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DUALITY IN THE OPTIMAL CONTROL PROBLEMS FOR NONLINEAR HYPERBOLIC DISTRIBUTED PARAMETER SYSTEMS*

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In this paper, we study the duality theory of nonlinear hyperbolic systems. The main objective is to prove the duality theorem under general conditions within an infinite-dimensional framework. An example of its application is also given.

Key Words : Duality; Optimal Control; Hyperbolic Systems; Nonlinear Distributed Control Systems; Duality Theorem

1. INTRODUCTION

Recently, Many authors have studied optimal control problems of various practical systems [1, 3, 4, 6]. In these developments, it is crucial to establish conditions which characterize an optimum of the optimal control problem. The optimal control problems is to find existence condition of infimum cost functional of primal systems. The purpose of duality theory prove that the infimum cost functional of given systems is equal to suprimum cost functional of dual systems. Thus we can solve optimal control problem of given systems by finding a solution of the optimal control problem of dual systems. Up to now, most of authors dealt with duality theory of linear systems. Chan[2] and Tanimoto[9] studied the duality theory for linear parabolic optimal control problem. Park and Lee[7&8] dealt with duality theory for the corresponding linear hyperbolic optimal control systems. Here, we consider the optimal control problems and the dual optimal control problems for hyperbolic nonlinear systems. The main purpose of this paper is to extend results to nonlinear hyperbolic optimal control systems and to prove the duality theorem under general conditions within an infinite-dimensional framework.

Our approach is based on an infinite dimensional version of the duality theorem for the optimality systems. To this end, we invoke necessary and sufficient conditions of optimality for distributed control systems given in [5]. We now explain the content of this paper. In Section 2, we present the preliminaries and hypotheses of the paper. In Section 3, we prove the duality theorem of a nonlinear hyperbolic distributed parameter systems. Finally, we give an example of its application.

2. PRELIMINARIES

Let $X_i\,(i=1,2)$ and H be separable Hilbert spaces. The norm on X_i will be denoted by $\|\cdot\|_{X_i}$. The

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norm on H will be denoted by $|\cdot|$ and the corresponding inner product by $(\cdot, \cdot)$. We assume that $X_i \subseteq H$ and the injection of X_i into H is continuous and that X_i is dense in H. We denote the dual space of X_i by X_i^* and the norm on it by $\|\cdot\|_{X_i}^*$. Identifying H with its dual space, we have $X_1 \subseteq X_2 \subseteq H \subseteq X_2^* \subseteq X_1^*$. The usual bilinear form on the dual pair (X_1, X_1^*) is denoted by $\langle \cdot, \cdot \rangle$. The equalities $\langle x, h \rangle = \langle x, h \rangle_{X_2, X_2^*}$ for $x \in X_1, h \in X_2^*$ and $\langle x, h \rangle = (x, h)$ for $x \in X_1, h \in H$ hold. Let T be a fixed positive number. We define a function space $W(0, T)$ by

$$W(0, T) = \{x \mid \dot{x} \in L^2(0, T; X_1), \ \dot{x} \in L^2(0, T; X_2), \ \ddot{x} \in L^2(0, T; X_1^*)\},$$

with an inner product

$$(x_1, x_2)_{W(0,T)} = \int_0^T \{(x_1(t), x_2(t))_{X_1} + (\dot{x}_1(t), \dot{x}_2(t))_{X_2} + (\ddot{x}_1(t), \ddot{x}_2(t))_{X_1}^*\} \, dt.$$

This becomes a Hilbert space with norm

$$\|x\|_{W(0,T)} = \left(\|x\|^2_{L^2(0,T;X_1)} + \|\dot{x}\|_{L^2(0,T;X_2)}{}^2 + \|\ddot{x}\|^2_{L^2(0,T;X_1^*)} \right)^{\frac{1}{2}},$$

where $\dot{x} = \dfrac{dx}{dt}$ and $\ddot{x} = \dfrac{d^2x}{dt^2}$.

