Indian J. pure appl. Math., 33(7) : 1109-1119, July 2002

FREE AXISYMMETRIC VIBRATION OF A STEPPED CIRCULAR PLATE

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(*Received* 4 *July* 2001; *accepted* 16 *October* 2001)

Free axisymmetric vibration of a circular plate whose thickness, density and elastic properties, along the radial direction, vary in any number of steps, is analyzed by classical plate theory. Numerical results for natural frequencies and normalized mode shapes for first four normal modes of vibration for clamped, simply-supported and free boundary conditions are plotted in graphs for a two stepped plate.

Key Words : Classical Theory of Plates; Stepped Variation in Thickness; Natural Frequency

1. INTRODUCTION

Many practical structures exhibit geometric steps. On several occasions fabrication and assembly considerations dictate such necessities. Attempts are continuously made to study the vibration characteristics of structures with such specified geometries. Some of the references dealing with the free vibration of circular plate of stepped thicknesses are cited below :

Gallego-Juarez[1] has analyzed the axisymmetric vibration of circular plates of single circular step using the classical uniform plate solution to each zone of different thickness. San Emeterio *et al*[2] have developed an approximate procedure, based on the Rayleigh method, for the study of an axisymmetric vibration mode of circular plate having any number of stepped variation in thickness. But they have imposed a restriction that the number of steps has to be equal to the number of nodal circles. Gutierrez & Laura[3] have given approximate approaches for stepped plate, applied only to the first mode of vibration. Natural frequencies corresponding to axisymmetric and antisymmetric modes of vibration of a circular plate with stepped thickness over a concentric circular region have been studied by Avalos *et al.*[4] using the optimized Rayleigh-Ritz approach. Gu & Wang[5] have used differential quadrature element method for the free vibration analysis of circular plate with stepped thickness over a concentric region. Avalos *et al.*[6] have also analyzed the annular plates of stepped thickness.

In this paper, free axisymmetric vibration of a circular plate whose thickness, density and elastic properties along the radial direction, vary in any number of steps is analyzed by classical plate theory. The plate is assumed to be made up of n concentric annular plate elements joined end to end. The plate elements are having in general different constant thicknesses, densities and elastic properties. In the present method the order of frequency determinant does not increase with the number of steps. It always remains of order two. Therefore, the method can be conveniently used for any number of steps. The arbitrary constants arising in the solution are solved by boundary and continuity conditions. Numerical results for natural frequencies and normalized mode shapes for first four normal modes computed for a plate made-up of three plate elements are plotted in graphs for clamped, simply-supported and free boundary conditions. The variations in radii, thicknesses and densities of the plate elements are so made that the radius, average thickness and average density of the whole plate remain constant. The frequencies computed as a particular case for a one stepped plate of uniform density and elastic properties are compared with that of Gallego-Juarez[1].

2. EQUATIONS OF MOTION

An isotropic circular plate of radius a, whose thickness, density and elastic properties along the radial direction, vary in steps, is considered using classical theory of plates. The plate is referred to cylindrical co-ordinates by taking origin at centre, z-axis along the thickness and r-axis along the radius of the plate. The middle plane and the circular boundary of the plate are $z = 0$ and $r = a$ respectively. The plate is assumed to be made up of n concentric plate elements joined end to end with their middle planes lying in plane $z = 0$. The innermost plate element is a solid circular plate and the other are annular plates. Their Young's moduli, densities, Poisson ratio, constant thicknesses, inner radii and outer radii are taken as $E_k, \rho_k, \nu_k, h_k, r_{k-1}$ and r_k ; $k = 1, 2, ..., n$; where $r_0 = 0$ and $r_n = a$ and $r_k - r_{k-1} = a_k$. A thickness profile of the plate is shown in Fig. 1.

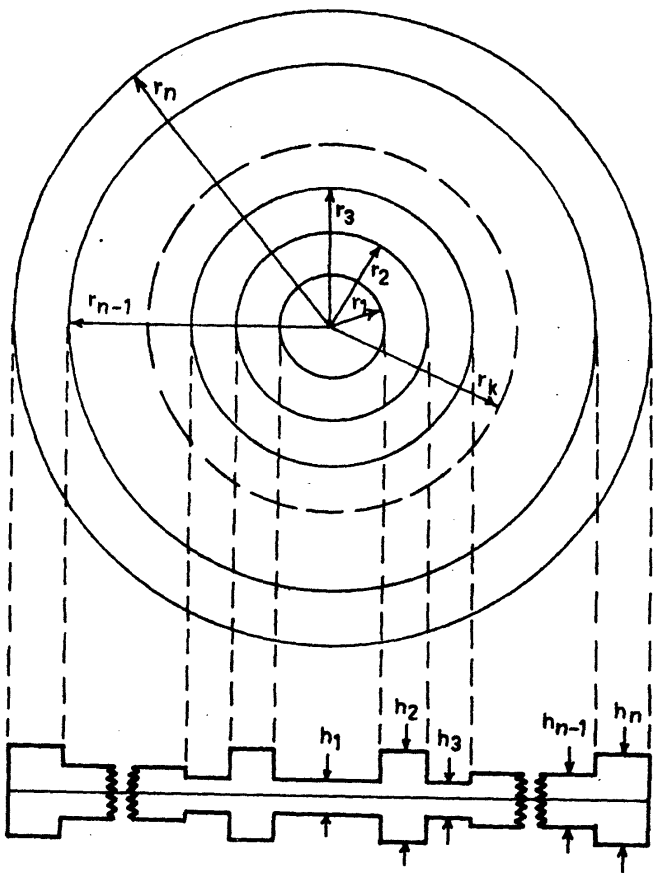

FIG. 1. Thickness profile of the plate

For free vibration analysis, the equations of motion of the k^{th} plate element of constant thickness are

