively. Thus Y is a regular space. Since regularity is a topological property and f is a homeomorphism, X is also a regular space.

Definition 5.2 — A function $f: X \to Y$ is said to be a *z-homeomorphism*, if f is a bijection such that both f and f^{-1} are z-supercontinuous.

Clearly, every z-homeomorphism is a homeomorphism but the converse is not necessarily true as is exhibited by the following example.

Example 5.3 — Let $X = Y$ be the real line endowed with the Smirnov's deleted sequence topology [17, Example 64, p. 86]. Let f denote the identity map from X onto Y. Then f is a homeomorphism but not a z-homeomorphism.

However, a homeomorphism between completely regular spaces is a z-homeomorphism. Converse of the same is also true as is well illustrated by the following result.

Theorem 5.4 — *Let $f: X \to Y$ be z-homeomorphism from a space X onto Y. Then the spaces X and Y are completely regular.*

PROOF : Let A be a closed set in Y and let $y \notin A$. Then $f^{-1}(y) = \{x\}$ is a singleton. So x is not in the z-closed set $f^{-1}(A)$. Let $f^{-1}(A) = \bigcap_{\alpha \in \Lambda} Z_\alpha$, where each Z_α is a zero set. So there exists an $\alpha_0 \in \Lambda$ such that $x \notin Z_{\alpha 0}$. Thus there exists a continuous function $g: X \to [0, 1]$ such that $g(x) = 1$ and $g(Z_{\alpha 0}) = 0$. Let $h = gof^{-1}$. Since f is a z-homeomorphism, h is well defined and f^{-1} is a z-supercontinuous function. Moreover, $h(y) = 1$ and $h(A) = g(f^{-1}(A)) = 0$. Thus Y is completely regular. Since complete regularity is a topological invariant and since f is a homeomorphism, X is also completely regular.

Theorem 5.5 — *Let $f: X \to Y$ be a z-supercontinuous, open and closed surjection. If either X or Y is a T-space, then Y is Hausdorff.*

PROOF : In view of closedness of f, in either case we may assume that Y is T_1.

First we show that set

$$E = \{(x_1, x_2) : f(x_1) = f(x_2)\}$$

is closed in $X \times X$. If $(x_1, x_2) \notin E$, then $x_1 \notin f^{-1}(f(x_2))$. Since Y is T_1 and since f is z-supercontinuous, in view of Theorem 3.1, $f^{-1}(f(x_2))$ is a z-closed set. Let $f^{-1}((f(x_2))) = \bigcap_{\alpha \in \Lambda} Z_\alpha$, where each Z_α is a zero set. Then there exist $\alpha_0 \in \Lambda$ such that $x_1 \notin Z_{\alpha 0}$. Hence x_1 and $Z_{\alpha 0}$ are functionally separated and so there exist disjoint open sets U and V containing x_1 and $Z_{\alpha 0}$, respectively. That is, in particular there exist disjoint open sets U and V containing x_1 and $f^{-1}(f(x_2))$, respectively. Since f is closed, by [3, Theorem 11.2, p. 86] there is an open set W containing $f(x_2)$ such that

$$f^{-1}(f(x_2)) \subset f^{-1}(W) \subset V.$$

Again, since f is continuous (infact z-supercontinuous), $U \times f^{-1}(W)$ is an open set containing (x_1, x_2) which does not intersects E. Thus E is closed in $X \times X$.

Now, suppose that $f(x_1)$ and $f(x_2)$ are distinct points of Y. Then $(x_1, x_2) \notin E$. Hence, there exist open sets U_1 and U_2 containing x_1 and x_2, respectively such that

$$(U_1 \times U_2) \cap E = \phi$$

Since f is open, $f(U_1)$ and $f(U_2)$ are disjoint open sets containing $f(x_1)$ and $f(x_2)$, respectively.