We consider the following optimal control problem :

$$(P) \begin{cases} \displaystyle\int_0^T (F(t, x(t)) + G(t, u(t)))\, dt \to \inf \\ \text{subject to} \\ \begin{cases} \ddot{x}(t) = -A_2(t)\dot{x}(t) - A_1(t)x(t) + B(t, u(t)) + f(t, x(t)) \text{ a.e.} \\ x(0) \in C, \dot{x}(0) \in H, u(t) \in U(t) \text{ a.e.} \end{cases} \end{cases} \quad \text{... (2.1)}$$

where $A_1(t)$ is the operator defined by a bilinear form on X_1, $A_2(t)$ is the operator defined by another bilinear form on X_2, $f(t, x(t))$ is a nonlinear forcing function, $U(t)$ is control space and C is a closed convex set of H. Here, $x(\cdot)$ means $x(u(\cdot))$. We assume that the control variable u takes its values in another separable Banach space Y, the norm of whose element is denoted by $\|\cdot\|_Y$. We denote by $S(Y)$ the set of nonempty closed, convex subset of Y. The set of admissible control is given by

$$\mathcal{U}_{ad} = \{u(\cdot) \mid u(\cdot) \in L^2(0, T; Y), u(t) \in U(t) \text{ a.e.}\}.$$

We will describe the assumptions which are imposed on the optimal control problem (P).

(A1) Let $a_i(t; \phi, \varphi), t \in [0, T], i = 1, 2$ be a bilinear form on $X_i \times X_i$ satisfying

(i) $a_i(t; \phi, \varphi) = a_i(t; \varphi, \phi), \ \forall\ \phi, \varphi \in X_i, \ \forall\ t \in [0, T]$,

(ii) $\exists\, c_{i1} > 0$ such that

$$| a_i(t; \phi, \varphi) | \le c_{i1} \| \phi \|_{X_i} \| \varphi \|_{X_i}, \ \forall\ \phi, \varphi \in X_i, \ \forall\ t \in [0, T]$$

and $\exists\, \alpha_i > 0, \lambda_i \in R$ such that

$$a_i(t; \phi, \phi) + \lambda_i | \phi |_H^2 \ge \alpha_i \| \phi \|_{X_i}^2, \ \forall\ \phi \in X_i, \ \forall\ t \in [0, T],$$

(iii) $t \to a_i(t; \phi, \varphi)$ is continuously differentiable in $[0, T]$ and $\exists\, c_{i2} > 0$ such that

$$\left| \frac{d}{dt} a_i(t; \phi, \varphi) \right| \le c_{i2} \| \phi \|_{X_i} \| \varphi \|_{X_i}, \ \forall\ \phi, \varphi \in X_i, \ \forall\ t \in [0, T].$$

Then we can define the operator $A_i(t) \in \mathcal{L}(X_i, X_i^*)$ for $t \in [0, T]$ defined by the relation

$$a_i(t; \phi, \varphi) = \langle A_i(t)\, \phi, \varphi \rangle_{X_i^*, X_i} \text{ for all } \phi, \varphi \in X_i.$$

(A2) The nonlinear term $f(t, x) : [0, T] \times X_2 \to X_2^*$

(i) $t \to f(t, x(t))$ is (strongly) measurable in X_2^* for all x and $f(\cdot, \cdot)$ is a convex function only of its second variable :

Let $P \subset X_2$ be a positive cone ad define its dual cone P^* by

$$P^* = \{p^* \in X_2^* : \langle p, p^* \rangle \ge 0, \ \forall p \in P\}.$$

Then the order $\ge_*$ in X_2^* is defined by

$$a^* \ge_* b^* \Leftrightarrow a^* - b^* \in P^*.$$

The convexity of the mapping $f : [0, T] \times X_2 \to X_2^*$ can be defined as follows

$$f(t, y) - f(t, x) \ge_* (t, x) \cdot (y - x)$$

for every $p \in P$.

(ii) there exists a $\alpha \in L^2(0, T; R^+)$ such that

$$\| f(t; x) - f(t, y) \|_{X_1} \le \alpha(t) \| x - y \|_{X_2} \text{ a.e. } t \text{ for } x, y \in X_2,$$

(iii) there exists a $\delta \in L^2(0, T; R^+)$ such that $\| f(t, 0) \|_{X_2}^* \le \delta(t)$ a.e. t.