$$E_k h_k^3 \left[w_{k,rrrr} + \frac{2}{r} w_{k,rrr} - \frac{1}{r^2} w_{k,rr} + \frac{1}{r^3} w_{k,r} \right] + 12 \rho_k h_k (1 - \nu_k^2) w_{k,tt} = 0$$

$$r_{k-1} \le r \le r_k;\ k = 1, 2, ..., n, \qquad \text{... (1)}$$

where w_k and t are displacement component of k^{th} beam element in the direction of z-axis and time, respectively. The comma followed by the variable suffix denotes differentiation with respect to that variable.

Making eqs. (1) non-dimensional, one gets

$$L_k \left[\overline{W}_{k,RRRR} + \frac{2}{R} \overline{W}_{k,RRR} - \frac{1}{R^2} \overline{W}_{k,RR} + \frac{1}{R^3} \overline{W}_{k,R} \right] + \gamma_k H_k \overline{W}_{k,TT} = 0$$

$$R_{k-1} \le R \le R_k;\ k = 1, 2, ..., n, \qquad \text{... (2)}$$

where $\overline{W}_k = w_k/a,\ R = r/a,\ H_k = h_k/a,\ R_k = r_k/a,\ e_k = E_k/E,$

$$\gamma_k = \rho_k/\rho_a,\ L_k = e_k H_k^3/12\,(1 - \nu_k^2),\ T = t\sqrt{E/a^2 \rho_a},\ R_0 = 0,\ R_n = 1.$$

ρ_a is the average density of the plate and E is the Young's modulus of some standard material.

3. Solution

For harmonic solution, $\overline{W}_k (R, T)$ are taken as

$$\overline{W}_k (R, T) = Q_k (R)\, e^{i\Omega T}. \qquad \text{... (3)}$$

where Ω is the circular frequency of vibration.

Substitution of solutions (3) in eqs. (2) gives

$$L_k \left[W_{k,RRRR} + \frac{2}{R} W_{k,RRR} - \frac{1}{R^2} W_{k,RR} + \frac{1}{R^3} W_{k,R} \right] - \gamma_k H_k \Omega^2 W_k = 0. \qquad \text{... (4)}$$

The solutions of eqs. (4) can be taken as

$$W_k (R) = S_k (R)\, D_k,\ S_1 (R) = [J_0 (\omega_1 R)\, I_0 (\omega_1 R)],\ D_1 = [d_{11}\ d_{21}],$$

where $S_k (R) = [J_0 (\omega_k R)\, I_0 (\omega_k R)\, Y_0 (\omega_k R)\, K_0 (\omega_k R)],$

and $D_k = [d_{1k}\, d_{2k}\, d_{3k}\, d_{4k}]'$ for k = 2, 3, ..., n.

Here, J_0, Y_0, I_0 and K_0 are Bessel's functions of order zero, d's are the arbitrary constant and $\omega_k^2 = \gamma_k H_k \Omega^2 / L_k$.

3.1 Continuity Conditions

The continuity conditions between the plate elements at $R = R_k$; $k = 1, 2, \ldots, n-1$; are taken as

$$W_l(R_k) = W_k(R_k),\; W_{l,R}(R_k) = W_{k,R}(R_k),$$

$$L_l\left[W_{l,RR}(R_k) + \frac{\nu_l}{R_k} W_{l,R}(R_k)\right] = L_k\left[W_{k\,RR}(R_k) + \frac{\nu_k}{R_k} W_{k,R}(R_k)\right],$$

$$L_l\left[W_{l,RRR}(R_k) + \frac{1}{R_k} W_{l,RR}(R_k) - \frac{1}{R_k^2} W_{l,R}(R_k)\right]$$

$$= L_k\left[W_{k,RRR}(R_k) + \frac{1}{R_k} W_{k,RR}(R_k) - \frac{1}{R_k^2} W_{k,R}(R_k)\right], \quad \ldots (6)$$

where $l = k + 1$.

