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NECESSARY AND SUFFICIENT CONDITIONS FOR NÖRLUND SUMMABILITY

W. T. SULAIMAN

Department of Mathematics, College of Science, University of Qatar, P. O. Box 2173, Doha-Qatar

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In this note we give the necessary and sufficient conditions satisfied by the sequence (ε_n) in order to have $\Sigma a_n \varepsilon_n$ summable $|N, q_n|_k$ whenever Σa_n is summable $|N, p_n|_k$, $k \geq 1$.

Key Word : Nörlund Summability

1. INTRODUCTION

Let (p_n) and (q_n) be sequences of positive number such that $P_n = p_1 + p_2 + \ldots + p_n \to \infty$ as $n \to \infty$.

An infinite series Σa_n is summable $|N, q_n|_k$, [2], if

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \frac{1}{P_n} \sum_{v=1}^{n} p_{n-v} v a_v \right|^k < \infty$$

For $p_n = A_n^{\alpha-1}$, $|N, p_n|_k$ summability reduces to $|C, \alpha|_k$ summability.

The series Σa_n is summable $|\bar{N}, p_n|_k$, $k \geq 1$, if [1], $\Sigma \left(\frac{P_n}{p_n} \right)^{k-1} |\Phi_n - \Phi_{n-1}|^k < \infty$, where

$$\Phi_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v s_v,$$

(s_v) is the sequence of partial sums of Σa_n.

We [3] have proved the following.

Theorem 1 — *Suppose that* $\Delta \lambda_n \geq 0$,

$$\rho_n Q_n = O(P_n q_n)$$

$$\frac{Q_n \Delta \lambda_n}{p_n P_n} \to 0 \text{ as } n \to \infty.$$

Then the necessary and sufficient conditions that $\Sigma a_n \lambda_n$ *be summable* $|\bar{N}, q_n|_k$ *whenever* Σa_n *is summable* $|\bar{N}, p_n|_k$, $k \geq 1$, *are*

$$\lambda_n = O\left(\frac{p_n Q_n}{P_n q_n}\right)^{1/k}, \Delta \lambda_n = O\left(\frac{P_n}{P_{n-1}}\right).$$

The object of this note is to give the following characterization in the type of $|N, p_n|_k$ summability:

Theorem — *Let* $(p_n), (q_n)$ *be a nonincreasing sequences of positive numbers.*

Let

$$T_n = \frac{1}{n^{1/k} Q_n} \sum_{\nu=1}^{n} \nu a_\nu q_{n-\nu} \varepsilon_\nu, \quad t_n = \frac{1}{n^{1/k} P_n} \sum_{\nu=1}^{n} \nu a_\nu p_{n-\nu}$$

$$W_\nu = \max_{1 \leq j \leq \nu} |t_j|$$

(a) Let $\sum_{n=\nu}^{\infty} \frac{|\Delta_\nu q_{n-\nu}|}{p_{n-\nu}} < \infty$

$$\sum_{n=\nu}^{\infty} \frac{q_{n-\nu-1}}{n Q_n^k} \left| \Delta_\nu \left(\frac{1}{p_{n-\nu}}\right) \right| = O\left(\frac{1}{\nu Q_\nu^k}\right)$$

$$\sum_{n=\nu}^{\infty} \frac{q_{n-\nu-1}}{n Q_n P_{n-\nu-1}^k} = O\left(\frac{1}{\nu p_\nu^k}\right)$$

Then sufficient conditions that $\sum_{n=1}^{\infty} |T_n|^k < \infty$ *whenever* $\sum_{n=1}^{\infty} W_n^k < \infty$ *are*

(i) $\varepsilon_n = O(Q_n/P_n)$, (ii) $\Delta \varepsilon_n = O\left(\frac{1}{P_n}\right)$

(b) Let $\Delta q_n = O\left(\frac{q_n}{n}\right)$, $\sum_{n=\nu}^{\infty} \frac{p_{n=\nu}^k}{n P_n^k} = O\left(\frac{1}{\nu P_\nu^k}\right)$.

