Indian J. pure appl. Math., 33(8) : 1301-1306, August 2002
© Printed in India.

ON GENERALISED KÖTHE-TOEPLITZ DUALS OF SOME SEQUENCE SPACES

PRABHAT CHANDRA

Department of Mathematics, Patna University, Patna 800 005, *India*

AND

BINOD CHANDRA TRIPATHY*

Mathematical Sciences Division, Institute of Advanced Study in Science and Technology, Khanapara, Guwahati 781 022, *India*
e-mail: tripathybc@yahoo.com

(*Received* 14 *March* 2001; *after revision* 20 *July* 2001; *accepted* 5 *November* 2001)

In this article we generalize the notion of the Köthe-Toeplitz dual of sequence spaces on introducing the concept of η-dual of sequence spaces. Further we establish some results involving the perfectness of different sequence spaces relative to η-dual.

Key Words : Dual Space; Perfect Space; l_r-Space; Converent Sequence; Bounded Variation.

1. INTRODUCTION

The idea of dual sequence space was introduced by Köthe and Toeplitz[6], whose main results concerned α-duals. An account of the duals of sequence spaces is found in Köthe[5]. One can find about different types of duals of sequence spaces in Maddox[11]. Cook[3], Kamthan and Gupta[4] and many others. Throughout the article $\omega, c, c_0, l_1, l_p, l_\infty, v, v_\infty, \varphi, bv$ denote the spaces of all, convergent, null, absolutely summable, p-absolutely summable, bounded, convergent series, series with bounded partial sums, finite and bounded variation sequence respectively.

The α-dual of a subset E of ω is defined as

$$E^\alpha = \{(y_k) \in \omega : (x_k y_k) \in l_1 \text{ for all } (x_k) \in E\}.$$

If we replace l_1 by v and v_∞ in the above definition, then we shall get the β and γ-duals of E respectively.

Lascarides[7], Basarir[2], Maddox[9, 10] and others have studied results involving α-dual and β-duals of different sequence spaces and their properties. Throughout the article sums without limit means the summation is from $k = 1$ to ∞.

*The work of the second author was carried under the Department of Science and Technology, Govt. of India Project No. DST/MS/IRHPA : 001/93

2. Definitions and Preliminaries

By σ we denote the space of all eventually alternating sequences i.e. if $(x_k) \in \sigma$, then there exists $k_0 \in N$ such that $x_k = -x_{k+1}$ for all $k > k_0$. It is well known that $bv_0 = bv \cap c_0$.

Definition — Let E be a non-empty subset of w and $r \geq 1$, then the η-dual of E is defined as

$$E^{\eta} = \{(y_k) \in \omega : (x_k y_k) \in l_r \text{ for all } (x_k) \in E\}.$$

A non-empty subset E of ω is said to be perfect or η-reflexive if $E^{\eta\eta} = E$. Taking $r = 1$ in the above definition we get the α-dual of E.

The proof of the following result is a routine work in view of the definition of η-dual.

Lemma 1 — (i) E^{η} is a linear subspace of ω for every $E \subset \omega$.

(ii) $E \subset F$ implies $E^{\eta} \supset F^{\eta}$ for every E, $F \subset \omega$.

(iii) $E^{\eta\eta} = (E^{\eta})^{\eta} \supset E$ for every $E \subset \omega$.

(iv) $\left(\bigcup_j E_j \right)^{\eta} = \bigcap_j E_j^{\eta}$ for every family $\{E_j\}$ with $E_j \subset \omega$ for all $j \in N$.

From the existing literature we have,

Lemma 2 — $c_0^{\alpha} = c^{\alpha} = bv^{\alpha} = bv_0^{\alpha} = l_1$.

3. Main Results

In this section we find the η-dual of some sequence spaces and establish whether they are perfect or not relative to η-dual.

Theorem 1 — *$l_r^{\eta} = l_{\infty}$, $l_{\infty}^{\eta} = l_r$ and the spaces l_r and l_{∞} are perfect spaces.*

PROOF : Let $(x_k) \in l_{\infty}$, then $\sum_k |x_k y_k|^r$ converges for every $(y_k) \in l_r$. Hence we have $l_{\infty} \subseteq l_r^{\eta}$.

Next let $(x_k) \notin l_{\infty}$, then (x_k) will have a subsequence (x_{k_i}) such that $x_{k_i} \geq i^s$, for some fixed s > 1, for all $i \in N$. Define (y_n) as $y_n = i^{-s}$ if $n = k_i$ and $y_n = 0$, otherwise.

Then $(y_n) \in l_r$ and $(x_n y_n) \notin l_r$. Hence $l_r^{\eta} = l_{\infty}$.

