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EXISTENCE RESULTS FOR DISCONTINUOUS IMPLICIT FUNCTIONAL STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

In this paper we consider the implicit functional Sturm-Liouville problem

$$\left.\begin{aligned} &L_1\, u(t) = f(t, u, L_1\, u) \qquad \text{for } a.e. \quad t \in J = [t_0, t_1], \\ &L_i\, u = C_i\,(L_2\, u, L_3\, u, u), \quad i = 2, 3, \end{aligned}\right\} \qquad \dots (1.1)$$

where

$$\left.\begin{aligned} &L_1\, u\,(t) = -\frac{d}{dt}\,(\mu\,(t)\, u'\,(t)) - g\,(t, u), \quad t \in J, \\ &L_2\, u = a_0\, u\,(t_0) - b_0\, u'\,(t_0), L_3\, u = a_1\, u\,(t_1) + b_1\, u'\,(t_1). \end{aligned}\right\} \qquad \dots (1.2)$$

Assuming that

(*A*) $\mu \in C\,(J, 0, \infty))$, $a_0, b_0, a_1, b_1 \in I\!R_+$, and $a_0\, a_1 + a_0\, b_1 + a_1\, b_0 > 0$,

we prove that problem (1.1), (1.2) has a solution if the functions $g : J \times C\,(J) \to I\!R, f : J \times C\,(J) \times L^1\,(J) \to I\!R$ and $C_i : I\!R \times I\!R \times C\,(J) \to I\!R$, $i = 2$ and 3, satisfy the following hypotheses when $C(J)$ is ordered pointwise and $L^1\,(J)$ a.e. pointwise.

(*g*) $g(t, u)$ is measurable in t for all $u \in C\,(J)$ and increasing in u for a.e. $t \in J$, and $|\, g\,(\cdot, u)\,| \le M \in L^1_+\,(J)$ for all $u \in C\,(J)$.

(*f*0) $f\,(t, u, \upsilon)$ is measurable in t for all $u \in C\,(J)$ and $\upsilon \in L^1\,(J)$, and increasing in u and in υ for *a. e.* $t \in J$.

(*f*1) $\|\, f\,(\cdot, u, \upsilon)\,\|_1 \le m + \lambda\, \|\, \upsilon\, \|_1$ for all $u \in C\,(J)$ and $\upsilon \in L^1\,(J)$, where $m \ge 0$ and $\lambda \in [0, 1)$.

(*C*0) The functions C_2 and C_3 are increasing in all their variables.

(*C*1) $|\, C_i\,(x_2, x_3, u)\,| \le c_i\, |\, x_i\,| + d_i$ for all $x_2, x_3 \in I\!R$ and $u \in C\,(J)$, where $d_i \ge 0$, and $c_i \in [0, 1)$, $i = 2, 3$.

To prove this result we reduce problem (1.1), (1.2) to an operator equation of the form $Lu = Nu$, and apply an existence result derived for equation $Lu = Nu$ in 2. We also give an algorithm which may be used to find a solution of (1.1), (1.2) when $g(\cdot, u) \equiv q \in L^1(J)$. An example is solved to illustrate the applicability of the algorithm.

No continuity hypotheses are imposed on the functions g, f, C_2 and C_3. Moreover, these functions have the unknown function u as their functional variable in (1.1) and (1.2), and f depends functionally also on $L_1 u$ in (1.1).

2. PRELIMINARIES

Throughout this section we assume that the hypotheses (A), (g), $(f0)$, $(f1)$, $(C0)$ and $(C1)$ hold. To reduce problem (1.1), (1.2) to an operator equation $Lu = Nu$, denote

$$X = L^1(J) \times I\!R \times I\!R \text{ and } Y = \{u \in C^1(J) \mid \mu u' \in AC(J)\}. \quad \text{... (2.1)}$$