Then necessary conditions that $\Sigma |T_n|^k < \infty$ *whenever* $\Sigma |t_n|^k < \infty$ *are*

$$\text{(i) } \varepsilon_n = O\left(\frac{Q_n}{P_n}\right), \text{ (ii) } \Delta\varepsilon_n = O\left(\frac{Q_\nu^{1-1/k}}{P_\nu q_\nu}\right).$$

2. LEMMAS

We need the following Lemmas :

Lemma 1 — (a) If (p_n) is positive nonincreasing, then for $1 \le \nu \le n$,

$$\left|\sum_{r=1}^{\nu} p_{n-r}\, ra_r\right| \le \max_{1\le j\le \nu}\left|\sum_{r=1}^{j} p_{j-r}\, ra_r\right|$$

(b) If (p_n) is positive nondecreasing, then for $1 \le \nu \le n$,

$$\left|\sum_{r=1}^{\nu} p_{n-r}\, ra_r\right| \ge \min_{1\le j\le \nu}\left|\sum_{r=1}^{j} p_{j-r}\, ra_r\right|$$

PROOF : (a)

$$\left|\sum_{r=1}^{\nu} p_{n-r}\, ra_r\right| = \left|\sum_{r=1}^{\nu} \frac{p_{n-r}}{p_{n-r-1}}\frac{p_{n-r-1}}{p_{n-r-2}}\cdots\frac{p_{m-r-1}}{p_{m-r}}\cdot p_{m-r}\, ra_r\right|$$

$$\le \frac{p_{n-\nu}}{p_{n-\nu-1}}\frac{p_{n-\nu-1}}{p_{n-\nu-2}}\cdots\frac{p_{m-\nu-1}}{p_{m-\nu}}\max_{\substack{1\le t\le \nu\\ 1\le m\le \nu\\ t\le m}}\left|\sum_{r=1}^{t} p_{m-r}\, ra_r\right|$$

$$\le \frac{p_{n-\nu}}{p_{m-\nu}}\max_{1\le t\le m\le \nu}\left|\sum_{r=1}^{t} p_{m-r}\, ra_r\right|$$

$$\le \max_{1\le m\le \nu}\left|\sum_{r=1}^{m} p_{m-r}\, ra_r\right|$$

The proof of (b) is similar

Lemma 2 — Let $k \ge 1, Q_n = q_0 + q_1 + \ldots + q_n \to \infty$ as $n \to \infty$, $\{q_n\}$ is nonincreasing, $\Delta q_n = O\left(\frac{q_n}{n}\right)$, then

$$\frac{q_n^m}{\nu^\rho Q_\nu^k} = O(1)\sum_{n=\nu}^{\infty}\frac{q_{n-\nu}^m}{n^\rho Q_n^{k+1}}.$$

3. Proof of the Theorem

We have

$$T_n = \frac{1}{n^{1/k} Q_n} \sum_{\nu=1}^{n} \nu p_{n-\nu} a_\nu \cdot \frac{q_{n-\nu} \varepsilon_\nu}{p_{n-\nu}}$$

$$= \frac{1}{n^{1/k} Q_n} \left[\sum_{\nu=1}^{n-1} \left(\sum_{r=1}^{\nu} r p_{n-r} a_r \right) \Delta_\nu \left(\frac{q_{n-\nu} \varepsilon_\nu}{p_{n-\nu}} \right) + \left(\sum_{\nu=1}^{n} \nu p_{n-\nu} a_\nu \right) \frac{q_0 \varepsilon_n}{p_0} \right] \quad \ldots (3.1)$$

$$|T_n| \leq \frac{1}{n^{1/k} Q_n} \left[\sum_{\nu=1}^{n-1} \left| \sum_{r=1}^{\nu} r p_{n-r} a_r \right| \right.$$

$$\left\{ \frac{|\Delta_\nu (q_{n-\nu})|\, |\varepsilon_\nu|}{p_{n-\nu}} + q_{n-\nu-1} |\varepsilon_\nu| \left| \Delta_\nu \left(\frac{1}{p_{n-\nu}} \right) \right| \right.$$

$$\left. \left. + \frac{q_{n-\nu-1} |\Delta \varepsilon_\nu|}{p_{n-\nu-1}} + \frac{p_0}{q_0} |\varepsilon_n| n^{1/k} P_n |t_n| \right\} \right].$$