Using above conditions, one gets

$$D_l = B^{(l)} D_k,\; B^{(l)} = A_l^{-1}(R_k) A_k(R_k), \quad \ldots (7)$$

where

$$A_1(R_1) = \begin{bmatrix} J_0(\omega_1 R_1) \\ -\omega_1 J_1(\omega_1 R_1) \\ L_1\left\{-\omega_1^2 J_0(\omega_1 R_1) + (1-\nu_1)\frac{\omega_1}{R_1} J_1(\omega_1 R_1)\right\} \\ L_1 \omega_1^3 J_1(\omega_1 R_1) \end{bmatrix.$$

$$\begin{matrix} I_0(\omega_1 R_1) \\ \omega_1 I_1(\omega_1 R_1) \\ L_1\left\{\omega_1^2 I_0(\omega_1 R_1) - (1-\nu_1)\frac{\omega_1}{R_1} I_1(\omega_1 R_1)\right\} \\ L_1 \omega_1^3 I_1(\omega_1 R_1) \end{matrix}\Bigg], \quad \ldots (8)$$

$$A_l(R_k) = \begin{bmatrix} J_0(\omega_l R_k) & I_0(\omega_l R_k) & Y_0(\omega_l R_k) & K_0(\omega_l R_k) \\ -\omega_l J_1(\omega_l R_k) & \omega_l I_1(\omega_l R_k) & -\omega_l Y_1(\omega_l R_k) & -\omega_l K_1(\omega_l R_k) \\ L_l\left(-\omega_l^2 J_0(\omega_l R_k) + \frac{\omega_l}{R_k}(1-\nu_l) J_1(\omega_l R_k)\right] & L_l\left(\omega_l^2 I_0(\omega_l R_k) - \frac{\omega_l}{R_k}(1-\nu_l) I_1(\omega_l R_k)\right] & L_l\left(-\omega_l^2 Y_0(\omega_l R_k) + \frac{\omega_l}{R_k}(1-\nu_l) Y_1(\omega_l R_k)\right] & L_l\left(\omega_l^2 K_0(\omega_l R_k) + \frac{\omega_l}{R_k}(1-\nu_l) K_1(\omega_l R_k)\right] \\ L_l \omega_l^3 J_1(\omega_l R_k) & L_l \omega_l^3 I_1(\omega_l R_k) & L_1 \omega_l^3 Y_1(\omega_l R_k) & -L_l \omega_l^3 K_1(\omega_l R_k) \end{bmatrix} \quad \dots (9)$$

The matrices $A_k(R_k)$ are obtained by replacing l by k in eqs. (9)

From eqs. (7), one gets

$$D_l = C^{(l)} D_1, \; C^{(l)} = B^{(l)} B^{(l-1)} \dots B^{(2)} = \left[c_{qr}^{(l)} \right]_{4\times 2} = [C_1 \; C_2] \text{ (say).} \quad \dots (10)$$

In this way the $4n - 2$ constant arising in solutions (5) are reduced to 2. It should be noted that if the thicknesses, densities and elastic properties of the n plate elements are taken to be the same, the matrices $B^{(l)}$ and $C^{(l)}$ reduces to unit matrices and the whole plate reduces to a uniform circular plate.

3.2 Boundary Conditions

The plate is subjected to following types of boundary conditions :

3.2.1. Clamped plate (C-plate)

For a plate clamped at the boundary, we have

$$W_n = W_{n,R} = 0 \text{ at } R = 1. \quad \dots (11)$$

3.2.2 Simply-Supported Plate (S-Plate)

For a plate simply supported at the boundary, we have

$$W_n = L_n\left\{ W_{n,RR} + \nu_n W_{n,R} \right\} = 0 \text{ at } R = 1.$$

3.2.3 Free Plate (F-Plate) — For a plate free at the boundary, we have

$$L_n\left\{ W_{n,RR} + \nu_n W_{n,R} \right\} = L_n\left\{ W_{n,RRR} + \frac{1}{R} W_{n,RR} - \frac{1}{R^2} W_{n,R} \right\} = 0 \text{ at R = 1.} \quad \dots (13)$$

Using relation (10) in solution (5) and then putting them in conditions (11) to (13) as the case may be, one gets

$$\left\{\begin{array}{l} s_{11}\, d_{11} + s_{12}\, d_{21} = 0 \\ s_{21}\, d_{11} + s_{22}\, d_{21} = 0 \end{array}\right\}, \qquad \dots (14)$$

where
$$s_{1i} = \begin{cases} S_n(1)\, C_i & \text{for } C\text{-plate and } S\text{-plate} \\ [S_{n,RR}(1) + \nu_n S_{n,R}(1)]\, C_i & \text{for } F\text{-plate} \end{cases}$$

and
$$s_{2i} = \begin{cases} S_{n,R}(1)\, C_i & \text{for } C\text{-plate} \\ [S_{n,RR}(1) + \nu_n S_{n,R}(1)]\, C_i & \text{for } S\text{-plate} \\ [S_{n,RRR}(1) + S_{n,RR}(1) - S_{n,R}(1)]\, C_i & \text{for } F\text{-plate} \end{cases}$$

3.3 Frequency Equation

For nontrivial solution of homogeneous eqs. (14),

$$s_{11}\, s_{22} - s_{21}\, s_{12} = 0. \qquad \dots (15)$$

The countably infinite roots of frequency equation are the natural frequencies Ω for various normal modes of vibrations of the plate.

