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ABSOLUTE CESÀRO SUMMABILITY FACTORS

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In this paper using δ-quasi-monotone sequences and any almost increasing sequences a theorem on $|C, \alpha, \beta; \delta|_k$ summability factors of infinite series, which generalizes a theorem of Mazhar[8] concerning $|C, 1|_k$ summability factors, has been proved.

Key Words : Quasi-monotone Sequences; Almost Increasing; Infinite Series

1. INTRODUCTION

A sequence (b_n) of positive numbers is said to be quasi-monotone if $n \Delta b_n \geq -\gamma b_n$ for $\gamma > 0$ and if a is said to be δ-quasi-monotone, if $0 < b_n \to 0$ ultimately and $\Delta b_n \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see[2]). Let Σa_n be a given finite series with partial sums (s_n). We denote by u_n^α and t_n^α the nth Cesáro means of order α with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, i.e.,

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \qquad \dots (1)$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \qquad \dots (2)$$

where
$$A_n^\alpha = \binom{n+\alpha}{n} = O(n^\alpha), \alpha > -1, A_0^\alpha = 1 \text{ and } A_{-n}^\alpha = 0 \text{ for } n > 0. \qquad \dots (3)$$

The series Σa_n is said to be summable $|C, \alpha|_k, k \geq 1$, if (see[5])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty. \qquad \dots (4)$$

But since $t_n^\alpha = n\,(u_n^\alpha - u_{n-1}^\alpha)$ (see [7]) condition (4) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^\alpha \mid^k < \infty. \qquad \ldots (5)$$

We say that the series $\Sigma\, a_n$ is said to be summable $\mid C, \alpha, \beta; \delta \mid_k$, $k \geq 1$, if (see[6])

$$\sum_{n=1}^{\infty} n^{\beta(\delta k + k - 1) - k} \mid t_n^\alpha \mid^k < \infty, \qquad \ldots (6)$$

where $\delta \geq 0$ and β is a real number.

Mazhar[8] proved the following theorem for $\mid C, 1 \mid_k$ summability factor of infinite series.

Theorem A — *Let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\Sigma\, n\, \delta_n \log n < \infty$, $\Sigma\, B_n \log n$ is convergent and $\mid \Delta \lambda_n \mid \leq \mid B_n \mid$ for all n. If*

$$\sum_{n=1}^{m} \frac{1}{n} \mid t_n^1 \mid^k = O\,(\log m) \text{ as } m \to \infty, \qquad \ldots (7)$$

where (t_n^1) is the nth (C, 1) mean of the sequence (na_n), then the series $\Sigma\, a_n \lambda_n$ is summable $\mid C, 1 \mid_k$, $k \geq 1$.

2. The aim of this paper is to generalize Theorem A for $\mid C, \alpha, \beta; \delta \mid_k$ summability factors under weaker conditions by using an almost increasing sequence. For this we need the concept of almost increasing sequence. A positive sequence (d_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq d_n \leq Bc_n$ (see[1]). Obviously every increasing sequence is almost increasing but the converse need not be true as can be seen from the example $d_n = ne^{(-1)^n}$. Since log n is increasing in Theorem A, we are weakening the hypotheses of the theorem replacing the increasing sequence by any almost increasing sequence.

Now, shall prove the following theorem.

Theorem — *Let (X_n) be an almost increasing sequence. Let $k \geq 1, \alpha \geq 1$ and $\delta_n \geq 0$. Let $\Sigma\, a_n$ be a given series with $a_n \geq 0$ for all $n \geq 1$. Let $\lambda_n \to 0$ as $n \to \infty$ and β be a real number such that $\alpha + k - \beta\,(\delta k + k - 1) \geq 0$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\Sigma\, n^\alpha\, \delta_n X_n < \infty$, $\Sigma\, B_n X_n$ is convergent and $\mid \Delta \lambda_n \mid \leq \mid B_n \mid$ for all n. If*

$$\sum_{n=1}^{m} n^{\alpha - 1} \mid B_{n+1} \mid X_n = O\,(1), \qquad \ldots (8)$$

$$\sum_{n=1}^{m} n^{\beta(\delta k+k-1)-k} (t_n^{\alpha})^k = O(X_n) \text{ as } m \to \infty, \qquad \dots (9)$$

then the series $\Sigma a_n \lambda_n$ *is summable* $|C, \alpha, \beta; \delta|_k$.

It should be noted that if we take $X_n = \log n$, $\beta = 1$, $\delta = 0$ and $\alpha = 1$ in this theorem, then we get Theorem A.

We need the following lemmas for the proof of our theorem.