As

$$\left| \sum_{r=1}^{\nu} r p_{n-r} a_r \right| \leq \max_{1 \leq j \leq \nu} \left| \sum_{r=1}^{j} p_{j-r} r a_r \right| = \max_{1 \leq j \leq \nu} |j^{1/k} P_j t_j|$$

$$\leq \nu^{1/k} P_\nu \max_{1 \leq j \leq \nu} |t_j| = \nu^{1/k} P_\nu W_\nu, \text{ (by Lemma 1 (a)),}$$

we have

$$T_n \leq \frac{1}{n^{1/k} Q_n} \left[\sum_{\nu=1}^{n-1} \nu^{1/k} P_\nu W_\nu \left\{ \frac{|\Delta_\nu 1_{n-\nu}|\, |\varepsilon_\nu|}{p_{n-\nu}} + q_{n-\nu-1} |\varepsilon_\nu| \left| \Delta_\nu \left(\frac{1}{p_{n-\nu}} \right) \right| \right. \right.$$

$$\left. \left. + \frac{p_0}{q_0} |\varepsilon_n| \frac{P_n}{Q_n} |t_n| \right] \right.$$

$$= T_1 + T_2 + T_3 + T_4, \text{ say.}$$

(a) To prove sufficiency, by Minkowki's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} |T_r|^k < \infty, \; r = 1, 2, 3, 4.$$

Applying Hölder's inequality,

$$\sum_{n=2}^{m+1} |T_1|^k = \sum_{n=2}^{m+1} \frac{1}{nQ_n^k} \left\{ \sum_{v=1}^{n-1} v^{1/k} P_v W_v \frac{|\Delta_v q_{n-v}|}{P_{n-v}^k} |\varepsilon_v|^k \right\}$$

$$\leq \sum_{n=2}^{m+1} \frac{1}{nQ_n^k} \sum_{v=1}^{n-1} v P_v W_v^k \frac{|\Delta_v q_{n-v}|}{P_{n-v}} |\varepsilon_v|^k \left\{ \sum_{v=1}^{n-1} \frac{|\Delta_v q_{n-v}|}{p_{n-v}} \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} v P_v^k W_v^k |\varepsilon_v|^k \sum_{n=v+1}^{m+1} \frac{|\Delta_v q_{n-v}}{nQ_n^k p_{n-v}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{Q_v} \right)^k W_v^k |\varepsilon_v|^k \sum_{n=v+1}^{m+1} \frac{|\Delta_n q_{n-v}|}{p_{n-v}}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{Q_v} \right)^k W_v^k |\varepsilon_v|^k$$

$$\sum_{n=2}^{m+1} |T_2|^k = \sum_{n=2}^{m+1} \frac{1}{nQ_n^k} \left\{ \sum_{v=1}^{n-1} v^{1/k} P_v W_v q_{n-v-1} |\varepsilon_v| \left| \Delta_v \left(\frac{1}{p_{n-v}} \right) \right| \right\}^k$$

$$\leq \sum_{n=2}^{m+1} \frac{1}{nQ_n^k} \sum_{v=1}^{n-1} v P_v^k W_v^k q_{n-v-1} |\varepsilon_v|^k$$

$$\left| \Delta_v \left(\frac{1}{p_{n-v}} \right) \right| \left\{ \sum_{v=1}^{n-1} q_{n-v-1} \left| \Delta_v \frac{1}{p_{n-v}} \right| \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} v P_v^k W_v^k |\varepsilon_v|^k \sum_{n=v+1}^{m+1} \frac{q_{n-v-1}}{nQ_n^k} \left| \Delta_v \left(\frac{1}{p_{n-v}} \right) \right|$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{Q_v} \right)^k W_v^k |\varepsilon_v|^k$$

$$\sum_{n=2}^{m+1} |T_3|^k = \sum_{n=2}^{m+1} \frac{1}{nQ_n^k} \left\{ \sum_{v=1}^{n-1} v^{1/k} P_v W_v \frac{q_{n-v-1}}{p_{n-v-1}} |\Delta \varepsilon_v| \right\}^k$$

$$\leq \sum_{n=2}^{m+1} \frac{1}{nQ_n} \sum_{v=1}^{n-1} v P_v^k W_v^k \frac{q_{n-v-1}}{p_{n-v-1}^k} |\Delta \varepsilon_v|^k \left\{ \sum_{v=1}^{n-1} \frac{q_{n-v-1}}{Q_n} \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} v P_v^k W_v^k |\Delta \varepsilon_v|^k \sum_{n=v+1}^{m+1} \frac{q_{n-v-1}}{nQ_n p_{n-v-1}^k}$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k W_v^k |\Delta \varepsilon_v|^k$$

$$\sum_{n=1}^{m} |T_4|^k = \sum_{n=1}^{m} \left\{ \frac{p_0}{q_0} |\varepsilon_n| \frac{P_n}{Q_n} |t_n| \right\}^k$$