4. Results and Discussion

The variations in radii, thicknesses and densities of the plate elements are taken in such a way that radius, average thickness and average density of the whole plate remain constant. For that we take

$$\alpha_k = a_k/a_1, \quad \beta_k = h_k/h_1, \quad \delta_k = \rho_k/\rho_1.$$

If h_a is the average thickness of the plate and $H_a = h_a/a$, then

$$H_k = \frac{\beta_k H_a}{\sum_{i=1}^{n} \beta_i (R_i^2 - R_{i-1}^2)}, \quad \gamma_k = \frac{\delta_k H_a}{\sum_{i=1}^{n} \beta_i\, \delta_i (R_i^2 - R_{i-1}^2)} \quad \text{and} \quad R_k = \frac{\sum_{i=1}^{k} \alpha_i}{\sum_{i=1}^{n} \alpha_i}.$$

Numerical results for frequencies and normalized mode shapes for first four normal modes of vibration for a plate made up of three plate element whose first and third elements are identical, i.e., for $\alpha_3 = \beta_3 = \delta_3 = 1.0$ by taking $\nu_1 = \nu_2 = \nu_3 = 0.3$, $\delta_2 = e_2 = e_3 = 1.0$ and $H_a = 0.06$ are plotted in graphs.

Ω vs β_2 for C-plate is plotted in Fig. 2. It is seen in first mode that as β_2 increases, Ω first increases, attains a maximum value and then decreases. The rate of decreases with the increase in β_2. As α_2 increases, Ω decreases for $\beta_2 > 1$. Therefore, maximum frequency is attained when the thickness and length of the middle element remain less than the other two elements. The frequency can be reduced by increasing the thickness and/or length of the middle element.

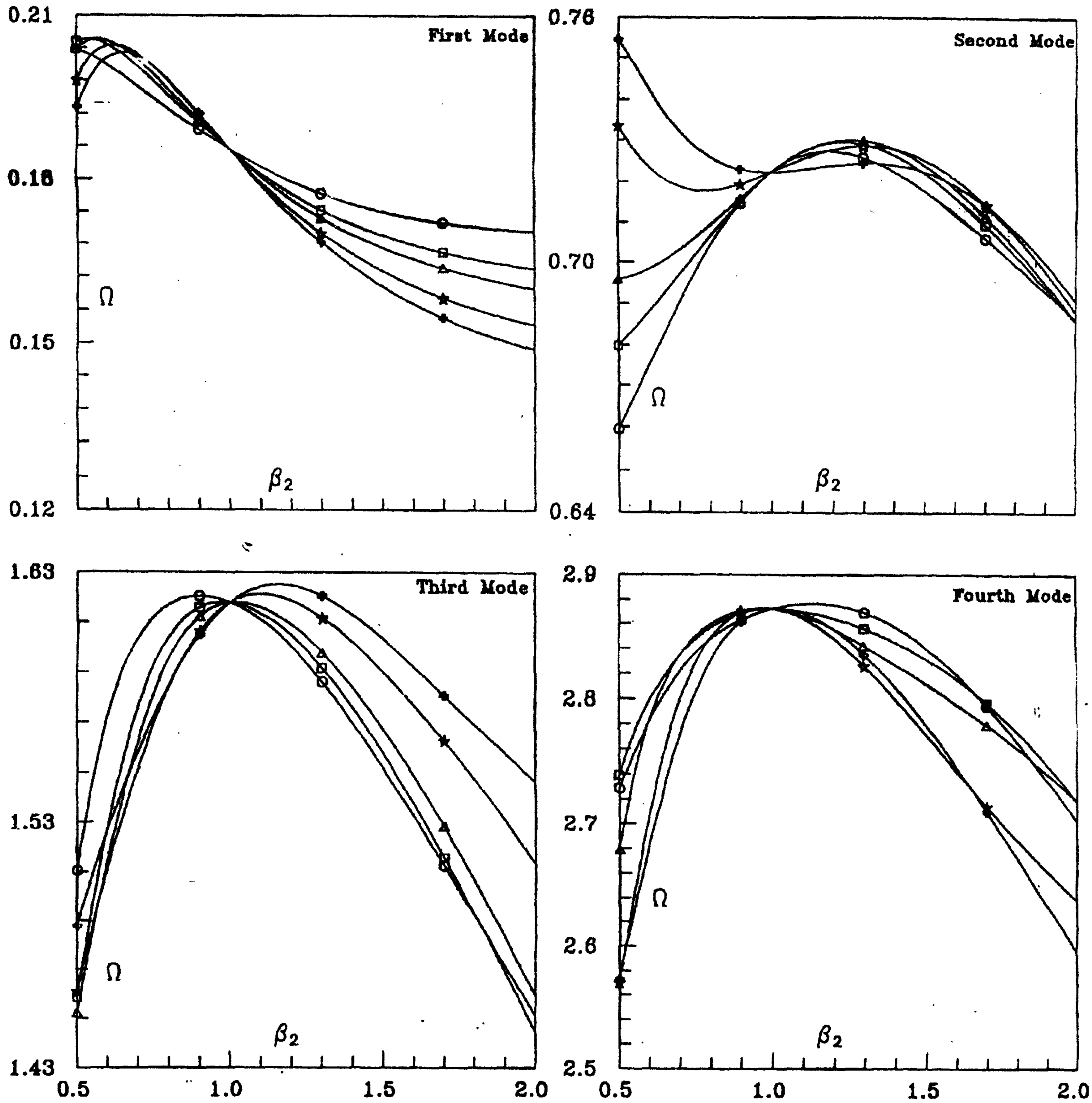

FIG. 2. Ω vs β_2 for C-plate for various values of α_2
—O—, 0.5; —□—, 0.8; —Δ—, 1.0; —*—, 1.5; — + —