Lemma 1[4] — If $\tau > -1$ and $\tau - \sigma > 0$, then for $k = 1, 2, \dots$

$$\sum_{n=k}^{\infty} \frac{A_{n-k}^{\sigma}}{nA_n^{\tau}} = \frac{1}{kA_k^{\tau-\sigma-1}}. \qquad \dots (10)$$

Lemma 2 — Let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\Sigma B_n X_n$ is convergent and $|\Delta \lambda_n| \le |B_n|$ for all n. then

$$|\lambda_n| X_n = O(1) \text{ as } n \to \infty. \qquad \dots (11)$$

PROOF : Proof of Lemma 2 is similar to proof of Lemma 2 of Bor (see[3] case $X_n = \log n$) and hence omitted.

Lemma 3 — Let $\alpha \ge 1$. If (B_n) is δ-quasi-monotone with $\Sigma n^{\alpha} \delta_n X_n < \infty$ and $\Sigma B_n X_n$ is convergent, then

$$m^{\alpha} B_m X_m = O(1) \text{ as } m \to \infty \qquad \dots (12)$$

$$\sum_{n=1}^{\infty} n^{\alpha} |\Delta B_n| X_n < \infty. \qquad \dots (13)$$

PROOF : Proof of Lemma 3 is similar to proof of Theorem 1 and 2 of Boas (see[2] case $\gamma = \alpha$, $a_n = B_n$ and $X_n = \log n$) and hence omitted.

Lemma 4[4] — Let (t_n^{α}) be the n-th Cesàro mean of order α with $\alpha \ge 1$, of the sequence (na_n) such that $a_n \ge 0$ for all $n \ge 1$ whenever $\alpha > 1$. If $n \ge v$, then

$$\sum_{p=1}^{v} A_{n-p}^{\alpha-1} pa_p \le A_{n-v}^{\alpha-1} A_v^{\alpha} t_v^{\alpha}. \qquad \dots (14)$$

PROOF OF THE THEOREM : Let (T_n^{α}) be the nth (C, α) mean of the sequence $(na_n \lambda_n)$. Then, by (2), we have

$$T_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v.$$

By Abel's transformation, we have

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} pa_p + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v .$$

So that, making use of Lemma 4, we get

$$|T_n^\alpha| \le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}|\Delta\lambda_v|\left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} pa_p\right| + \frac{\lambda_n}{A_n^\alpha}\left|\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v\right|$$

$$\le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}|\Delta\lambda_v|A_{n-v}^{\alpha-1} A_v^\alpha t_v^\alpha + |\lambda_n| t_n^\alpha$$

$$= T_{n,1}^\alpha + T_{n,2}^\alpha, \text{ say.}$$

Since $|T_{n,1}^\alpha + T_{n,2}^\alpha|^k \le 2^k\left(|T_{n,1}^\alpha|^k + |T_{n,2}^\alpha|^k\right)$, to complete the proof of the theorem it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\beta(\delta k+k-1)-k}|T_{n,r}^\alpha|^k < \infty, \text{ for } r = 1, 2, \text{ by (6).}$$

Since $n^\alpha |B_n| = O(1/X_n) = O(1)$, by (12) and $|\Delta v^\alpha| = O(v^{\alpha-1})$, now, when $k > 1$, applying Hölder's inequality with indices k and k', where $1/k + 1/k' = 1$, we have

$$\sum_{n=2}^{m+1} n^{\beta(\delta k+k-1)-k}|T_{n,1}^\alpha|^k \le \sum_{n=2}^{m+1} n^{\beta(\delta k+k-1)-k}$$

$$\left\{\frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}|B_v|A_v^\alpha A_{n-v}^{\alpha-1} t_v^\alpha\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1} n^{\beta(\delta k+k-1)-k}\frac{1}{(A_n^\alpha)^k}\left\{\sum_{v=1}^{n-1} v^\alpha |B_v| A_{n-v}^{\alpha-1} t_v^\alpha\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1} n^{\beta(\delta k+k-1)-k}\frac{1}{(A_n^\alpha)^k}\left\{\sum_{v=1}^{n-1} v^\alpha\, |B_v|\,(A_{n-v}^{\alpha-1^{1/k}})\,(A_{n-v}^{\alpha-1})^{1/k} t_v^\alpha\right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} n^{\beta(\delta k+k-1)-k} \frac{1}{(A_n^\alpha)^k} \sum_{v=1}^{n-1} (v^\alpha | B_v | (A_{n-v}^{\alpha-1})^{1/k} t_v^\alpha)^k$$

$$x \left\{ \sum_{v=1}^{n-1} \left((A_{n-v}^{\alpha-1})^{1/k} \right)^{k'} \right\}^{\frac{k}{k'}}$$