$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{Q_n} \right)^k |W_n^k| |\varepsilon_n|^k.$$

(b) (*Necessity*) — Suppose that $\Sigma |T_n|^k < \infty$ whenever $\Sigma |t_n|^k < \infty$. Therefore,

$$\sum_{n=1}^{m} |T_n|^k = O(1) \sum_{n=1}^{m} |t_n|^k \qquad \ldots (3.2)$$

$$\sum_{n=1}^{m} \left| \frac{1}{n^{1/k} Q_n} \sum_{v=1}^{n} v q_{n-v} a_v \varepsilon_v \right|^k = O(1) \sum_{n=1}^{m} \left| \frac{1}{n^{1/k} P_n} \sum_{v=1}^{n} v p_{n-v} a_v \right|^k$$

Let $a_m = 1, a_v = 0$ for $v \neq m$. We have

$$\left| \frac{1}{m^{1/k} Q_m} m q_0 \varepsilon_m \right|^k = O(1) \left| \frac{1}{m^{1/k} P_m} m p_0 \right|^k$$

This gives

$$\varepsilon_m = O\left(\frac{Q_m}{P_m} \right)$$

From (3.1) and (3.2), we have

$$\sum_{n=1}^{m}\frac{1}{nQ_n^k}\left|\sum_{v=1}^{n-1}\left(\sum_{r=1}^{v}rp_{n-r}a_r\right)\frac{q_{n-v}}{P_{n-v}}\Delta\varepsilon_v+\left(\sum_{r=1}^{v}rp_{n-r}\right)\right.$$

$$\left\{\frac{\Delta_v(q_{n-v})}{p_{n-v}}+q_{n-v}\Delta_v\left(\frac{1}{p_{n-v}}\right)\right\}\varepsilon_{v+1}$$

$$\left.+\left(\frac{p_0}{q_0}\frac{P_n}{Q_n}t_n\right)\varepsilon_n\right|^k=O(1)\sum_{n=1}^{m}\frac{1}{nP_n^k}\left|\sum_{v=1}^{n-1}vp_{n-v}a_v\right|^k.$$

As ε_v and $\Delta\varepsilon_v\approx\frac{d}{dv}\varepsilon_v$ have behaviour as x^2 and its derivative $2x$ which are linearly independent. Therefore the linear independence of ε_v and $\Delta\varepsilon_v$ implies

$$\sum_{n=1}^{m}\frac{1}{nQ_n^k}\left|\sum_{v=1}^{n-1}\left(\sum_{r=1}^{v}rp_{n-r}a_r\right)\frac{q_{n-v}}{p_{n-v}}\Delta\varepsilon_v\right|^k$$

$$=O(1)\sum_{n=1}^{m}\frac{1}{nP_n^k}\left|\sum_{v=1}^{n-1}vp_{n-v}a_v\right|^k$$

For $v\geq 1$, let $a_v=1, a_\rho=0$ for $\rho\neq v$, we have

$$\sum_{n=v}^{m}\frac{1}{nQ_n^k}\left|vp_{n-v}\frac{q_{n-v}}{p_{n-v}}\Delta\varepsilon_v\right|^k=O(1)\sum_{n=v}^{m}\frac{1}{nP_n^k}|vp_{n-v}|^k$$

$$v^k|\Delta\varepsilon_v|^k\sum_{n=v}^{m}\frac{q_{n-v}^k}{nQ_n^k}=O(v^k)\sum_{n=v}^{m}\frac{p_{n-v}^k}{nP_n^k}$$

$$v^k|\Delta\varepsilon_v|^k\frac{q_v^k}{vQ_v^{k-1}}=O\left(\frac{v^{k-1}}{P_v^k}\right)$$

$$\Delta\varepsilon_v=O\left(\frac{Q_v^{1-1/k}}{pP_vq_v}\right)$$

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