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OSCILLATION CRITERIA OF SOLUTIONS FOR A CLASS OF IMPULSIVE PARABOLIC DIFFERENTIAL EQUATION

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In this paper we study oscillation of solution of a class of Impulsive parabolic differential equation. Several oscillation criteria are obtained.

Key Words : Impulsive; Parabolic Differential Equation; Oscillation

1. INTRODUCTION

Literature[1] studied a class of impulsive parabolic differential equation and got several compare criteria. Literature[2] studied periodic boundary value problem of impulsive hyperbolic differential equation. Literature[3] investigated the oscillatory properties of solutions of impulsive hyperbolic differential equations with fixed moments of impulse effects and established several oscillation criteria. In this paper we study oscillation of solution for the following parabolic differential equation:

$$\begin{cases} u = a(t)\,\Delta u - p(t,x)\,u - q\,(t,x) f(u), & t \neq t_k, \\ u\,(t_k^+, x) - u\,(t_k^-, x) = g\,(t_k, x, u), & k = 1, 2, \ldots, \end{cases} \qquad \ldots (1)$$

where $u = u\,(t, x)$, $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$, $(t, x) \in G = R_+ \times \Omega$, Ω is a bounded domain in R^n with a piecewise continuous smooth boundary $\partial\Omega$, $R_+ = [0, +\infty)$.

We consider two kinds of boundary conditions :

$$\frac{\partial u}{\partial n} + \beta(x)\,u = h_1'(t, x), \quad (t, x) \in R_+ \times \partial\Omega, \quad t \neq t_k, \qquad \ldots (B1)$$

$$u = h_2\,(t, x), \quad (t, x) \in R_+ \times \partial\Omega, \quad t \neq t_k, \qquad \ldots (B2)$$

where $\beta \in PC[R_+ \times \partial\Omega, R_+]$, $h_1, h_2 \in PC[R_+ \times \partial\Omega, R]$, n denotes the unit exterior vector normal to $\partial\Omega$.

We assume throughout this paper that the following conditions hold :

(*H*1) $0 < t_1 < t_2 < \ldots < t_k \ldots$, and $\lim_{k \to +\infty} t_k = +\infty$;

(*H*2) $p, q \in PC[R_+ \times \bar{\Omega}, R_+]$, $a \in PC[R_+, R_+]$, $f \in C[R, R]$, where PC denotes a piecewise continuous function classes with the following properties: such that the function is the first kinds noncontinuous at only $t = t_k$, $k = 1, 2, \ldots$. But left continuous at $t = t_k$.

$$p_0(t) = \min_{x \in \bar{\Omega}} p(t, x), \quad q_0(t) = \min_{x \in \bar{\Omega}} q(t, x)$$

(*H*3) $u(t_k^-, x) = u(t_k, x)$, $u(t_k^+, x) = u(t_k, x) + g(t_k, x, u(t_k, x))$,

$$g : R_+ \times \bar{\Omega} \times R \to R, \ g(t_k, x, -u(t_k, x)) = -g(t_k, x, u(t_k, x)), \ k = 1, 2, \ldots,$$

and
$$\int_\Omega g(t_k, x, u(t_k, x))\, dx \le \alpha_k \int_\Omega u(t_k, x))\, dx, \ k = 1, 2, \ldots,$$

where $\alpha_k > 0$ is a constant.

(*H*4) $f(u)$ is a convex function in R_+ and $f(-u) = -f(u) < 0, u \in R_+$.

Definition 1 — A non-zero solution $u(t, x)$ of the problem (1), (Bi) ($i = 1, 2$) is called oscillatory in the domain G if for each positive number T there exists a point $(t_0, x_0) \in [T, +\infty) \times \Omega$ such that the condition $u(t_0, x_0) = 0$ holds, otherwise nonoscillatory.

2. Main Results

Lemma 1 — If $\beta \in C(\partial\Omega, (0, +\infty))$, and λ_1 is the smallest eigenvalue of the Robin eigenvalue problem

$$\begin{cases} \Delta u + \lambda u = 0 & x \in \Omega \\ \dfrac{\partial u}{\partial n} + \beta(x)\, u = 0 & x \in \partial\Omega \end{cases} \qquad \ldots (2_1)$$

and $\Phi_1(x)$ is the corresponding eigenfunction, then $\lambda_1 > 0$ and $\Phi_1(x) > 0$. (see Theorem 3.3.22 of [4]).