In second mode, as β_2 increases when $\alpha_2 < 1$, Ω first in attains a maximum value when $\beta_2 > 1$ and then decreases. As β_2 increases when $\alpha_2 > 1$. Ω decreases, then increases and finally decreases. As α_2 increases, Ω increases when $\beta_2 < 1$. Therefore the frequency cannot be increased beyond a certain value by varying the thickness of the middle element when its length is less than or equal to the other two elements, whereas the frequency can be increased considerably by decreasing its thickness when its length is greater than the other two.

In third and fourth modes, as β_2 increases, Ω first increases, attains a maximum value near $\beta_2 > 1$ and then decreases for all values of α_2. In third mode, Ω is maximum when both β_2 and

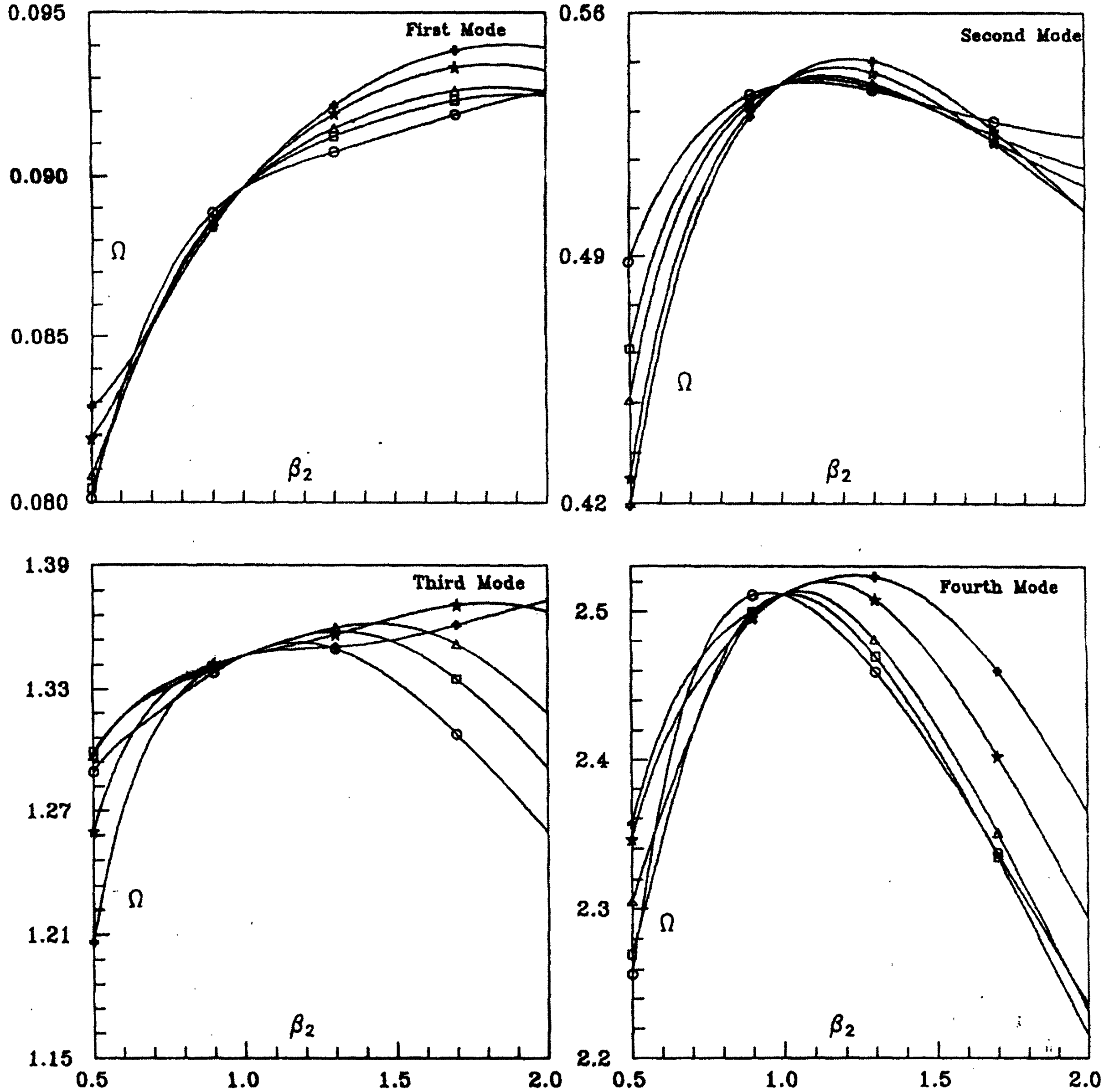

FIG. 3. Ω vs β_2 for S-plate for various values of α_2 :
—O—, 0.5; —□—, 0.8; —Δ—, 1.0; —*—, 1.5; — + —, 2.0

α_2 are either greater or less than 1 whereas in fourth mode it happens when one is less and the other is greater than 1.

