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RELATIVELY INVERTIBLE ENDOMORPHISMS OF A SEMILATTICE OF GROUPS

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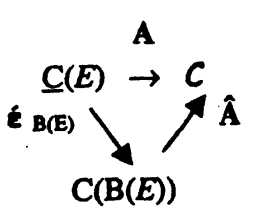
For a semilattice of Groups G, in this paper we prove the inverse semigroup of all relatively invertible endomorphisms of G^3 is isomorphic to the inverse semigroup of partial isomorphisms of G^6.

Key Words : Semilattice of Groups; Relatively Invertible Endomorphisms; Inverse Semi Group of Partial Isomorphisms

1. INTRODUCTION

We assume familiarity with the notation and results of[6]. For any regular biordered set E, $G(E)$ denote the ordered groupoid of E-chains of E and $B(E)$ the universal regular idempotent generated semigroup of E[8]. The category $\underline{C}(E)$ is defined as follows. The objects of $\underline{C}(E)$ are the elements of E and a morphism from e to f is a pair (e, c), where $c = c(e_0, e_1, ..., e_n) \in G(E)$ is an E-chain such that $e \geq e_0$ and $e_n = f$. Let $C(B(E))$ be the category whose objects are idempotents of $B(E)$ and a morphism from e to f is a triple $(e, x, x') : e \to f$, where $x \in B(E), x' \in V(B(E))$ are such that $x'x = f$ and $e \geq xx'$. If $(e, x, x') : e \to f, (f, y, y') f \to g$ are two morphisms, then $(e, x, x')(f, y, y') = (e, xy, y', x')$ If we view the inductive groupoid $G(B(E))$ of $B(E)$ as a subcategory of $C(B(E))$ via the embedding $(x, x') \to (xx', x, x')$ then the evaluation map $\varepsilon_{B(E)} : c(e_0, e_1, ..., e_n) \to (e_0 e_1 \cdots e_n, e_n\ e_{n-1} \cdots e_0) : G(E) \to G(B(E))$ extends to a functor $\varepsilon'_{B(E)} : \underline{C}(E) \to C(B(E))$ such that $\varepsilon'_{B(E)}(e, c) = (e, \varepsilon_{B(E)}(c))$ for every morphism (e, c) of $\underline{C}$(E). An E-diagram in an arbitrary category C we mean a functor $A : \underline{C}(E) \to C$ which factors through $C(B(E))$. That is, there exists a (necessarily) unique functor $\hat{A} : C(B(E)) \to C$ such that the diagram



commutes.

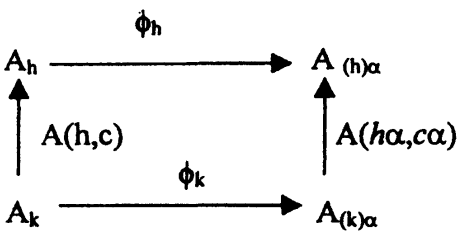
Let A be an E-diagram in C. For each $e \in E$, let A^e denote the composite

$$A^e : \underline{C}\ (\omega(e)) \xrightarrow{i_e} C(E) \xrightarrow{A} C.$$

Define $G(A)$ to be the category whose objects are the elements of E. A morphism $e \to f$ is a pair (α, ϕ) consisting of an ω-isomorphism $\alpha : \omega(e) \to \omega(f)$ and a natural isomorphism $\phi : A^e \to A^f \underline{C}\ (\alpha)$, where $\underline{C}\ (\alpha) : \underline{C}\ (\omega(e)) \to \underline{C}\ (\omega(f))$ is the functor defined by the ω-isomorphism α, and $A^f \underline{C}\ (\alpha)$ is the composite

$$\underline{C}\ (\omega(e)) \xrightarrow{C_{(\alpha)}} \underline{C}\ (\omega(f)) \xrightarrow{A^f} C.$$

Note that the natural isomorphism ϕ assigns to each object h in $\underline{C}\ (\omega(e))$, an isomorphism $\phi_h : A_h \to A_{(h)\alpha}$ such that, for any morphism $(h, c) : h \to k$ in $\underline{C}\ (\omega(e))$, the following diagram



commutes.

The composition of two morphisms $(\alpha, \phi) : e \to f, (\beta, \psi) : f \to g$ is given by

$(\alpha, \phi)(\beta, \psi) = (\alpha\beta, \phi \underline{C}\ (\alpha)\psi)$, where $\alpha\beta$ is the composite $\omega(e) \xrightarrow{\alpha} \omega(f) \xrightarrow{\beta} \omega(g)$ and the natural isomorphism $\phi\ (\underline{C}(\alpha)\psi) : A^e \to A^g \underline{C}\ (\alpha\beta)$ is defined by $(\phi\ \underline{C}\ (\alpha)\psi)_h = \phi_h \circ \psi_{(h)\alpha} : A_h \to A_{(h)\alpha\beta}$ for all $h \in \omega(e)$. Then by[6] and [8], $G(A)$ is an inductive groupoid and the associated regular semigroup is called the regular semigroup of partial isomorphisms of A, and denoted by $Reg_E(A)$.