$$= O(1) \sum_{n=2}^{m+1} n^{\beta(\delta k+k-1)-k} \frac{1}{(A_n^\alpha)^k} \sum_{v=1}^{n-1} (v^\alpha | B_v |)^k (A_{n-v}^{\alpha-1}) (t_v^\alpha)^k x \left\{ \sum_{v=1}^{n-1} A_{n-v}^{\alpha-1} \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} n^{\beta(\delta k+k-1)-k} \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} (v^\alpha | B_v |)^k A_{n-v}^{\alpha-1} (t_v^\alpha)^k \times \left\{ \sum_{v=1}^{n-1} \frac{A_{n-v}^{\alpha-1}}{A_n^\alpha} \right\}^{k-1}$$

$$= O(1) \sum_{y=1}^{m} (v^\alpha | B_v |)^{k-1} (v^\alpha | B_v |) (t_v^\alpha)^k \sum_{n=v+1}^{m+1} \frac{A_{n-v}^{\alpha-1}}{n^{-\beta(\delta k+k-1)+k} A_n^\alpha}$$

$$= O(1) \sum_{v=1}^{m} v^\alpha | B_v | v^{\beta(\delta k=k-1)-k} (t_v^\alpha)^k,$$

by Lemma 1. By Abel transformation we discover

$$\sum_{n=2}^{m+1} n^{\beta(\delta k+k-1)-k} | T_{n,1}^\alpha |^k = O(1) \sum_{v=1}^{m-1} | \Delta (v^\alpha | B_v |) / \sum_{p=1}^{v} p^{\beta(\delta k+k-1)-k} (t_p^\alpha)^k$$

$$+ O(1) m^\alpha | B_m | \sum_{v=1}^{m} v^{\beta(\delta k+k-1)-k} (t_v^\alpha)^k$$

$$= O(1) \sum_{v=1}^{m-1} | \Delta (v^\alpha | B_v |) | X_v + O(1) m^\alpha | B_m | X_m$$

$$= O(1) \sum_{v=1}^{m-1} v^\alpha | \Delta B_v | X_v + O(1) \sum_{v=1}^{m-1} | \Delta (v^\alpha) | | B_{v+1} | X_v + O(1) m^\alpha | B_m | X_m$$

$$= O(1) \sum_{v=1}^{m-1} v^\alpha | \Delta B_v | X_v + O(1) \sum_{v=1}^{m-1} v^{\alpha-1} | B_{v+1} | X_v + O(1) m^\alpha | B_m | X_m$$

$$= O(1) \text{ as } m \to \infty,$$

by conditions (9), (13) and (8) of the Theorem and Lemma 3.

Again, since $\lambda_n = O(1)$ and $\Delta|\lambda_n| \le |\Delta\lambda_n|$, we have that

$$\sum_{n=1}^{m} n^{\beta(\delta k+k-1)-k} |T_{n,2}^{\alpha}|^k = \sum_{n=1}^{m} n^{\beta(\delta k+k-1)-k} |\lambda_n||\lambda_n|^{k-1} (t_n^{\alpha})^k$$

$$= O(1) \sum_{n=1}^{m} n^{\beta(\delta k+k-1)-k} |\lambda_n| (t_n^{\alpha})^k$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| \sum_{p=1}^{n} p^{\beta(\delta k+k-1+k)} (t_p^{\alpha})^k + O(1)|\lambda_m| \sum_{n=1}^{m} n^{\beta(\delta k+k-1)-k} (t_n^{\alpha})^k$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m = O(1) \sum_{n=1}^{m-1} |B_n| X_n + O(1)|\lambda_m| X_m$$

$$= O(1) \text{ as } m \to \infty,$$

by conditions (9) and (11) of Lemma 2 and the Theorem.

Therefore, we get that

$$\sum_{n=1}^{m} n^{\beta(\delta k+k-1)-k} |T_{n,r}^{\alpha}|^k = O(1) \text{ as } m \to \infty \text{ for } r = 1, 2.$$

This completes the proof of the theorem.

REFERENCES

1. L. S. Aljancic and D. Arandelovic, *Publ. Inst. Math.* **22** (1977) 5-22.
2. R. P. Boas, *Proc. London math. Soc.* **14A** (1965) 38-46.
3. H. Bor, *Atti. Sem. Math. Fis. Univ. Modena*, **XLII** (1994) 135-40.
4. H. C. Chow, *J. London Math. Soc.* **29** (1954) 459-76.
5. T. M. Flett, *Proc. London Math. Soc.* **7** (1957) 113-41.
6. A. N. Gürkan, *J. Anal.* **7** (1999) 133-38.
7. E. Kogbetliantz, *Bull. Sci. Math.* **49** (1925) 234-56.
8. S. M. Mazhar, *Indian J. pure. appl. math.* **8** (1977) 784-90.