(iv) For $t \in [0, T], f(t, x)$ is Frechet differentiable with respect to x at $x = \phi \in X_2$ if there exists an operator $f_x(t, \phi) \in \mathcal{L}(X_2, X_2^*)$ such that

$$\frac{\| f(t, \phi+\lambda) - f(t, \phi) - f_x(t, \phi)\lambda \|^*_{X_2}}{\| \lambda \|_{X_2}} \to 0 \text{ as } \| \lambda \|_{X_2} \to$$

Here $f_x(t, \phi)$ is called the Fréchet derivative of $f(t, x)$ with respect to $x = \phi$, furthermore, $f_x(\cdot, x(\cdot)) \in L^2(0, T; \mathcal{L}(X_2, X_2^*))$.

(A3) $B : [0, T] \times Y \to X_1^*$ is a measurable mapping such that $B(t, \cdot)$ is continuous for each t.

(A4) $F : [0, T] \times H \to R$ is a function such that $F(\cdot, x)$ is a measurable function of t for each $x \in H$ and $F(t, \cdot)$ is a convex and continuously Gateaux differentiable function of x for each t.

(A5) $G : [0, T] \times Y \to R$ is a measurable function such that $G(t, \cdot)$ is continuous for each t.

A6) $U(\cdot)$ is a multifunction $[0, T] \to S(Y)$ such that the function

$t \to \sup \{\| u \|_Y \mid u \in u(t)\}$ belongs to $L^2(0, T)$ and such that its graph is a measurable subset of the product measurable space generated by the Borel σ-field of $[0, T]$ and the Borel σ-field of Y.

(A7) $C \subseteq H$ is a closed and convex set with nonempty interior.

We associate another optimal control problem to (P), which is called the dual problem of (P). In order to describe it, we need a function $K : [0, T] \times H \to R$ defined by

$$K(t, p(t)) = \begin{cases} \inf\limits_{u \in U(t)} (G(t, u) + \langle p, B(t, u) \rangle), & p \in P \subset X_1 \\ 0, & p \in H \setminus P. \end{cases}$$

We make the following assumption on the function K.

(A8) $K(\cdot, p(\cdot)) \in L^1(0, T)$ for all $p(\cdot) \in W(0, T)$.

We call the following optimal control problem of the dual problem :

$$\begin{cases} \int_0^T (K(t, p(t)) - \langle p(t), \dot{x} + A_2(t)\dot{x}(t) + A_1(t)x(t) - f(t, x(t)) \rangle \\ + F(t, x(t))\, dt \to \sup \\ \text{subject to} \\ \begin{cases} \dot{p}(t) = A_2^*(t)\dot{p}(t) - (A_1^*(t) - \dot{A}_2^*(t))p(t) + f_x^*(t, x(t))p(t) + F_x(t, x(t)) \text{ a.e.} \\ p(T) = 0, \dot{p}(T) = 0, \ x \in W(0, T), \\ p(0) = 0, \ (\dot{p}(0), x(0) - c) \geq 0, \\ (p(0), A_2(0)(c - x(0)) \geq 0, h \in H, \ c \in C \\ \text{where } c - x(0) \in X_2 \end{cases} \end{cases} \quad \ldots (2.2)$$

where $A_1^*(t), A_2^*(t)$ are the adjoint operator of $A_1(t), A_2(t), f_x^*(t, x)$ of $f_x(t, x, x \in X_2$ and $F_x(t, x)$ the

derivative of $F(t, x)$ with respect to x. It should be noted that the control variable of this problem is x. Here, $x(\cdot)$ satisfies the differential equation in (P) and $p(\cdot)$ means $p(x(\cdot))$. As for the differential equation in (2.2), we restrict ourselves to only solutions which belong to $W(0, T)$. Hence, if there exists no solution $(p, x) \in W(0, T) \times W(0, T)$ in (2.2), we define the supremum of problem (D) to be $-\infty$.

3. DUALITY

We first prove a weak duality theorem saying that the infimum of (P) is equal to or greater than the supremum of (D).