Lemma 2[5] — Assume that λ_2 is the smallest eigenvalue of the Dirichlet problem

$$\begin{cases} \Delta u + \lambda u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \qquad \ldots (2_2)$$

and $\Phi_2(x)$ is the corresponding eigenfunction, then $\lambda_2 > 0$ and $\Phi_2(x) > 0$ ($x \in \Omega$).

Let
$$R_1(t) = \int_{\partial\Omega} h_1(t, s)\, \Phi_1(s) ds, \quad R_2(t) = -\int_{\partial\Omega} h_2(t, s) \frac{\partial \Phi_2}{\partial n} ds$$

$$H_1(t) = a(t) R_1(t) \left(\int_{\Omega} \Phi_1(x)\, dx \right)^{-1},$$

$$H_2(t) = a(t) R_2(t) \left(\int_{\Omega} \Phi_2(x)\, dx \right)^{-1}.$$

Theorem — *Assume condition (H1-H4) hold. If the following impulsive differential inequalities*

$$\begin{cases} V'(t) + [\lambda_i a(t) + p_0(t)] V(t) + q_0(t) f(V(t)) \le H_i(t), & t \ne t_k \\ V(t_k^+) \le (1 + \alpha_k) V(t_k) & k = 1, 2, \ldots \end{cases} \quad \ldots (3)$$

$$\begin{cases} V'(t) + [\lambda_i a(t) + p_0(t)] V(t) + q_0(t) f(V(t)) \le -H_i(t), & t \ne t_k \\ V(t_k^+) \le (1 + \alpha_k) V(t_k) & k = 1, 2, \ldots \end{cases} \quad \ldots (4)$$

have no eventually positive solution, then the every non-zero solution of (1)-(Bi) is oscillatory in G. (i = 1, 2).

PROOF : Assume that $u(t, x)$ is a non-zero solution of the problem (1)-(Bi). Let $u(t, x)$ is a eventually positive solution of the problem (1)-(Bi) in $[T + \infty) \times \Omega$, for $T \ge 0$.

When $t \ne t_k$, Multiplying both side of eq. (1) by eigenfunction $\Phi_i(x)$ of the eigenvalue problem (2_i) and integrating with respect to x over the domain Ω, we have

$$\frac{d}{dt}\left[\int_{\Omega} \Phi_i(x)\, u(t, x) dx \right] = a(t) \int_{\Omega} \Phi_i(x)\, \Delta u(t, x) dx - \int_{\Omega} p(t, x)\, \Phi_i(x)\, \Phi_i(x)\, u(t, x)\, dx$$

$$- \int_{\Omega} q(t, x)\, \Phi_i(x) f(u(t, x))\, dx, \; t \ne t_k, \; t \ge T, \; i = 1, 2. \quad \ldots (5)$$

Using the Green formula, boundary condition $(B1)$ and the Robin eigenvalue problem (2_1), we have

$$\int_{\Omega} \Phi_1(x)\, \Delta u(t, x)\, dx = \int_{\partial\Omega} \left(\Phi_1 \frac{\partial u}{\partial n} - u \frac{\partial \Phi_1}{\partial n} \right) ds + \int_{\Omega} u(t, x)\, \Delta \Phi_1(x)\, dx$$

$$= \int_{\partial\Omega} h_1(t, s)\, \Phi_1(s)\, ds - \lambda_1 \int_{\Omega} \Phi_1(x)\, u(t, x)\, dx,$$

$$= R_1(t) - \lambda_1 \int_{\Omega} \Phi_1(x)\, u(t, x)\, dx, \; t \ne t_k, \; t \ge T. \quad \ldots (6_1)$$

Using the Green formula, boundary condition $(B2)$ and the Dirichlet eigenvalue problem (2_2), we have

$$\int_{\Omega} \Phi_2(x)\, \Delta u(t, x)\, dx = \int_{\partial\Omega} \left(\Phi_2 \frac{\partial u}{\partial n} - u \frac{\partial \Phi_2}{\partial n} \right) ds$$

$$+ \int_{\Omega} u(t,x)\, \Delta\, \Phi_2(x)\, dx$$

$$= -\int_{\partial\Omega} h_2(t,s) \frac{\partial \Phi_2}{\partial n} ds - \lambda_2 \int_{\Omega} \Phi_2(x)\, u(t,x)\, dx$$

$$= R_2(t) - \lambda_2 \int_{\Omega} \Phi_2(x)\, u(t,x)\, dx, \quad t \neq t_k, t \geq T. \quad \ldots (6_2)$$