Define a partial ordering and a norm on X by

$$(h_0, x_0, y_0) \le (h_1, x_1, y_1) \text{ iff } h_0 \le h_1, x_0 \le x_1 \text{ and } y_0 \le y_1, \quad \text{... (2.2)}$$

and

$$\| (h, x, y) \| = \| h \|_1 + |x| + |y|. \quad \text{... (2.3)}$$

Lemma 2.1 — Let L_i, $i = 1, 2, 3$, be given by (1.2). Denoting

$$N_1 u = f(\cdot, u, L_1 u), \; N_i u = C_i (L_2 u, L_3 u, u), i = 2, 3, \quad \text{... (2.4)}$$

we get mappings $L = (L_1, L_2, L_3)$ and $N = (N_1, N_2, N_3)$ from Y to X. Moreover, $u \in Y$ is a solution of the BVP (1.1), (1.2) if and only if $Lu = Nu$.

PROOF : The assertions are obvious consequences of the given hypotheses and eqns. (1.1), (1.2) and (2.4). □

We also need the following existence and comparison result for eqn. $Lu = h$.

Lemma 2.2 — Equation $Lu = h$ has for each $h = (h_0, c_2, c_3) \in X$ the least and the greatest solution in Y, and they are increasing with respect to h.

PROOF : Equation $Lu = h = (h_0, c_2, c_3)$ is equivalent to the BVP

$$-\frac{d}{dt}(\mu(t)\, u'(t)) = g(t, u) + h_0(t) \text{ for a.e. } t \in J, \; L_i u = c_i, \; i = 2,3. \quad \text{... (2.5)}$$

This problem has by [1, Lemma 4.2.2] for each $h = (h_0, c_2, c_3) \in X$ least and greatest solutions which are increasing with respect to h_0, c_2 and c_3. □

The proof of our existence result is based on the following Lemma which is a consequence of [2, Proposition 3.1].

Lemma 2.3 — Given a nonempty subset V of a set Y, a subset P of an ordered normed space X, and mappings $L, N : V \to P$, assume that

(H0) P contains a sup-center (or inf-center) a i.e. $\sup\{a, y\} \in P$ (or $\inf\{a, y\} \in P$) for each $y \in P$.

(H1) $L[V] = P$, and if $u, \upsilon \in V$ and $Lu \leq L\upsilon$, then $Nu \leq N\upsilon$.

(H2) Monotone sequences of $N[V]$ converge in P.

Then the operator equation $Lu = Nu$ has a solution.

Hints to the proof in the case when a is a sup-centre of P. Relation

$$Gx = y, \text{ where } \{y\} = N[L^{-1}[\{x\}]],\ x \in P, \qquad \text{... (2.6)}$$

defines an increasing mapping $G : P \to P$ and, $G[P] = N[V]$. The union C of those well-ordered chains A of P for which

$$x = \sup\{a, G[\{y \in P \mid y < x\}]\} \text{ for all } x \in A,$$

is well-ordered, $b = \max C$ exists and $Gb \leq b$. The union D of those inversely well-ordered chains B (each nonempty subset of B has a maximum) of P for which

$$b = \max B, \text{ and if } b > x \in B, \text{ then } x = \inf G[\{y \in P \mid y > x\}],$$

is inversely well-ordered, $x = \min D$ exists and $Gx = x$. In view of (2.6) we have $\{x\} = N[L^{-1}[\{x\}]]$. If $u \in L^{-1}[\{x\}]$, then $x = Lu$ and $x = Nu$, so that $Lu = Nu$. □

3. An Existence Result

In this section we prove an existence result for problem (1.1), (1.2) by assuming that the hypotheses (A), (g), (f 0), (f 1), (C0) and (C1) hold. It suffices to find a subset V of Y and a subset P of X such that the operators $L = (L_1, L_2, L_3)$ and $N = (N_1, N_2, N_3)$, defined in V by (1.2) and (2.4), satisfy the hypotheses of Lemma 2.3. This ensures that the equation $Lu = Nu$ has a solution $u \in V$, which is by Lemma 2.1 also a solution of problem (1.1), (1.2).