Theorem 3.1 — *Under the assumptions (A1)-(A8), if* $F(\cdot, x(\cdot)) \in L^1(0, T)$ *and* $F_x(\cdot, x(\cdot)) \in L^2(0, T; X_1^*)$ *for all* $x \in W(0, T)$ *and* $G(\cdot, u(\cdot)) \in L^1(0, T)$ *and* $B(\cdot, u(\cdot)) \in L^2(0, T; X_1^*)$ *for all* $u \in \mathcal{U}_{ad}$, *then the infimum of (P) is equal to or greater than the supremum of (D).*

PROOF : Let $u(\cdot) \in \mathcal{U}_{ad}$ be any admissible control and fix it for a moment. Due to Ha and Nakagiri [2; Ch. 4], there exists a solution $x \in W(0, T)$ of (2.1) with any initial condition $x(0) = x_0 \in C$ by the assumption $B(\cdot, u(\cdot)) \in L^2(0, T; X_1^*)$. Let $\bar{x}$ be any one of such solutions corresponding to u. That is,

$$\dot{\bar{x}}(t) = -A_2(t)\bar{x}(t) - A_1(t)\bar{x}(t) + B(t, u(t)) + f(t, \bar{x}(t)) \qquad \ldots (3.1)$$

$$\bar{x}(0) \in C, \bar{x}(0) \in (H).$$

To this u we associate the following problem (D):

$$\int_0^T \{F(t, x(t)) + G(t, u(t))$$

$$- \langle p(t), \dot{x}(t) + A_2(t)\dot{x}(t) + A_1(t)x(t) - f(t, x(t)) - B(t, u(t)) \rangle\} \, dt \to \sup$$

subject to (2.2).

If there is no solution $(p, x) \in W(0, T) \times W(0, T)$ satisfying (2.2), we define the supremum of (D) to be $-\infty$. If $(p^\circ(\cdot), x^\circ(\cdot)) \in W(0, T) \times W(0, T)$ is an arbitrary solution of (2.2), then we can show that the value

$$\int_0^T \{F(t, \bar{x}(t)) + G(t, u(t))\} \, dt - \int_0^T \{F(t, x^\circ(t)) + G(t, u(t))$$

$$- \langle p^\circ(t), \dot{x}^\circ(t) + A_2(t)\dot{x}^\circ(t) + A_1(t)x^\circ(t) - f(t, x^\circ(t)) - B(t, u(t)) \rangle\} \, dt \qquad \ldots (3.2)$$

is nonnegative. To do this, note that

$$B(t, u(t)) = \dot{\bar{x}}(t) + A_2(t)\bar{x}(t) + A_1(t)\bar{x}(t) - f(t, \bar{x}) \text{ a.e.,}$$

Since F, f are convex and differentiable function with respect to x. We see that the value (3.2) is equal to or greater than

$$\int_0^T \langle \bar{x}(t) - x^\circ(t), F_x(t, x^\circ(t)) \rangle dt + \int_0^T \langle p^\circ(t), f_x(t, x^\circ(t))(\bar{x}(t) - x^\circ(t)) \rangle dt$$

$$+ \int_0^T \langle p^\circ(t)\dot{x}^\circ(t) - \dot{\bar{x}}(t) \rangle dt + \int_0^T \langle p^\circ(t), A_2(t)\dot{x}^\circ(t) - A_2(t)\bar{x}(t) \rangle dt$$

$$+ \int_0^T \langle p^\circ(t), A_1(t)x^\circ(t) - A_1(t)x^\circ A_1(t)\bar{x}(t) \rangle dt.$$

Since $p(T) = 0, \dot{p}(T) = 0, (p(0), h - \dot{x}(0)) \geq 0, (\dot{p}(0), x(0) - c) \geq 0$

and $(p(0), A_2(0)(c - x(0))) \geq 0$, we get

$$\int_0^T \langle p^\circ(t), \dot{x}^\circ(t) - \dot{\bar{x}}(t) \rangle dt + \int_0^T \langle p^\circ(t), A_2(t)\dot{x}^\circ(t) - A_2(t)\bar{x}(t) \rangle dt$$

$$+ \int_0^T \langle p^\circ(t), A_1(t)(x^\circ(t) - \bar{x}(t)) \rangle dt$$

$$\geq \int_0^T \langle \dot{p}^\circ(t), x^\circ(t) - \bar{x}(t) \rangle dt + \int_0^T \langle A_1^*(t)p^\circ(t), x^\circ(t) - \bar{x}(t) \rangle dt$$

$$- \int_0^T \langle \dot{A}_2^*(t)p^\circ(t), x^\circ(t) - \bar{x}(t) \rangle dt - \int_0^T \langle A_2^*(t)\dot{p}^\circ(t), x^\circ(t) - \bar{x}(t) \rangle dt.$$