Using Jensen inequality, we have

$$\int_{\Omega} q(t,x)\, \Phi_i(x) f(u(t,x))\, dx$$

$$\geq q_0(t) f\left[\left(\int_{\Omega} \Phi_i(x)\, dx \right)^{-1} \int_{\Omega} \Phi_i(x)\, u(t,x)\, dx \right]$$

$$\int_{\Omega} \Phi_i(x) dx, \quad t \neq t_k, t \geq T, \; i = 1, 2. \quad \ldots (7)$$

Therefore, from (5)-(7), we have

$$\frac{d}{dt}\left[\int_{\Omega} \Phi_i(x)\, u(t,x)\, dx \right] \leq a(t)\, R_i(t)$$

$$-a(t)\, \lambda_i \int_{\Omega} \Phi_i(x)\, u(t,x)\, dx - p_0(t) \int_{\Omega} \Phi_i(x)\, u(t,x)\, dx$$

$$-q_0(t) f\left[\left(\int_{\Omega} \Phi_i(x) dx \right)^{-1} \int_{\Omega} \Phi_i(x)\, u(t,x)\, dx \right] \int_{\Omega} \Phi_i(x) dx,$$

$$t \neq t_k, \; t \geq T, \; i = 1, 2. \quad \ldots (8)$$

Let

$$V_i(t) = (S_{\Omega} \; \phi_i(x)\, d(x)^{-1} \, ((x)\, 4(X) dX,$$

we have

$$V_i(t) + [\lambda_1\, a(t) + P_0^{r}(t)]\; V_i(t) + g_0(t) f(V_i(t)$$

$$t \geq T, i = 1, 2, \leq H_i(t), t \neq t_k \quad \ldots (9)$$

From condition $(H3)$, we have

$$\int_{\Omega} [u(t_k^+, x) - u(t_k, x)]\, \Phi_i(x) = \int_{\Omega} g(t_k, x, u(t_k, x))\, \Phi_i(x)\, dx$$

$$\leq \alpha_k \int_{\Omega} u(t_k, x)\, \Phi_i(x)\, dx, \quad k = 1, 2, \ldots, \; i = 1, 2.$$

i.e.,
$$\int_{\Omega} u(t_k^+, x)\, \Phi_i(x) dx \leq (1 + \alpha_k) \int_{\Omega} u(t_k, x)\, \Phi_i(x)\, dx, \quad k = 1, 2, \ldots.\; i = 1, 2.$$

We have

$$V_i(t_k^+) \leq (1 + \alpha_k)\, V_i(t_k), \quad k = 1, 2, \ldots, \; i = 1, 2. \qquad \ldots (10)$$

From formula (9) and (10), we know that $V_i(t)$ is an eventually positive solution of the impulsive differential inequality (3), which contradicts to the condition of Theorem 1.

If $u(t, x) < 0, (t, x) \in [T, +\infty) \times \Omega$, let $v(t, x) = -u(t, x)$, then using the above-mentioned method, we can also get that

$$\overline{V}_i(t) = \left(\int_{\Omega} \Phi_i(x)\, dx \right)^{-1} \int_{\Omega} \Phi_i(x)\, v(t, x)\, dx$$

is an eventually positive solution of the impulsive differential inequality (4), which contradicts the condition of Theorem 1 as well. This completes the proof of theorem 1.

Theorem 2 — *Assume condition (H1-H4) hold, and* $h_i(t, x) = 0$ *(i = 1, 2)*

If the following impulsive differential inequalities

$$\begin{cases} V'(t) + [\lambda_i a(t) + p_0(t)]\, V(t) + q_0(t) f(V(t))\, 0, & t \neq t_k \\ V(t_k^+) \leq (1 + \alpha_k)\, V(t_k) & k = 1, 2, \ldots \end{cases}$$

has no eventually positive solution, then the every non-zero solution of (1)-(Bi) is oscillatory in G. (i = 1, 2.)

The proof of Theorem 2 is similar to that of Theorem 1 and hence is omitted.