Lemma 3.1 — If $u \in Y$ is a solution of problem (1.1), (1.2), then Lu belongs to the set

$$P = \left\{ (h_1, x_2, x_3) \in X \mid \| h_1 \|_1 \leq \frac{m}{1-\lambda}, |x_2| \leq \frac{d_2}{1-c_2}\ |x_3| \leq \frac{d_3}{1-c_3} \right\}. \qquad \text{... (3.1)}$$

PROOF : Assume that $u \in Y$ is a solution of problem (1.1), (1.2). Applying the hypotheses (f1) and (C1) we get

$$\begin{cases} \| L_1 u \|_1 = \| f(\cdot, u, L_1 u) \|_1 \leq m + \lambda \| L_1 u \|_1, & \text{i.e. } \| L_1 u \|_1 \leq \dfrac{m}{1-\lambda}, \\ |L_i u| = |C_i(L_2 u, L_3 u, u)| \leq d_i + c_i |L_i u|, & \text{i.e. } |L_i u| \leq \dfrac{d_i}{1-c_i}, i = 2, 3. \end{cases}$$

□

These results and (3.1) imply that $Lu \in P$.

Lemma 3.2 — If $u \in Y$ and $Lu \in P$, then $Nu \in P$.

PROOF : Assume that $u \in Y$ and $Lu \in P$. Then

$$\| L_1 u \|_1 \leq \frac{m}{1-\lambda} \text{ and } |L_i u| \leq \frac{d_i}{1-c_i},\ i = 2,3.$$

Applying these inequalities and the hypotheses (f1) and (C1) we get

$$\begin{cases} \| N_1 u \|_1 = \| f(\cdot, u, L_1 u) \|_1 \le m + \lambda \| L_1 u \|_1, \le m + \lambda \dfrac{m}{1-\lambda} = \dfrac{m}{1-\lambda}, \\ | N_i u | = | C_i (L_2 u, L_3 u, u) | \le d_i + c_i | L_i u |, \le d_i + c_i \dfrac{d_i}{1-c_i}, = \dfrac{d_i}{1-c_i} \quad i = 2, 3. \end{cases}$$

These results imply by (3.1) that $Nu \in P$. □

Lemma 3.3 — $a = (0, 0, 0)$ is a sup-center and an inf-center of P, defined by (3.1).

PROOF : If $h = (h_1, x_2, x_3) \in P$, then $h_1 \in L^1(J)$ and $h_i \in I\!R$, $i = 2, 3$. Because sup $\{0, h_1\}$ $= t \mapsto \max\{0, h_1(t)\}$ belongs to $L^1(J)$, and sup $\{0, x_i\} = \max\{0, x_i\}$, then sup $\{a, h\}$ = (sup $\{0, h_1\}$, max $\{0, x_2\}$, max $\{0, x_3\}$) exists in X. Moreover,

$$\| \sup\{0, h_1\} \|_1 = \int_J | \max\{0, h_1(t)\} | \, dt \le \int_J | h_1(t) | \, dt = \| h_1 \|_1 \le \frac{m}{1-\lambda},$$

and
$$| \max\{0, x_i\} | \le | x_i | \le \frac{d_i}{1-c_i}, \; i = 2, 3.$$

These relations and (3.1) imply that sup $\{a, h\} \in P$, where a is a sup-center of P. The proof that a is an inf-center of P is similar. □

Lemma 3.4 — Defining

$$V = \{ u \in Y \mid u \text{ is the least solution of } Lu = h \text{ for some } h \in P \}, \quad \ldots (3.2)$$

then the following properties are valid :

(a) $L[V] = P$ and $N[V] \subseteq P$.

(b) If $u, v \in V$, and $Lu \le Lv$, then $Nu \le Nv$.

(c) Monotone sequences of $N[V]$ converge in P, in the norm of X defined by (2.3).

PROOF : *a)* If $u \in V$, then $Lu \in P$ by (3.2), and if $h \in P$, then equation $Lu = h$ has the least solution $u \in Y$ by Lemma 2.2. This proves that $L[V] = P$.