Let S be a semilattice of groups with $E(S) = E$. Then by Clifford's theorem S induces a functor $S : C(E) \to Gr$ where Gr denotes the category of groups. For each $e \in E(S)$, S_e is equal to H-class of E and for each $e \geq f, S(e, f) : S_e \to S_f$ is the homomorphism given by $x\,S(e, f) = xf$. Conversely, if $S : \underline{C}\ (E) \to Gr$ is a functor, then $S = \bigcup_{e \in E(S)} S_e$ becomes a semilattice of groups with respect to the multiplication

$$xy = x\,S(e, ef)\,y\,S(f, ef), \quad x \in S_e, y \in S_f.$$

Thus a semilattice of groups with idempotents E may be viewed as a functor from $\underline{C}(E)$ to the category of groups or as an inverse semigroup with central idempotent

We recall the following definition from[3]

Definition 1.1 — Let G be a semilattice of groups and **End** G denote the endomorphisms of G. Then $\alpha \in$ **End** G is said to be *relatively invertible*, if there exists $\alpha \in$ **End** G, $e_\alpha \in E(G)$ such that

(*i*) $g\,\alpha\,\alpha'' = g\,e_{\alpha'}$ for all $g \in G$

(*ii*) $g\,\alpha''\,\alpha = g\,(e_\alpha\,\alpha)$ for all $g \in G$

(*iii*) $e_\alpha\,\alpha$ is a right identity on $G\,\alpha$, e_α is a right identity on $G\,\alpha$

By[3] the set of all relatively invertible endomorphisms of G is an inverse semigroup and is denoted by **end***G*.

2. MAIN THEOREM

Theorem 2.1 — *Let G be a semilattice of groups and let* $\hat{G} : \underline{C}(E) \to Gr$ *be the associated functor. Then* $Reg_E(\hat{G})$ *is isomorphic to end G.*

PROOF : Let $\mu \in$ **end***G*. Then there exists $\mu' \in$ **End***G* and $e \in E(G)$ such that

$$g\,\mu\,\mu' = ge \text{ for all } g \in G$$

$$g\,\mu'\,\mu = g\,(e\,\mu), \text{ for all } g \in G \qquad \ldots (2.1)$$

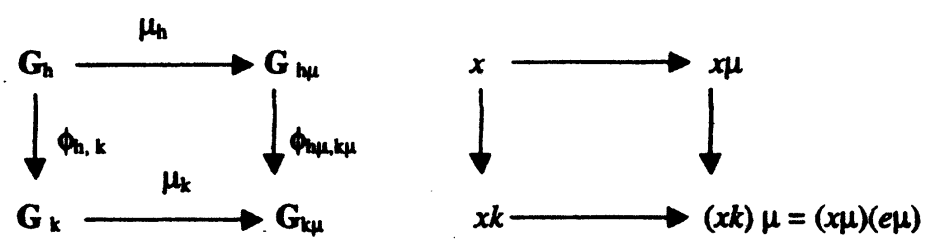
Since μ is a homomorphism, $\alpha_\mu = \mu/\omega(e) : \omega(e) \to \omega(e\,\mu)$ is a homomorphism and (2.1) implies α_μ is an isomorphism with inverse $\alpha_{\mu'}$. For $g \in \omega(e)$,

$$g\,\alpha_\mu\,\alpha_{\mu'} = (g\mu)\,\mu' = ge = g$$

and for $h \in \omega(e\,\mu)$,

$$h\,\alpha_{\mu'}\,\alpha_\mu = h\,\mu'\,\mu = h\,(e\,\mu) = h$$

Hence α_μ is an isomorphism. For $\mu \in$ **End***G*, $\mu/E(G) : E(G) \to E(G)$ and for $x \in G$, if eHx, then $e\,\mu\,H\,x\,\mu$. That is $(G_e)\,\mu \subseteq G_{e\,\mu}$. Now to show $\mu^{\#} = \{\mu_e\}_{e \in E(G)} : G^e \to G^{e\,\mu} \circ \underline{C}(\alpha_\mu)$ is a natural equivalence. For $h \le e$, $\{\mu_h\}_{h \le e} : G_h \to G_{h\,\mu}$ and for $k \le h \le e$ the diagram



commutes. Suppose $(\mu'_{h\,\mu})_{h\,\mu \le e} : G_{h\,\mu} \to G_h$. Then for $x \in G_h$,

$$x\,\mu_h\,\mu'_{h\,\mu} = (x\,\mu)\,\mu'_{h\,\mu} = x\,\mu\,\mu' = xe = x \text{ since } h \le e$$

and for $y \in G_{h\,\mu}$,

$$y\mu'_{h\,\mu}\,\mu_h = (y\,\mu')\,\mu = y\,(e\,\mu) = y$$

Hence $\mu^{\#}$ is a natural equivalence. So we define a map η from end G to $Reg_E(\hat{G})$ by $(\mu)\eta = (\alpha_\mu, \mu^{\#})$. To show η is a homomorphism. Let $\mu, \nu \in$ **end** G be such that $(\mu)\,\eta = (\alpha_\mu, \mu^{\#})$ and $(\nu)\,\eta = (\beta_\nu, \nu^{\#})$. Then $\alpha_\mu = \mu/\omega(e_\mu) : \omega(e_\mu) \to \omega(e\,\mu)$ and $\beta_\nu = \nu/\omega(e_\nu) : \omega(e_\nu) \to \omega(e\,\nu)$ are isomorphisms. We have the composite as

$$\omega((e\,\mu\,e_\nu)\,\alpha_\mu^{-1}) \xrightarrow{\alpha_\mu} \omega(e\,\mu\,e_\nu) \xrightarrow{\beta_\nu} \omega((e\,\mu\,e_\nu)\,\beta_\nu)$$