Theorem 5.6 — *If X is Hausdorff and Y is obtained by identifying a single zero set Z in X, then Y is Hausdorff.*

PROOF : Let y_1 and y_2 be any two distinct points in Y. Let $f: X \rightarrow Y$ denote the quotient map. Then the sets $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are either singletons or a point and the zero set Z. In either case there are disjoint open sets U and V containing $f^{-1}(y_1)$ and $f^{-1}(y_2)$ respectively. Since the quotient map f is closed, by [3, Theorem 11.2, p. 86] there exists disjoint open sets W_1 and W_2 is Y such that $f^{-1}(y_1) \subset f^{-1}(W_1) \subset U$ and $f^{-1}(y_2) \subset f^{-1}(W_2) \subset V$.

Since U and V are disjoint, W_1 and W_2 are disjoint open sets containing y_1 and y_2, respectively. Thus Y is Hausdorff.

6. COMPLETE REGULARIZATION

In this section we show that if the domain of a z-supercontinuous function is retopologized in an appropriate way, then f is simply a continuous function.

6.1 Complete Regularization

Let (X, τ) be a topological space and let β denote the collection of all cozero subsets of (X, τ). Since the intersection of two cozero sets is a cozero set, the collection β is a base for a topology τ_z on X called the complete regularization of τ. Clearly, $\tau_z \subset \tau$. The space (X, τ) is completely regular if and only $\tau_z = \tau$.

Throughout the section, the symbol τ_z will have the same meaning as in the above paragraph.

Remark 6.2 : Any topological property which is invariant under continuous bijections will be carried over from (X, τ) to (X, τ_z). In particular, if (X, τ) is compact or countably compact, Lindelöf or pseudocompact or quasicompact, separable, connected or pathwise connected, then so is (X, τ_z). Moreover, any topological property which is honoured by an enlargement of topology such as T_0, T_1, T_2, Uryshon space, functionally Hausdorff, ultra-Hausdorff etc. is enjoyed by (X, τ), whenever it is reflected in (X, τ_z).

Theorem 6.3 — *A function $f: (X, \tau) \rightarrow (Y, \mathcal{J})$ is z-supercontinuous if and only if $f: (X, \tau_z) \rightarrow (Y, \mathcal{J})$ is continuous.*

Theorem 6.4 — *Let (X, τ) be a topological space. Then the following statements are equivalent.*

(a) (X, τ) is completely regular.

(b) Every continuous function from (X, τ) into a space $(Y, \mathcal{J})$ is z-supercontinuous.

PROOF : (a) $\Rightarrow$ (b) is obvious.

(b) ⇒ (a) : Take $(Y, \mathcal{J}) = (X, \tau)$. Then the identity function I_x on X is continuous and hence z-supercontinuous. Thus by Theorem 6.3, $I_x : (X, \tau_z) \to (X, \tau)$ is continuous. Since $U \in \tau$ implies $I_x^{-1}(U) = U \in \tau_z$, $\tau \subset \tau_z$. Therefore, $\tau_z = \tau$, and so (X, τ) is completely regular.

Many of the results studied in Section 3 follow now from Theorem 6.3 and the corresponding standard properties of continuous functions.

Theorem 6.5 — *Let* $f : (X, \tau) \to (Y, \mathcal{J})$ *be a function. Then*

(a) f is z-continuous if and only if $f : (X, \tau) \to (Y, \mathcal{J}_z)$ is continuous.

(b) f is z-open if and only if $f : (X, \tau_z) \to (Y, \mathcal{J})$ is open.

In the light of Theorems 6.3 and 6.5, Theorem 3.9 can be restated as follows :

If $f : (X, \tau_z) \to (Y, \mathcal{J})$ is a continuous open surjection and $g : (Y, \mathcal{J}) \to (Z, \nu)$ is a function, then g is continuous if and only if gof is continuous and Theorem 3.17 is simply the result that the composition gof of the continuous functions $f : (X, \tau_z) \to (Y, \mathcal{J})$ and $g : (Y, \mathcal{J}) \to (Z, \nu)$ is continuous. Moreover, z-quotient topology on Y determined by $f : (X, \tau) \to Y$ in section 4 coincides with the usual quotient topology on Y determined by $f : (X, \tau_z) \to Y$.

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