If $u \in V$, then $Lu \in P$ by (3.2), whence $Nu \in P$ by Lemma 3.2. Thus $N[V] \subseteq P$.

b) If $u, v \in V$, and $Lu \le Lv$, then $u \le v$ by Lemma 2.2. These inequalities, (2.2), (2.4), and the hypotheses (f 0) and (C 0) imply that

$$\begin{cases} N_1 u(t) = f(t, u, L_1 u) \le f(t, v, L_1 v) = N_1 v(t) \quad \text{a.e. in } J, \\ N_2 u = C_2 (L_2 u, L_3 u, u) \le C_2 (L_2 v, L_3 v, v) = N_2 v, \\ N_3 u = C_3 (L_2 u, L_3 u, u) \le C_3 (L_2 v, L_3 v, v) = N_3 v, \end{cases}$$

or equivalently, that $Nu \le Nv$.

c) Assume that $(Nu_n)_{n=0}^{\infty}$ is a monotone sequence in $N[V]$. In view of *a*) and (3.1) we have $\| N_1 u_n \|_1 \le \frac{m}{1-\lambda}$ for each $n \in \mathbb{N}$. Thus by the monotone convergence theorem there is a function $h_0 \in L^1(J)$ such that

$$\| N_1 u_n - h_0 \|_1 \to 0 \text{ as } n \to \infty. \qquad \text{... (3.3)}$$

Since the sequences $\left(N_i u_n\right)_{n=0}^{\infty}$, $i = 2, 3$, are bounded by *a*) and monotone, there exist $x_0, y_0 \in \mathbb{R}$ such that

$$| N_2 u_n - x_0 | \to 0, | N_3 u_n - y_0 | \to 0 \text{ as } n \to \infty. \qquad \text{... (3.4)}$$

The limes relations (3.3) and (3.4) imply by (2.3) that

$$\| Nu_n - (h_0, x_0, y_0) \| \to 0 \text{ as } n \to \infty.$$

The above proof shows that monotone sequences of $N[V]$ converge in X. It follows from (2.3) and (3.1) that P is a closed subset of X. Since monotone sequences of $N[V]$ are contained in P by *a*), then their limits belong to P. □

Now we are ready to prove our main existence result for problem (1.1), (1.2).

Theorem 3.1 — *If the hypotheses (A), (g), (f0), (f1), (C0) and (C1) hold, then problem (1.1), (1.2) has a solution.*

PROOF : Let P and V be defined by (3.1) and (3.2), respectively. According to Lemma 3.3 the set P has a sup-center, whence the hypothesis (H0) of Lemma 2.3 is valid. The results of Lemma 3.4 imply that eqs. (1.2) and (2.4) define operators $L = (L_1, L_2, L_3) : V \to P$ and $N = (N_1, N_2, N_3) : V \to P$, having the following properties :

(H1) $L[V] = P$, and if $u, v \in V$ and $Lu \le Lv$, then $Nu \le Nv$.

(H2) Monotone sequences of $N[V]$ converge in P.

It then follows from Lemma 2.3 that eq. $Lu = Nu$ has a solution $u \in V$. In view of Lemma 2.1, u is also a solution of problem (1.1), (1.2). □

4. ALGORITHMIC METHODS

Consider next the problem

$$\begin{cases} L_1 u(t) = f(t, u, L_1 u) & \text{for a.e. } t \in J, \\ L_i u = C_i(L_2 u, L_3 u, u), & i = 2, 3, \end{cases} \qquad \text{... (1.1)}$$

where

$$\begin{cases} L_1 u(t) = -\frac{d}{dt}(\mu(t) u'(t)) - q(t) \quad t \in J, \\ L_2 u = a_0 u(t_0) - b_0 u'(t_0), L_3 u = a_1 u(t_1) + b_1 u'(t_1). \end{cases} \qquad \text{... (4.1)}$$

Assuming that

(B) $\mu \in C(J, 0, \infty)), q \in L^1(J), a_0, b_0, a_1, b_1 \in I\!R_+$, and $a_0 a_1 + a_0 b_1 + a_1 b_0 > 0$,

we have the following result.