The ratio of Ω vs β_2 for S-plate is plotted in Fig. 3. It is seen in all the four modes that as β_2 increases, Ω first increases, attains a maximum value and then decreases for all values of α_2. The maximum in first mode is attained at a higher value of β_2 in comparison to other modes except in third mode when $\alpha_2 > 1$. In firs modem as α_2 increases, Ω increases when $\beta_2 < 0.6$ and when $1 < \beta_2 < 2$. In second mode, as α_2 increases, Ω decreases when $\beta_2 < 1$ and when $\beta_2 > 1.5$ but it increases when $1 < \beta_2 < 1.5$. In third mode, as α_2 increases, Ω first increases and then

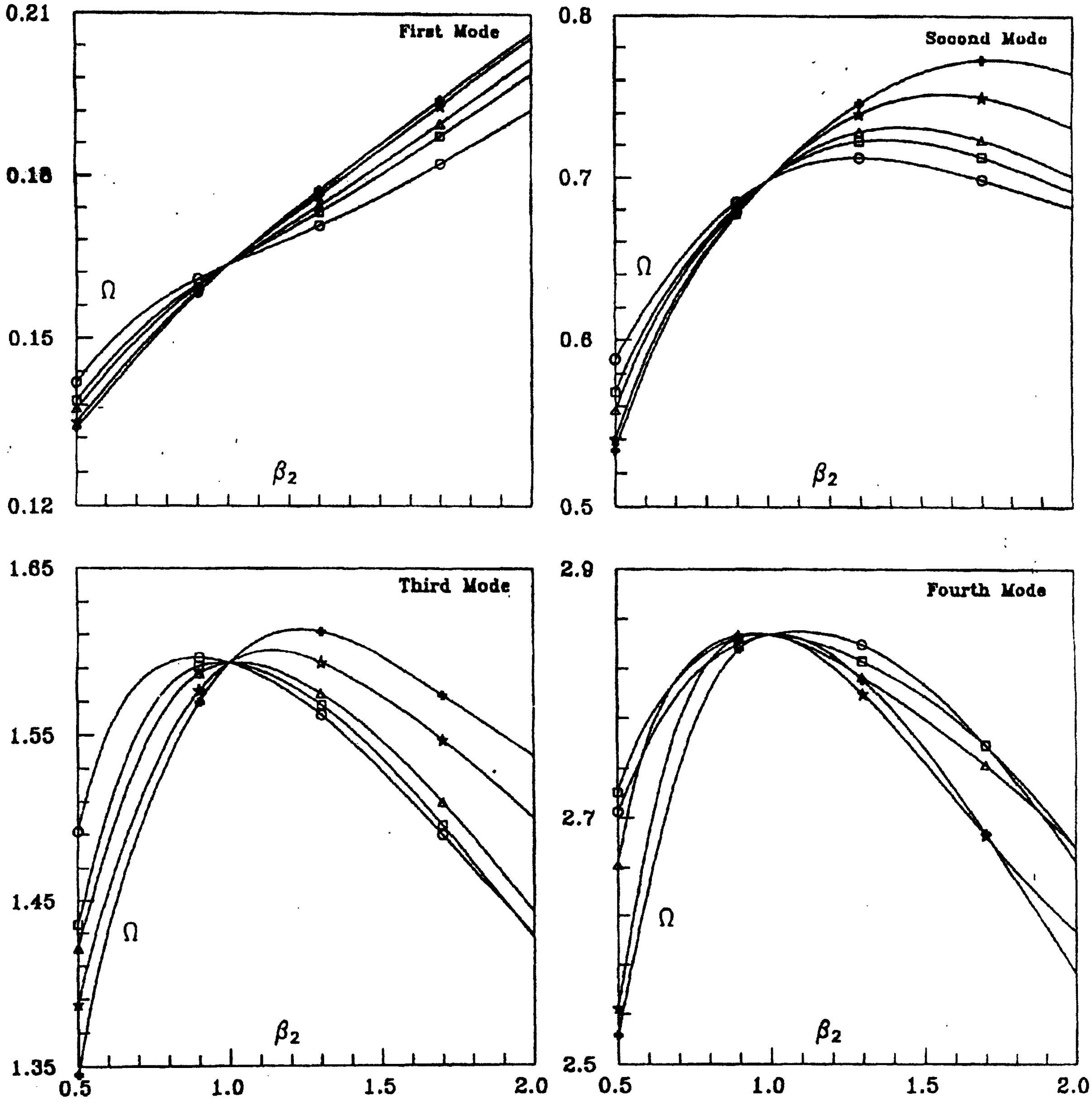

FIG. 4. Ω vs β_2 for F-plate for various values of α_2 :
—O—, 0.5; —□—, 0.8; —Δ—, 1.0; —*—, 1.5; — + —, 2.0

decreases when $\beta_2 < 0.7$ but the process gets reversed when $\beta_2 > 1.5$. In fourth mode, as α_2 increases, Ω increases when $\beta_2 < 0.6$ and when $\beta_2 > 1.0$.