Consequently, we have

$$\int_0^T \{F(t, \bar{x}(t)) + G(t, u(t))\} dt - \int_0^T \{F(t, x^\circ(t)) + G(t, u(t))$$

$$- \langle p^\circ(t), \dot{x}^\circ(t) + A_2(t)\dot{x}^\circ(t) + A_1(t)x^\circ(t) - f(t, x^\circ) - B(t, u(t)) \rangle\} dt$$

$$\geq \int_0^T \langle \bar{x}(t) - x^\circ(t), F_x(t, x^\circ(t)) + f_x^*(t, x^\circ(t))p^\circ(t)$$

$$+ \dot{A}_2^*(t)\, p^\circ(t) + A_2^*(t)\, \dot{p}^\circ(t)$$

$$- A_1^*(t)\, p^\circ(t) - \ddot{p}^\circ(t) \rangle\, dt = 0,$$

because (p°, x°) is a solution of (2.2). Let us denote by $\boldsymbol{p}(\bar{x}\, u)$ the value

$$\int_0^T \{F'(t, \bar{x}(t) + G(t, u(t))\}\, dt,$$

and by $\boldsymbol{q}(x^\circ, u)$ the supremum of problem (D). Then the above argument show that

$$\boldsymbol{p}(\bar{x}, u) \geq \boldsymbol{q}(x^\circ, u) \qquad \ldots (3.3)$$

holds for every $u \in \mathcal{U}_{ad}$ and every $\bar{x}$ satisfying (3.1) together with u. If we denote by $J(p, x, u)$ the objective function of (D), that is, if we put

$$J(p, x, u) = \int_0^T \{F(t, x(t)) + G(t, u(t)) + \langle p(t), B(t, u(t)) \rangle$$

$$- \langle p(t), \ddot{x}(t) + A_2(t)\, \dot{x}(t) + A_1(t)\, x(t) - f(t, x(t)) \rangle\}\, dt$$

it follows from (3.3) and a well-known inequality of game theory[10] that

$$\inf_{\bar{x}, u} \boldsymbol{p}(\bar{x}, u) \geq \inf_u \boldsymbol{q}(x^\circ, u) = \inf_u \sup_{p, x} J(p, x, u)$$

$$\geq \sup_{p, x} \inf_u J(p, x, u). \qquad \ldots (3.4)$$

For a given $u(\cdot) \in \mathcal{U}_{ad}$, we have

$$\int_0^T \{G(t, u(t)) + \langle p(t), B(t, u(t)) \rangle\}\, dt \geq \int_0^T K(t, p(t))\, dt,$$

since $G(t, u(t)) + \langle p(t), B(t, u(t)) \rangle \geq K(t, p(t))$ a.e. by definition of the function K. Hence, it follows that

$$\inf_u J(p, x, u) \geq \int_0^T \{k(t, p(t)) - \langle p(t), \ddot{x}(t) + A_2(t)\, \dot{x}(t) + A_1(t)\, (x(t) - f(t, x(t)) \rangle$$

$$+ F(t, x(t))\}\, dt$$

for every solution (p, x) of (2.2). If we denote by inf (P) the infimum of (P) and by sup (D) the supremum of (D), it is obvious that $\inf_{\bar{x}, u} \boldsymbol{p}(\bar{x}, u) = \inf(P)$ and $\sup_{p,x} \inf_u j(p, x, u) \geq \sup(D)$ by the above inequality. Therefore, we conclude from (3.4) that inf $(P) \geq$ sup (D). This completes the proof.

We prove next the duality theorem that under certain conditions the infimum of (P) coincides with the supremum of (D). We make the following assumptions:

(A3') $B(t): Y \to X_1^*$ is a continuous linear operator for each $t \in [0, T]$ such that $B(\cdot)u$ is a continuous mapping of t and $\| B(t)u \|_* \le \beta(t) + b \| u \|_Y$ a.e. with $\beta \in L^2(0, T)$, b is constant.