Lemma 4.1 — Assume that (*B*) holds and that $L = (L_1, L_2, L_3)$ is defined by (4.1). The equation $Lu = h$ has for each $h = (h_0, x_2, x_3) \in X$ a unique solution u in Y, which is increasing with respect to h, and can be represented as

$$u(t) = \frac{x_2 y_1(t) + x_3 y_0(t)}{D} + \int_{t_0}^{t_1} k(t, s)(q(s) + h_0(s))\, ds, \quad t \in J, \qquad \text{... (4.2)}$$

where

$$\begin{cases} y_0(t) = \int_{t_0}^{t} \frac{a_0}{\mu(s)} ds + \frac{b_0}{\mu(t_0)}, \quad D = \int_{t_0}^{t_1} \frac{a_0 a_1}{\mu(s)} ds + \frac{a_0 b_1}{\mu(t_1)} + \frac{a_1 b_0}{\mu(t_0)}, \\ y_1(t) = \int_{t}^{t_1} \frac{a_1}{\mu(s)} ds + \frac{b_1}{\mu(t_1)}, \quad k(t, s) = \begin{cases} \frac{y_1(t) y_0(s)}{D}, & t_0 \le s \le t, \\ \frac{y_0(t) y_1(s)}{D}, & t \le s \le t_1 \end{cases} \end{cases} \qquad \text{... (4.3)}$$

PROOF : Equation $Lu = h$ is now equivalent to the BVP

$$\begin{cases} -\frac{d}{dt}(\mu(t) u'(t)) = q(t) + h_0(t) \quad \text{a.e. in } J, \\ a_0 u(t_0) - b_0 u'(t_0) = x_2, \quad a_1 u(t_1) + b_1 u'(t_1) = x_3. \end{cases} \qquad \text{... (4.4)}$$

This problem has by [1, Lemma 4.1.1] a unique solution which is increasing with respect to h_0, x_2 and x_3, and a representation (4.2). This implies the assertions. □

Assuming that the hypotheses (B), (f0), (f1), (C0) and (C1) hold, then properties (H1) and (H2) given in the proof of Theorem 3.1 are valid. Thus $G = N \circ L^{-1}$ is an increasing selfmapping of P, defined by (3.1). Since P has a sup-centre by Lemma 3.3, the proof of Lemma 2.3 ensures that G has a fixed point x. Consequently, $u = L^{-1} x$ satisfies equation $Lu = Nu$, so that u is a solution of (1.1), (4.1).

It can be shown that the first elements of the chains C and D in the proof of Lemma 2.3 are of the form :

$$\begin{cases} x_0 = a, \quad x_{n+1} = \sup(a, Gx_n) \quad \text{as long as } Gx_n \not\leq x_n, \text{ and} \\ y_0 = b, \quad y_{k+1} = Gy_k \quad \text{as long as } Gy_k < y_k, \end{cases}$$

where a denotes a sup-center of P. If C and D are finite, they can be combined to one sequence, whose last member is a fixed point of G, by the following algorithm :

(i) $x_0 = a$. For n from 0 while $x_n \neq Gx_n$, $x_{n+1} = Gx_n$ if $Gx_n < x_n$ else $x_{n+1} = \sup\{a, Gx_n\}$.

Denoting $u_n = L^{-1} x_n$, then $x_n = Lu_n$ and $Gx_n = N(L^{-1} x_n) = Nu_n$, whence a solution of equation $Lu = Nu$ is the last member of the finite sequence (u_n) obtained by rewriting the algorithm (i) in the following form.

(ii) $Lu_0 = a$. For n from 0 while $Lu_n \neq Nu_n$, $Lu_{n+1} = Nu_n$ if $Nu_n < Lu_n$ else Lu_{n+1} = sup $\{a, Nu_n\}$.