Since μ, ν are homomorphisms, the composite $\beta_\nu\,\alpha_\mu$ is a homomorphism. As $\nu\,\mu \in$ **end G** there exists $\delta \in$ **End G** and $e_\mu \in E(G)$ such that

(h) $\nu\,\mu\,\delta = h\,e_\mu$, for all $h \in G$

(h) $\delta\,\nu\,\mu = h\,(e_\mu\,\nu\,\mu)$, for all $h \in G$ ... (2.2)

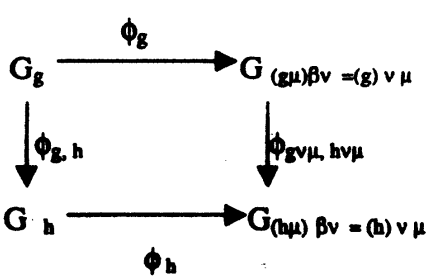
Hence by (2.2) we conclude that $\beta_\nu\,\alpha_\mu$ is an isomorphism with inverse $(\beta_\nu\,\alpha_\mu) = \delta$.

Next the composite

$$G^{(e\,\mu\,e\,\nu)\,\alpha_\mu - 1} \xrightarrow{\mu^{\#}} G^{(e\,\mu\,e_\nu)} \circ \underline{C}\,(\alpha_\mu) \xrightarrow{\nu^{\#}\,C(\alpha_\mu)} G^{(e\,\mu\,e_\nu)}\,\beta_\nu \circ \underline{C}\,(\beta_\nu\,\alpha_\mu)$$

is shown to be a natural equivalence.

For $g \le (e\,\mu\,e_\nu)\,\alpha_\mu^{-1}$ we have $\phi_g = (\nu^{\#}\,\underline{C}\,(\alpha_\mu)\,\mu^{\#})\,(g) : G_g \to G_{(g\,\mu)\,\beta\,\nu}$ and for $h \le g$ the diagram commutes. For $(\delta_{(g)\,\nu\mu})_{(g)\,\nu\,\mu \le (e\,\mu)\,\nu\,\mu} : G_{(g)\,\nu\,\mu} \to G_g$ and $x \in G_g$



$$x\,\phi_g\,(\delta_{(g)\,\nu\,\mu}) = ((x)\,\nu\,\mu)\,(\delta_{(g)\,\nu\,\mu}) = (x)\,\nu\,\mu\,\delta = x\,e_\mu = x$$

and for $y \in G_{(g)\, \nu\mu}$,

$$y\,(\delta_{(g)\,\nu\mu})\,\phi_g = (y\,\delta)\,\phi_\gamma = (y\,\delta)\,\nu\mu = y\,(e_\mu\,\nu\mu) = y$$

Hence $\nu^{\#} \subseteq (\alpha_\mu)\,\mu^{\#} : G^{(e\,\mu\,e\,\nu)\,\alpha_\mu - 1} \to G^{(e\,\mu\,e_\nu)}\,\beta_\nu \circ \subseteq (\beta_\nu\,\alpha_\mu)$ is a natural equivalence. So η is a homomorphism. Take any $(\alpha, \phi) \in Reg_E(\hat{G})$ and let $\alpha : \omega(e) \to \omega(f)$. Define $\mu : G \to G$ by $\mu(g) = (ge)\,\phi_{e,f}$ where $\phi_{e,f} : G_e \to G_f$ is an isomorphism and $\mu^{\#} : G \to G$ by (g) $\mu^{\#} = (gf)\,\phi_{f,e}^{-1}$. Clearly μ and $\mu^{\#}$ are homomorphisms of G. For $g \in G_e$,

$$(g)\,\mu\mu^{\#} = (ge)\,\phi_{e,f}\mu^{\#} = (ge)\,\phi_{e,f}\phi_{f,e}^{-1} = ge = g$$

and for $h \in G_f$,

$$(h)\,\mu^{\#}\mu = (hf)\,\phi_{f,e}^{-1}\,\phi_{e,f} = hf = h$$

It is clear that e is the right identity of $G\,\mu^{\#}$ and $e\,\mu$ the right identity of $G\,\mu$. So we define a mapping $\tilde{n} : Reg_E(\hat{G}) \to \mathbf{end}\, G\,(\alpha_\mu, \phi_\mu) \to \mu$. Clearly $\tilde{n}$ is a homomorphism. Hence

$Reg_E(\hat{G})$ is isomorphic to $\mathbf{end}G$.

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