(A5') $G: [0, T] \times Y \to R$ is a measurable function such that $G(t, \cdot)$ is convex and continuously Gateaux differentiable with respect to u for each $t \in [0, T]$.

(A9) The interior of $\mathcal{U}_{ad}$ is nonempty.

Theorem 3.2 — *Assume (A1), (A2), (A3'), (A4), (A5')-(A9) and thatF* $(\cdot, x(\cdot)) \in L^1(0, T)$ *and* $F_x(\cdot, x(\cdot)) \in L^2(0, T; H)$ *for all* $x \in W(0, T)$ *and* $G(\cdot, u(\cdot)) \in L^1(0, T)$ *for all* $u \in \mathcal{U}_{ad}$. *If a triple* $(x^\circ(0), x^\circ, u^\circ) \in \times W(0, T) \times \mathcal{U}_{ad}$ *attains the infimum of (P), then there exists* $p^\circ \in W(0, T)$ *such that* (p°, x°) *attains the supremum of (D). Furthermore, the infimum of (P) is equal to supremum of (D).*

PROOF : If $(x^\circ(0), x^\circ, u^\circ) \in C \times W(0, T) \times \mathcal{U}_{ad}$ attains the infimum of (P). By Ha and Nakagiri[2], there exists $p^\circ \in W(0, T)$, $p^\circ(+) \in P$ satsfying

$$\dot{p}^\circ(t) = A_2^*(t) p^\circ(t) - (A_1^*(t) - \dot{A}_2^*(t)) p^\circ(t) + f_x^*(t, x^\circ(t)) p^\circ(t) + F_x(t, x^\circ(t)) \quad \text{... (3.5)}$$

a.e. $p(T) = 0, \dot{p}(T) = 0,$

$$(G_u(t, u^\circ(t)) + B^*(t) p^\circ(t), v - u^\circ(t))_{Y, Y^*} \ge 0. \quad \text{... (3.6)}$$

for all $v \in U(t)$ a.e., and

$$(p(0), h - \dot{x}(0)) \ge 0, (\dot{p}(0), x(0) - c) \ge 0,$$

$$(p(0), A_2(0)(c - x(0))) \ge 0 \quad \forall\, h \in H, c \in C, \quad \text{... (3.7)}$$

where $c - x(0) \in X_2$,

where $B^*(t)$ denotes the adjoint operator of $B(t)$, $(\cdot, \cdot)_{Y, Y^*}$ the bilinear form on the dual pair (Y, Y^*) and G_u the derivative of $G(t, u)$ with respect to u. By convexity of G, (3.6) implies $G(t, v) + \langle p^\circ(t), B(t) v \rangle \ge G(t, u^\circ(t)) = \langle p^\circ(t), B(t) u^\circ(t) \rangle$ for all $v \in u(t)$ a.e., from which we have

$$K(t, p^\circ(t)) = G(t, u^\circ(t)) + \langle p^\circ(t), B(t) u^\circ(t) \rangle \text{ a.e.} \quad \text{... (3.8)}$$

From (3.5) and (3.7) we see that (p°, x°) is a solution of (2.2). Thus from (3.8) that we obtain

$$K(t, p^\circ(t)) \langle p^\circ \dot{x}^\circ(t) + A_2(t) \dot{x}^\circ(t) + A_1(t) x^\circ(t) - f(t, x^\circ(t)) \rangle + F(t, x^\circ(t))$$

$$= G(t, u^\circ(t)) = \langle p^\circ(t), B(t) u^\circ(t) - \dot{x}^\circ(t) - A_2(t) \dot{x}^\circ(t) - A_1(t) x^\circ(t) = f(t, x^\circ(t)) \rangle$$

$$+ F(t, x^{\circ}(t))$$

$$= G(t, u^{\circ}(t)) + F(t, x^{\circ}(t)) \text{ a.e.,}$$

since (x°, p°) satisfies $\dot{x}^{\circ}(t) = -A_2(t)\dot{x}(t) - A_1(t)x^{\circ}(t) + B(t)u^{\circ}(t) + f(t, x^{\circ}(t))$ a.e..