Ω vs β_2 for F-plate is plotted in Fig. 4. It is seen that as β_2 increases, Ω first increases, attains a maximum value and then decreases for all values of α_2 in all four modes. The value of β_2 for maximum Ω is very high in firsy mode and therefore not visible in the figure. This value of β_2 decreases with the increase in mode number as well as with the increase in α_2. As α_2 increases, Ω decreases for $\beta_2 < 1$ but increase for $\beta_2 > 1$, in first three modes.

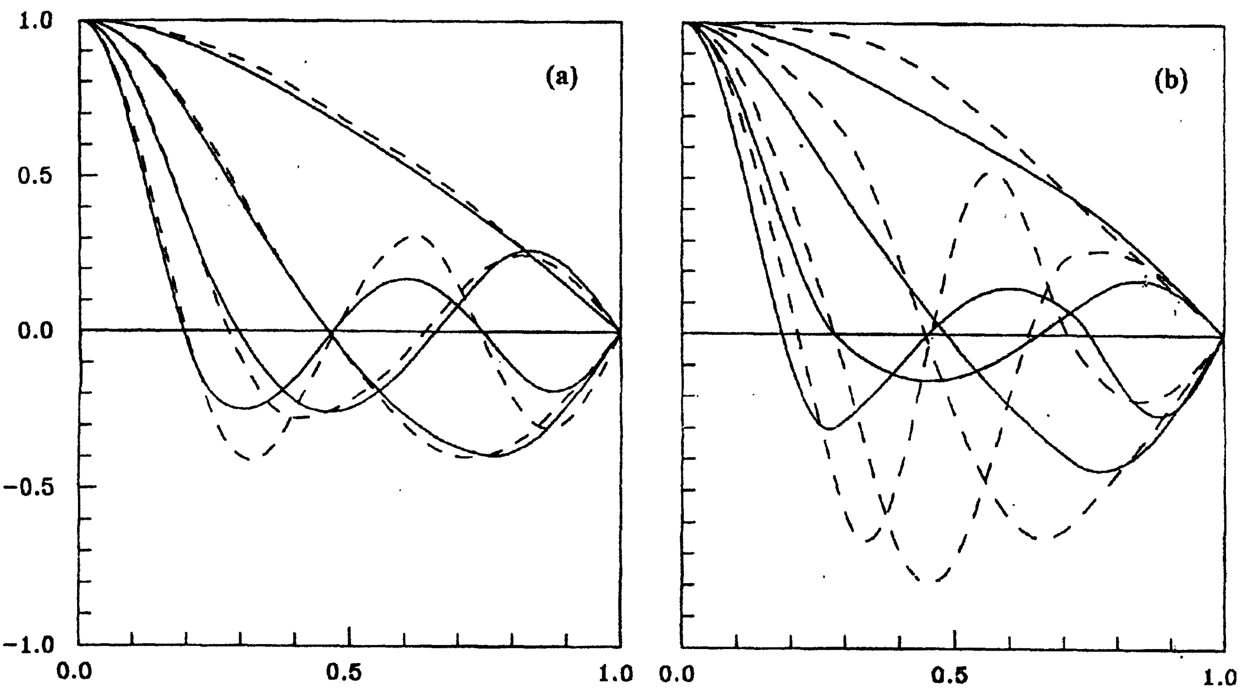

FIG. 5. Normalized transverse delfection in the first four normal modes of vibration for C-plate (a) $\alpha_2 = 0.5$; (b) $\alpha_2 = 2.0$ for $\beta_2 = 2.0$ (—) and $\beta_2 = 0.5$ (- - - -)