Lemma 3[6] — Assume $0 \leq t_0 \leq t_1 \leq t_2 \leq \ldots. \leq t_k \leq \ldots$ and $\lim_{k \to \infty} t_k = +\infty$;

$m \in PC^1[R_+, R], \varphi \in PC[R_+, R], b_k,\; k = 1, 2, \ldots$ is a constant.

If $m'(t) \leq \varphi(t), t \neq t_k, t \geq t_0, m(t_k^+) \leq (1 + b_k)\, m(t_k),\; k = 1, 2, \ldots$, then

$$m(t) \leq \prod_{t_0 < t_k < t} (1 + b_k)\, m(t_0) + \int_{t_0}^{t} \prod_{s < t_k < t} (1 + b_k)\, \varphi(s)\, ds, \quad t \geq t_0.$$

Theorem 3 — *Suppose that (H1-H4) hold, and*

$$\sum_{k=1}^{\infty} \alpha_k < +\infty \quad \text{... (11)}$$

If
$$\liminf_{t\to+\infty} \left\{ \left[\prod_{T<t_k<t} (1+\alpha_k) V_i(T) \right]^{-1} \int_T^t \prod_{s<t_k<t} (1+\alpha_k) H_i(s)\, ds \right\} = -\infty \quad \text{... (12)}$$

$$\liminf_{t\to+\infty} \left\{ \left[\prod_{T<t_k<t} (1+\alpha_k) V_i(T) \right]^{-1} \int_T^t \prod_{s<t_k<t} (1+\alpha_k) H_i(s)\, ds \right\} = +\infty \quad \text{... (13)}$$

hold for any sufficiently large $T > 0$, then the every non-zero solution of (1)-(Bi) is oscillatory in G. (i = 1, 2).

PROOF : From Theorem 1, we proof only the impulsive differential inequality (3) and (4) have no eventually positive solution.

Let $V_i(t)$ $(i = 1, 2.)$ is an eventually positive solution of the impulsive differential inequality (3), then there exists $T > 0$ such that $V_i(t) > 0, t \geq T$ and $f(V_i(t)) > 0$, $(i = 1, 2)$. We have

$$V_i'(t) \leq H_i(t), \quad t \neq t_k, \quad t \geq T. \ (i = 1, 2).$$

$$V_i(t_k^+) \leq (1+\alpha_k) V_i(t_k), k = 1, 2, \ldots, (i = 1, 2)$$

From Lemma 3, we have

$$V_i(t) \leq \prod_{T<t_k<t} (1+\alpha_k) V_i(T) + \int_T^t \prod_{s<t_k<t} (1+\alpha_k) H_i(s) ds, t \geq T. \ (i = 1, 2.)$$

$$V_i(t) \left[\prod_{T<t_k<t} (1+\alpha_k) V_i(T) \right]^{-1}$$

$$\leq V_i(T) + \int_T^t \prod_{s<t_k<t} (1+\alpha_k) H_i(s) ds \left[\prod_{T<t_k\ t} (1+\alpha_k) V_i(T) \right]^{-1}, \ t \geq T. \ (i = 1, 2)$$

From the condition (12), we have

$$\liminf_{t\to+\infty} \left\{ V_i(t) \left[\prod_{T<t_k<t} (1+\alpha_k) V_i(T) \right]^{-1} \right\} = -\infty, \ (i = 1, 2.) \quad \text{... (14)}$$

Since $V_i(t) > 0\ (t \geq T)$ and $\prod_{T<t_k<t} (1+\alpha_k) V_i(T) > 0$ $(i = 1, 2.)$, which contradicts to the formula (14). Thus the impulsive differential inequality (3) has no eventually positive solution. Using the formula (13), we have

$$\liminf_{t\to+\infty}\left\{\left[\prod_{T<t_k<t}(1+\alpha_k)\,V_i(T)\right]^{-1}\int_T^t\prod_{s<t_k<t}(1+\alpha_k)\,H_i(s)\,ds\right\}$$

$$= -\limsup_{t\to+\infty}\left\{\left[\prod_{T<t_k t}(1+\alpha_k)\,V_i(T)\right]^{-1}\int_T^t\prod_{s<t_k<t}(1+\alpha_k)\,H_i(s)\,ds\right\}=-\infty,\ (i = 1,\ 2)$$

Thus we can also get that the impulsive differential inequality (4) has no eventually positive solution. This completes the proof of Theorem 3.

3. References

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