Recalling the definitions (2.4), (4.1) and (3.1) of N, L and P, and noticing that a = (0, 0, 0) is a sup-centre of P by Lemma 3.3, the algorithm (ii) can be rewritten as follows :

(iii) $u_0(t) \equiv 0$. For n from 0 while

$$-u_n'' \neq q + f(\cdot, u_n, L_1 u_n) \text{ or } L_i u_n \neq C_0(L_1 u_n, L_2 u_n, u_n),\ i = 2 \text{ or } 3,$$

define $u_{n+1} \in Y$ by

$$u_{n+1}(t) = \begin{cases} \dfrac{C_2(L_2 u_n, L_3 u_m, u_n)\, y_1(t) + C_3(L_2 u_n, L_3 u_n, u_n)\, y_0(t)}{D} \\ + \displaystyle\int_{t_0}^{t_1} k(t, s)\,(q(s) + f(s, u_n, L_1 u_n))\, ds, \end{cases}$$

if

$$-u_n'' \geq q + f(\cdot, u_n, L_1 u_n) \text{ and } L_i u_n \geq C_i(L_1 u_n, L_2 u_n, u_n)\ i = 2,\ 3,$$

otherwise by

$$u_{n+1}(t) = \begin{cases} \dfrac{\max(0, C_2(L_2 u_n, L_3 u_n, u_n))\, y_1(t) + \max(0, C_3(L_2 u_n, L_3 u_n, u_n))\, y_0(t)}{D} \\ + \displaystyle\int_{t_0}^{t_1} k(t, s)\,(q(s) + \max(0, f(s, u_n, L_1 u_n)))\, ds. \end{cases}$$

If the so obtained sequence (u_n) is finite, its last member is a solution of problem (4.1).

Another algorithm which may yield a solution of problem (1.1), (4.1) is obtained by observing that a = (0, 0, 0) is also an inf-centre of P, given by (3.1). It differs from algorithm (iii) so that maximums are replaced by minimums and inequalities are reversed.

Example 4.1 — Let $[x]$ denote the greatest integer $\leq x$. Consider the BVP

$$\begin{cases} -u''(t) = \dfrac{[u(-t)-t^2]}{1+|[u(-t)-t^2]|} + \dfrac{[3(u'(0)-u'(1))]}{12} \text{ a.e in } J = [-1, 1], \\[2ex] u(-1) - 4u'(-1) = 1 + \dfrac{[u(-1) - 4u'(-1)]}{2}, \\[2ex] u(1) + 4u'(1) = 1 + \dfrac{[u(1) + 4u'(1)]}{2}. \end{cases} \qquad \dots (4.5)$$

Problem (4.5) is of the form (1.1), (4.1), where

$$\begin{cases} q(t) = 0, \quad f(t, u, v) = \dfrac{[u(-t)-t^2]}{1+|[u(-t)-t^2]|} + \dfrac{\left[3\displaystyle\int_0^1 v(s)\,ds\right]}{12}, \\[2ex] C_i(L_2 u, L_3 u, u) = 1 + \dfrac{[L_i u]}{2}, \quad i = 2, 3. \end{cases}$$

Obviously, the hypotheses (f0), (C0) and (C1) are valid. Moreover,

$$\|f(\cdot, u, v)\|_1 = \int_{-1}^{1} \left| \frac{[u(-t)-t^2]}{1+|[u(-t)-t^2]|} + \frac{\left[3\displaystyle\int_0^1 [v(s)]\,ds\right]}{12} \right| dt$$

$$\leq \int_{-1}^{1} 1\,dt + \int_{-1}^{1} \frac{\|v\|_1 + 1}{4}\,dt \leq 3 + \frac{\|v\|_1}{2}, \quad u \in C(J), \; v \in L^1(J_1).$$

Thus also the hypothesis (f1) holds. It then follows from Theorem 3.1 that problem (4.5) has a solution.