Therefore, we conclude that

$$\int_0^T \{F(t, x^{\circ}(t)) + G(t, u^{\circ}(t))\}\, dt$$

$$= \int_0^T \{K(t, p^{\circ}(t)) - \langle p^{\circ}(t), \dot{x}^{\circ}(t) + A_2(t)\dot{x}^{\circ}(t) + A_1(t)x^{\circ}(t)$$

$$+ A_1(t)x^{\circ}(t) - f(t, x^{\circ}(t))\rangle$$

$$+ F(t, x^{\circ}(t))\}\, dt.$$

Thus Theorem 3.1 implies that the infimum of (P) is equal to the supremum of (D) and that (p°, x°) attains the supremum of (D). This completes the proof.

4. APPLICATION

In this section we apply Theorem 3.2 (duality theorem) to a hyperbolic distributed system. We restrict ourselves to give one application to the Sine-Gordon equation. Let $T > 0$ and Ω be a bounded domain in R^n with a smooth boundary $\partial\Omega = \Gamma$. Let x be a bounded solution of hyperbolic distributed system such that $0 \le x(t, z) \le \pi$, a.e. $z \in \Omega$. On $[0, T], \times \Omega$ we consider the control problem (P') :

$$\int_0^T \int_{\Omega} \{h(t, z, x(t, z)) + g(t, z, u(t, z))\}\, dzdt \to \inf$$

subject to

$$\frac{\partial^2 x(t, z)}{\partial t^2} = -\alpha \frac{\partial x(t, z)}{\partial t} + \Delta x(t, z) - \beta \sin x(t, z) + u(t, z) \text{ on } [0, T] \times \Omega$$

$$x(t, z) = 0 \text{ on } [0, T] \times \Gamma, x(0, \cdot) \in C \subseteq L^2(\Omega), \dot{x}(0, \cdot) \in H,$$

where $\alpha > 0$

$$\left(\int_{\Omega} |u(t, z)|^2\, dz\right)^{\frac{1}{2}} \le M.$$

Taking $X_1 = H_0^1(\Omega), X_2 = H = L^2(\Omega), X_1^* = H^{-1}(\Omega)$, and $P = L^2(\Omega)^+$, the triple (X_1, H, X_1^*) enjoys the properties mentioned in Section 2. Let us introduce two bilinear form

$$a_1(\phi, \psi) = \int_\Omega \nabla \phi \cdot \nabla \psi dz, \quad \forall\ \phi, \psi \in H_0^1(\Omega)$$

$$a_2(\phi, \psi) = \int_\Omega \alpha \phi \psi dz, \quad \forall\ \phi, \psi \in L^2(\Omega)$$

on $X_1 \times X_1$. We take $\mathcal{U} = L^2(0, T; L^2(\Omega))$, $B = I$ and $f(x) = -\beta \sin x$. Then, we can easily verified that $f(x)$ satisfies the assumptions (A_2) ((i),(ii), (iii), (iv)). (ii) follows from that

$$\int_\Omega (\sin \xi(x) - \sin \zeta(x))^2 dx \le \int_\Omega (\xi(x) - \zeta(x))^2 dx$$

The control space is $Y = L^2(\Omega)$ and $U(t)$ is defined by $U(t) = \{u \in L^2(\Omega); \| u \|_2 \le M\}$ for $t \in [0, T]$, where $\| u \|_2 = \left(\int_\Omega | u(z) |^2 dz \right)^{\frac{1}{2}}$ and $M \ge 0$. Then (A8) is fulfilled. Let $C \subseteq L^2(\Omega)$ be a closed, convex subset with nonempty interior. Moreover, we assume that the functions h and g in (P') satisfy

(i) $h(\cdot, \cdot, \cdot) : [0, T] \times \Omega \times R \to R$ is a measurable function such that $h(t, z, \cdot)$ is convex and continuously differentiable with respect to x. $\int_0^T \int_\Omega h(t, z, x(t, z)) dz\, dt$ is finite for every $x \in L^\infty(0, T; L^2(\Omega))$.