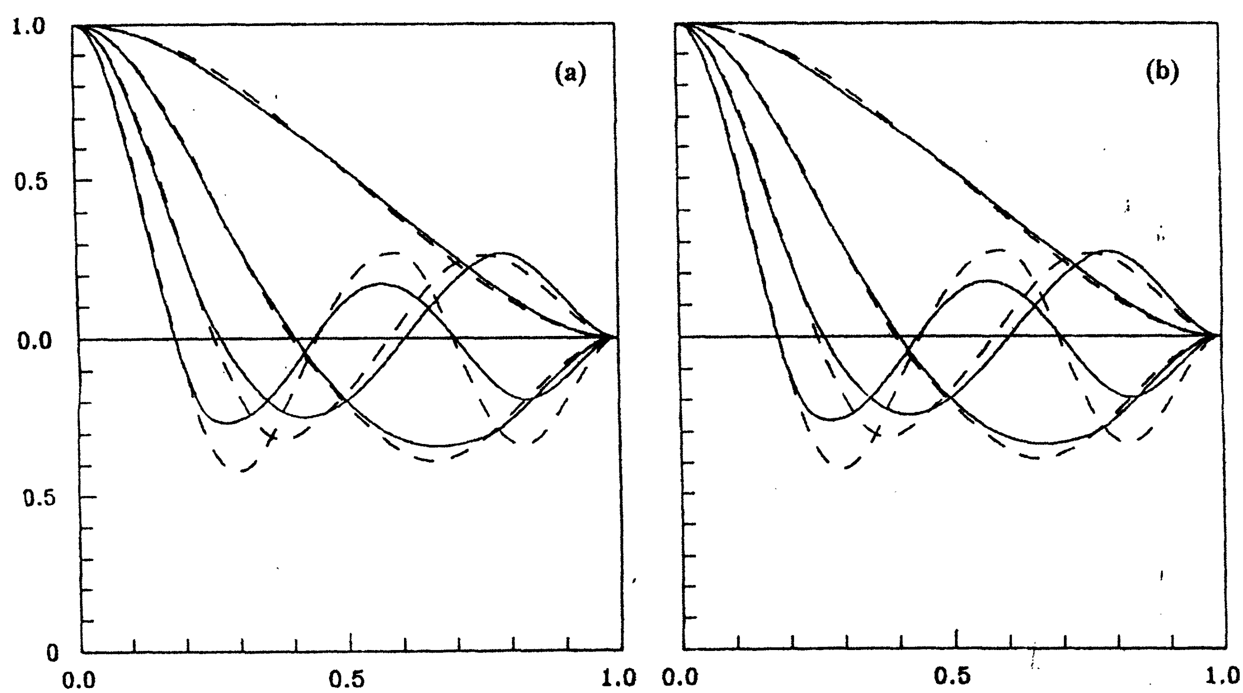

FIG. 6. Normalized transverse delfection in the first four normal modes of vibration for S-plate (a) $\alpha_2 = 0.5$; (b) $\alpha_2 = 2.0$ for $\beta_2 = 2.0$ (—) and $\beta_2 = 0.5$ (- - - -)

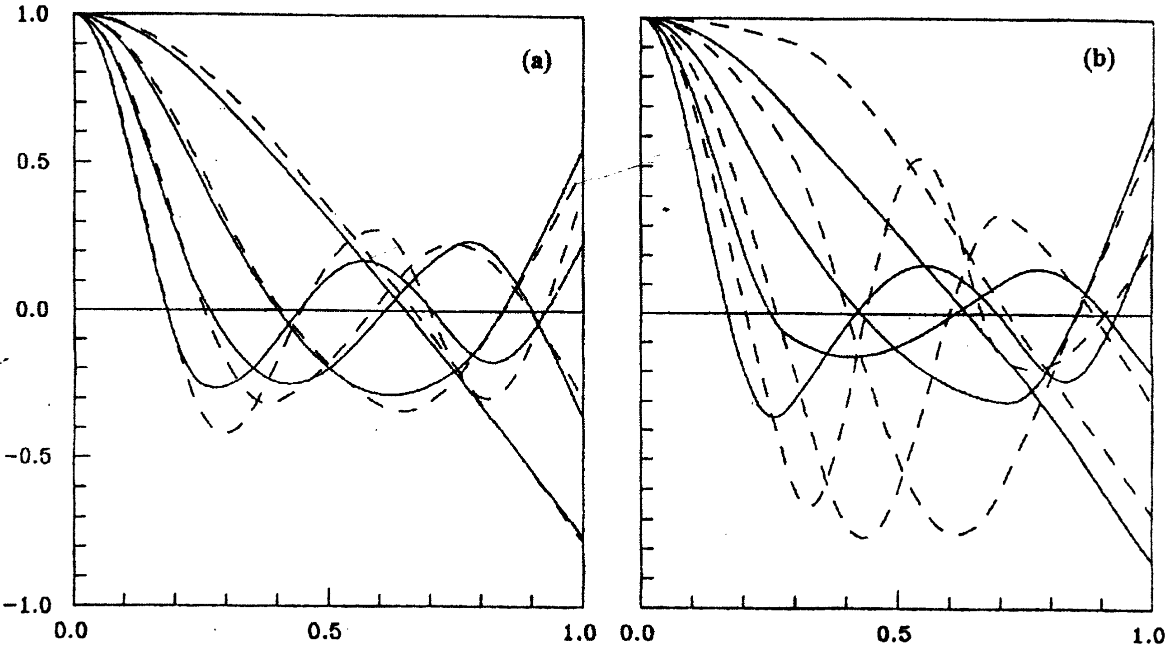

FIG. 7. Normalized transverse delfection in the first four normal modes of vibration for F-plate (a) α_2 = 0.5; (b) α_2 = 2.0 for β_2 = 2.0 (—) and β_2 = 0.5 (- - - -)

Normalized mode shapes for first four normal modes for C-plate, S-plate and F-plate are plotted in Figures 5, 6 and 7 respectively. Figs. 5(a) and (b) shows that the peaks of maximum deflection are higher in the case when the length and thickness of the middle element is lower than other two elements. This difference in maximum deflection is more prominent when the thickness of the middle element is lwoer than other two elements (Fig. 5(b)). Same type of variation can be seen for S-plates and F-plates.

To check the validity of the numerical results, fundamental frequency for a circular plate of single step is compared with the values reported by Gallego-Juarez[1] by taking

(a) $\alpha_2 = \beta_2 = \beta_2 = e_2$ = 1.0; $v_1 = v_2 = v_3$ = 0.25; $\delta_3 = e_3$ = 1.0; α_3 = 8, 3, 4/3 and 2/3; β_3 = 0.33/0.20 and 0.20/0.33

(b) Removing the restriction that average thickness and density of the plate remain constant.

An exact agreement in all the results is observed.

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