Applying algorithm (iii) calculating integrals by Simpson rule one obtains the following approximation for a solution of (4.5) :

$$u(t) \approx \begin{cases} -.48533333\,t^2 + .024\,t + 5.981, & -1 \leq t < -.806, \\ -.5t^2 + 5.972, & -.806 \leq t \leq .806, \\ -4.8333333\,t^2 - .025t + 5.982, & .806 < t \leq 1. \end{cases} \qquad \dots (4.6)$$

In view of this one can infer that the greatest solution of (4.5) is of the form

$$u(t) = \begin{cases} -\dfrac{29}{60}t^2 + at + b, & -1 \leq t < -d, \\[1ex] -\dfrac{1}{2}t^2 + c, & -d \leq t \leq d, \\[1ex] -\dfrac{29}{60}t^2 - at + b & d < t \leq 1. \end{cases} \qquad \dots (4.7)$$

It remains to determine a, b, c and d. The boundary conditions imply that $b = \frac{351}{60} + 5a$. Because u' is continuous at d, we get the equation

$$u'(d) = -x = -\frac{29}{30} d - a \Rightarrow a = \frac{d}{30}.$$

At a point $t = d$ the second derivative of u has a jump, and the approximation (4.6) implies that it is caused by the jump of $[u(-t) - t^2]$, from the value 5 to the value 4 at d, whence $u(-d) - d^2 = 5$. This and (4.7) yield

$$-\frac{3}{2} d^2 + c = 5 = -\frac{89}{60} d^2 - ad + b.$$

Recalling that $a = \frac{d}{30}$ and $b = \frac{351}{60} + 5a$ we see that $c = 5 + \frac{3}{2} d^2$, and that d is the positive solution of equation

$$91 x^2 - 10x - 51 = 0.$$

In view of this result and the above formulae for a, b and c we get the following exact and approximate values for the constants in the expression (4.7) of u :

$$\begin{cases} d = \dfrac{5 + \sqrt{4666}}{91} \approx .8055837961, \\[2ex] a = \dfrac{d}{30} = \dfrac{5 + \sqrt{4666}}{2730} \approx .0268527932, \\[2ex] b = \dfrac{351}{60} + 5a = \dfrac{31991 + 10\sqrt{4666}}{5460} \approx 5.984263966, \\[2ex] c = 5 + \dfrac{3}{2} d^2 = \dfrac{925 + 3\sqrt{4666}}{182} \approx 5.973447879. \end{cases} \qquad \text{... (4.8)}$$

Thus the function

$$u(t) = \begin{cases} -\dfrac{29}{60} t^2 + \dfrac{5 + \sqrt{4666}}{2730} t + \dfrac{31991 + 10\sqrt{4666}}{5460}, & -1 \le t < -\dfrac{5 + \sqrt{4666}}{2730} \\[2ex] -\dfrac{1}{2} t^2 + \dfrac{925 + 3\sqrt{4666}}{182}, & -\dfrac{5 - \sqrt{4666}}{2730} \le t \le \dfrac{5 + \sqrt{4666}}{2730}, \\[2ex] -\dfrac{29}{60} t^2 - \dfrac{5 + \sqrt{4666}}{2730} t + \dfrac{31991 + 10\sqrt{4666}}{5460}, & \dfrac{5 + \sqrt{4666}}{2730} < t \le 1 \end{cases}$$

... (4.9)

is an exact solution of problem (4.5).

Remark 4.1 : In [1, Section 4.2] existence and comparison results are derived for extremal solutions of the implicit Strum-Liouville boundary value problem

$$\begin{cases} L_1 u(t) = f(t, u, L_1 u(t)) & \text{for a. e } t \in J = [t_0, t_1], \\ L_1 u = C_i (L_2 u, L_3 u, u), & i = 2, 3, \end{cases} \quad \text{... (4.10)}$$

where

$$\begin{cases} L_1 u(t) = -\frac{d}{dt}(\mu(t) u'(t)) - g(t, u, u(t), u'(t)), \quad t \in J, \\ L_2 u = a_0 u(t_0) - b_0 u'(t_0), \; L_3 u = a_1 u(t_1) + b_1 u'(t_1). \end{cases} \quad \text{... (4.11)}$$

As for other results for explicit and implicit Strum-Liouville problems see, e.g. [3, Chapter 3] and the references listed in [1, Section 4.6]

REFERENCES

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3. S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker, Inc., New York, 1994.