(ii) $g(\cdot, \cdot, \cdot) : [0, T] \times \Omega \times R \to R$ is a measurable function such that $g(t, z, \cdot)$ is convex and continuously differentiable with respect to u. $\int_0^T \int_\Omega g(t, z, u(t, z)) dz\, dt$ is finite for every $u \in L^2([0, T] \times \Omega)$.

If $F : [0, T] \times L^2(\Omega) \to R$ is given by $F(t, x) = \int_\Omega h(t, z, x(t, z)) dz$ and $G : [0, T] \times L^2(\Omega) \to R$ by $G(t, u) = \int_\Omega g(t, z, u(t, z)) dz$, our optimal control problem (P') reduces to the abstract form

$$\int_0^T \{F(t, x(t)) + G(t, u(t))\}\ dt \to \inf$$

subject to

$$\ddot{x} = -A_2(t)\dot{x}(t) - A_1(t)x(t) + u(t) + f(t, x(t)) \text{ a.e.}$$

$$x(0) \in C, \dot{x}(0) \in H, u(t) \in U(t) \text{ a.e.}$$

The dual problem of (P') can be formulated via the dual problem of this abstract one. In order to obtain it, we need to evaluate $K(t, p(t))$, which is given by

$$k(t, p(t)) = \begin{cases} \inf\limits_{u \in U(t)} (G(t, u) + \langle p, u \rangle), & p \in P \cap X_1 \\ 0, & p \in H \; P \cap X_1. \end{cases}$$

In this case it is calculated as

$$K(t, p(t)) = \begin{cases} \inf\limits_{\| u \|_2 \le M} \int\limits_{\Omega} (g(t, z, u(z)) + p(z)\, u(z))\, dz, & p \in H_0^1(\Omega) \\ 0, & \text{otherwise} \end{cases}$$

Hence the dual problem written in the form becomes (D') :

$$\int_0^T K(t, p(t))\, dt - \int_0^T \int_{\Omega} p(t)\, (\ddot{x}(t) + A_2(t)\dot{x}(t) + A_1(t)x(t) - f(t, x(t, z)))\, dz dt$$

$$- \int_0^T \int_{\Omega} h(t, z, x(t, z))\, dt dz \to \sup$$

subject to

$$\ddot{p}(t) = A_2^*(t)\dot{p}(t) - (A_1^*(t) - \dot{A}_2^*(t))\, p(t) + f_x^*(t, x(t, z)\, p(t) + h_x(t, z, x(t, z))$$

on $[0, T] \times \Omega$,

$p(t, z) = 0$ on $[0, T] \times \Gamma$, $p(T, z) = 0, \dot{p}(T, z) = 0$, for $z \in \Omega$,

$$x \in W(0, T) = \{x \mid x \in L^2(0, T; X_1), \dot{x} \in L^2(0, T; H), \ddot{x} \in L^2(0, T; X_1^*)\},$$

$$\int_{\Omega} (h(z) - \dot{x}(0, z))\, p(0, z)\, dz \ge 0 \text{ for all } h(\cdot) \in H,$$

$$\int_{\Omega} (x(0, z) - c)\, \dot{p}(0, z)\, dz \ge 0 \text{ for all } c(\cdot) \in C,$$

$$\int_{\Omega} (A_2(0)(c - x(0)))\, p(0, z)\, dz \ge 0 \text{ for all } c(\cdot) \in C, \text{ where } c - x(0) \in X_2,$$

where $p(t)$ in $K(t, p(t))$ means $p(t) = p(t, \cdot) \in W(0, T)$, $A_1^*(t), A_2^*(t)$ are the formal adjoint of $A_1(t), A_2(t)$. Thus the following duality theorem is immediate from Theorem 3.2.

Theorem 4.1 — *Assume that* $(t,z) \to h_x(t,z,x(t,z)) \in L^2([0,T]\times\Omega)$ *for every* $x \in W(0,T)$. *Under the above assumptions, if* $(x^\circ(0), x^\circ, u^\circ) \in C \times W(0,T) \times \mathcal{U}_{ad}$ *attains the infimum of (P'), then there exists* $p^\circ \in W(0,T)$ *such that* (p°, x°) *attains the supremum of (D'). Furthermore, the infimum of (P') is equal to the supremum of (D').*

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