The proof of $l_{\infty}^{\eta} = l_r$ is a routine work. Thus the spaces are perfect.

Theorem 2 — *$\sigma^{\eta} = l_r$ and σ is not perfect.*

PROOF : We have $\sigma \subset l_{\infty}$. Hence by Theorem 1 and Lemma 1 we have

$$l_r \subseteq \sigma^{\eta}.$$

Next let $(y_n) \in \sigma^{\eta}$. Then $\sum_k |x_k y_k|^r < \infty$ for all $(x_k) \in \sigma$. Let the sequence (x_k) be defined by $x_{2k-1} = 1 = -x_{2k}$ for all $k \in N$. Then $\sum_k |x_k y_k|^r = \sum_k |y_k|^r < \infty$ implies $(y_n) \in l_r$. Hence $\sigma^{\eta} \subseteq l_r$. Thus $\sigma^{\eta} = l_r$. The rest follows from Theorem 1.

Theorem 3 — *Let $p > r \geq 1$, then $l_p^{\eta} = l_q$, where $p^{-1} + q^{-1} = r^{-1}$. The space l_p is perfect.*

PROOF : Let $(x_k) \in l_p$ and $(y_k) \in l_q$. Then by Hölder's inequality we have

$$\sum_k |x_k y_k|^r \leq \left(\sum_k |x_k|^p \right)^{r/p} \left(\sum_k |y_k|^q \right)^{r/q}$$

Hence we have $l_q \subset l_p^{\eta}$.

Suppose $(y_k) \notin l_q$, then we can find a sequence of naturals (n_i) with $0 = n_0 < n_1 < \ldots < n_i < n_{i+1} < \ldots$ such that

$$\sum_{k=n_i+1}^{n_{i+1}} |y_k|^q = M_{i+1} > 1$$

for $i = 0, 1, 2, 3, 4, \ldots$.

Define the sequence (x_k) as

$$x_k = \left\{ (t+1)^{1/r} M_{i+1}^{1/r} \right\}^{-1} y_k^{(q-r)/r}$$

for $n_i < k \leq n_{i+1}$, i = 0, 1, 2, 3, ...

We have

$$\sum_{k=n_i+1}^{n_{i+1}} |x_k|^p = \frac{1}{(i+1)^{p/r} M_{i+1}^{p/r}} \sum_{k=n_i+1}^{n_{i+1}} |y_k|^q \quad \text{since } q = \frac{p(q-r)}{r}$$

$$= \frac{1}{(i+1)^{p/r} (M_{i+1})^{p/r}} - 1 \, M_{i+1}.$$

$$= \frac{1}{(i+1)^{\frac{p}{2}} M_{i+1}^{\frac{p}{2}-1}}$$

$$< \frac{1}{(i+1)^{p/r}},$$

for $i = 0, 1, 2, \ldots$.

Hence $$\sum_{k=1}^{\infty} |x_k|^p < \sum_{i=0}^{\infty} \frac{1}{(i+1)^{p/r}}$$

is convergent, since $\frac{p}{r} > 1$. Thus $(x_k) \in l_p$. But

$$\sum_{k=1}^{\infty} |x_k y_k|^r = \sum_{i=0}^{\infty} \left\{ \sum_{k=n_i+1}^{n_{i+1}} |x_k y_k|^r \right\}$$

$$= \sum_{i=0}^{\infty} \left\{ \sum_{k=n_i+1}^{n_{i+1}} |y_k|^p \frac{1}{(i+1) M_{i+1}} \right\}$$

$$= \sum_{i=0}^{\infty} \frac{1}{i+1} = \infty.$$

$$\Rightarrow (y_k) \notin l_p^{\eta}.$$

Hence $l_p^{\eta} \subseteq l_q$.

Thus $l_p^{\eta} = l_q$.

Similarly we have $l_q^{\eta} = l_p$ and as such l_p is perfect.

Taking p = 2r in the above theorem we have the following result.

Corollary 1 — Let $r \geq 1$, then $l_{2r}^{\eta} = l_{2r}$. The space l_{2r} is perfect and it is the only space whose η-dual is itself.

Theorem 4 — $c_0^{\eta} = c^{\eta} = l_r$ *and the sequence spaces* c_0 *and* c *are not perfect.*

PROOF : We have $c_0 \subset l_{\infty}$. Hence $c_0^{\eta} \supseteq l_r$ by Theorem 1 and Lemma 1.

Next let $(x_k) \in c_0^{\eta}$. Then $\sum_k |x_k y_k|^r$ converges for every $(y_k) \in c_0$. This implies that $\sum_k |x_k^r z_k|$ converges for every $(z_k) \in c_0$, where $z_k = y_k^r$ for all $k \in N$. Hence

$$(x_k^r) \in c_0^{\alpha} = l_1, \text{ which implies } (x_k) \in l_r. \text{ Thus } c_0^{\eta} \subseteq l_r.$$

Hence $c_0^{\eta} = l_r$.

The rest follows from the inclusion $c_0 \subset c \subset l_{\infty}$ and Theorem 1.

Theorem 5 — $(bv)^{\eta} = l_r = (bv_0)^{\eta}$ *and the spaces* bv *and* bv_0 *are not perfect.*

PROOF : We have $bv \subset l_\infty$. Thus $(bv)^{\eta} \supseteq l_r$. Next let $(x_k) \in (bv)^{\eta}$, then $\sum_k |x_k y_k|^r$ converges for every $(y_k) \in bv$. Consider the sequence (y_k) defined by $y_k = 1$ for all $k \in N$. Then $\sum_k |x_k|^r$ converges. This implies $(x_k) \in l_r$. Hence $(bv)^{\eta} \subseteq l_r$. Thus $(bv)^{\eta} = l_r$.

Now we show that $(bv_0)^{\eta} = l_r$. It is clear that $(bv_0)^{\eta} \supseteq l_r$.

Suppose that $(y_n) \in (bv_0)^{\eta}$ but $(y_n) \notin l_r$. Then we have $\sum_{n=1}^{\infty} |x_n y_n|^r = \infty$ for every $(x_n) \in (bv_0)$. Since $(y_n) \notin l_r$, so we have $\sum_{n=1}^{\infty} |y_n|^r$. Hence we can find a sequence (n_k) of natural numbers (with $n_1 = 1$) such that $\sum_{i=n_k}^{n_{k+1}-1} |y_i|^r > k^r$ for all k = 1, 2, 3, ...

Now define the sequence (x_n) as $x_i = k^{-1}$ if $n_k \le i \le n_{k+1} - 1$ for all k = 1, 2, 3, ...

Then
$$\sum_{n=1}^{\infty} |\Delta x_n| = \sum_{k=1}^{\infty} \sum_{i=n_k}^{n_{k+1}-1} |\Delta x_i|$$

$$= \sum_{k=1}^{\infty} |x_{n_{k+1}} - x_{n_k}|$$

$$= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} < \infty.$$

Hence $(x_n) \in bv_0$. Further we have

$$\sum_{n=1}^{\infty} |x_n y_n|^r = \sum_{k=1}^{\infty} \sum_{i=n_k}^{n_{k+1}-1} |x_i y_i|^r$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^r} \sum_{i=n_k}^{n_{k+1}-1} |y_i|^r$$

$$> \sum_{k=1}^{\infty} \frac{1}{k^r} k^r = \infty.$$

We arrive at a concentration. Thus $(bv_0)^\eta \subseteq l_r$. Hence $(bv_0)^\eta = l_r$. The rest is clear from Theorem 1. This completes the proof of the theorem. The proof of the following result is a routine work.

Corollary 2 — $\varphi^\eta = w, w^\eta = \varphi$ and the spaces w and φ are perfect.

Remark : Taking $r = 1$ in the above results one will get the corresponding α-duals of the respective sequence spaces.

REFERENCES

1. Z. U. Ahmed and Mursaleen, *J. Indian Math. Soc.*, **53** (1998) 177-80.
2. M. Basarir, *Indian J. pure appl. Math.* **26**(10) (1995) 1003-10.
3. R. G. Cooke, *Infinite Matrices and Sequence Spaces*; Dover Publ. 1955.
4. P. K. Kamthan and M. Gupta, *Sequence Spaces, Lecture Notes in Pure and Applied Math.* **65**, Marcel Dekker 1981.
5. G. Köthe, *Topological Vector Spaces* I (English); Springer-Verlag, 1969.
6. G. Köthe and O. Toeplitz, *J. F. reine u. angew Math.* **171** (1934) 193-26.
7. C. G. Lascarides, *Pac. J. Math.* **38(2)** (1971) 487-500.
8. C. G. Lascarides and I. J. Maddox, *Proc. Camb phil. Soc.* **68** (1970) 99-104.
9. I. J. Maddox, *Proc. Camb. Phil. Soc.* **65** (1969) 431-35.
10. I. J. Maddox, *J. London math. Soc.* **2** 1 (1969) 316-22.
11. I. J. Maddox, *Infinite Matrices of Operators; Lecture Notes in Mathematics*, No. 786; Springer Verlag, 1980.
12. J. I. Okutoyi, *Acta Math. Hungarica*, **59** (No. 3-4) (1992) 291-99.