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A SPACE OF ANALYTIC FUNCTIONS WITH MONTEL PROPERTY

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A space of analytic functions with Montel property is developed in this paper.

Key Words : Modulus Function; Hardy-Orlicz Space; Bergman Space; Bergman-Orlicz Type Space; Montel Property; Topologically Compact Operator

1. INTRODUCTION

Let ω be a non-constant continuous, non-decreasing subadditive real-valued function defined on $[0, \infty)$ such that ω is zero only at zero. Such a function ω is called a modulus function. For $0 < p \leq 1$, $\omega(x) = x^p$, $\omega(x) = \log(1+x)$ are trivial examples of modulus functions. Let $D = \{Z \in \mathbb{C} : |z| < 1\}$ and let $H(D)$ denote the space of analytic functions in D. If

$$H(\omega) = \left\{ f : f \in H(D) \text{ and } \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \omega(|f(re^{i\theta})|)\, d\theta < \infty \right\},$$

then $H(\omega)$ is a topological vector space with distance function defined by

$$\|f-g\|_\omega = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \omega(|f(re^{i\theta}) - g(re^{i\theta})|)\, d\theta.$$

$H(\omega)$ is called a Hardy-Orlicz space. For $0 < p < \infty$, let

$$A^p(D) = \left\{ f \in H(D) : \int_D \int |f(z)|^p\, dA(z) < \infty \right\},$$

where dA is the normalized area measure on D. Then $A^p(D)$ is a Banach space for $1 \leq p < \infty$ and is called the Bergman space. The Bergman-Orlicz type space is defined as follows :

Let ω be a modulus function. Then $A(\omega)$, defined by

$$A(\omega) = \left\{ f : f \in H(D) \text{ and } \int_D \int \omega(|f(z)|)\, dA(z) < \infty \right\},$$

is known as Bergman-Orlicz type space. $A(\omega)$ is a topological vector space with metric

$$\|f\|_A = \int_D \int \omega(|f(z)|)\, dA(z) \text{ for all } f \in A(\omega).$$

The space $A(\omega)$ is not necessarily complete. However, if $\omega(|f|)$ is subharmonic in D for any holomorphic f on D, then $A(\omega)$ becomes an F-space. In view of Lemma 5.1 of Hasumi and Kataoka[4], the above condition on ω is equivalent to the following :

"$\omega(e^t)$ is a convex function".

Proposition 1.1 — Let ω be a strictly increasing modulus function such that $\omega(|f|)$ is subharmonic for all $f \in A(\omega)$. Then

$$\omega(|f(z)|) \leq \frac{\|f\|_A}{(1-|z|)^2} \text{ for all } z \in D.$$

PROOF : Let $r = 1 - |z|$. Then since $\omega(|f|)$ is subharmonic, by the Mean Value Theorem (See Duren [3, Page 7]),

$$\omega(|f(z)|) \leq \left(\frac{1}{2\pi}\right) \int_0^{2\pi} w(|f(z) + \rho e^{i\theta}|)\, d\theta.$$

Multiplying by $\rho, 0 < \rho < 1$ and integrating from 0 to r with respect to ρ, we get

$$\frac{r^2}{2}\, \omega(|f(z)|) \leq \frac{1}{2} \int\int_{|z-\xi|<r} \omega(|f(\xi)|)\, dA(\xi)$$

or

$$\omega(|f(z)|) \leq \frac{1}{r^2} \int\int_D \omega(|f(\xi)|)\, dA(\xi).$$

This completes the proof.

By an application of the above Proposition, one can show that $A(\omega)$ is a complete meterizable space. The proof is similar to that of Theorem 3 of Deeb and Marzuq [2,Page 300]. Throught this paper we will assume that our modulus function satisfies the following three properties :

(*i*) $\omega(|f|)$ is a subharmonic function whenever $f \in H(D)$,

(*ii*) ω is strictly increasing,

(*iii*) $\omega(x, y) \leq \omega(x) + \omega(y)$ for all $x, y \in [0, \infty)$.

For instance, the function, defined as $\omega(x) = \log(1+x)$ for every $x \in [0, \infty)$, is a modulus function which satisfies properties (*i*) - (*iii*) given above.

In this paper we show that $A(\omega)$ has the Montel property. It may recalled that a topological vector space V is said to have Montel property or Borel property if the topologically compact subsets of V are relatively compact.

2. $A(\omega)$ WITH MONTEL PROPERTY

Since $A(\omega)$ is non-normable, we have two types of boundedness in $A(\omega)$, namely boundedness in metric sense and topological boundedness.

Definition — A set $E \subset A(\omega)$ is said to be metric bounded if there exist $M > 0$ such that $\| f-g \|_A < M$ for all $f, g \in E$, and E is said to be topologically bounded if for every neighbourhood V of the origin there exist $t_0 > 0$ such that $E \in tV$ for all $t \geq t_0$.

It is clear that every topologically bounded subset of $A(\omega)$ is metric bounded.

The following theorem is a nice characterization of boundedness of a subset of $A(\omega)$ in the topologically sense :

Theorem 2.1 — *A set $E \subset A(\omega)$ is topologically bounded if and only if*

(i) E is uniformly bounded in the metric of $A(\omega)$ i.e. there exists $M(E)$ such that

$$\int\int_D \omega(|f(z)|)\, dA(z) < M \text{ for all } f \in E.$$

(ii) For all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $S \subset D$ with $dA(S) < \delta$,

$$\int\int_S \omega(|f(z)|)\, dA(z) < \varepsilon \text{ for all } f \in E.$$

PROOF : Let $\bigcup(\eta) = \{g \in A(\omega) : \| g \|_A < \eta\}$ be a neighbourhood of the zero function in $A(\omega)$. Choose ε so that $\omega(\varepsilon) + \varepsilon < \eta$. Then by the condition (ii) there exists $\delta > 0$, which without loss of generality, we may take less than ε, such that for any $S \subset D$, whenever $dA(S) < \delta$,

$$\int\int_S \omega(|f|)\, dA < \varepsilon \qquad \ldots (2.1)$$

for all $f \in E$. For $f \in E$, let

$$S_f = \left\{ \xi \in D : \omega(|f(\xi)|) < \frac{M}{\delta} \right\}.$$

Then by the condition (i)

$$M > \int\int_D \omega(|f|)\, dA$$

$$\geq \int\int_{D/S_f} \omega(|f|)\, dA$$

$$\geq \frac{M}{\delta} dA(D \setminus S_f),$$

which implies that $dA(D \setminus S_f) < \delta$ and so by *(ii)*

$$\geq \int\int_{D \setminus S_f} \omega(|f|)\, dA < \varepsilon.$$

Also,

$$|f(\xi)| < \omega^{-1}\left(\frac{M}{\delta}\right) = L(\delta) \text{ on } S_f.$$

Let α be such that $0 < \alpha < \min\{1, \varepsilon/L\}$. Then

$$\|\alpha f\|_A = \int\int_D \omega(|\alpha f|)\, dA$$

$$= \int\int_{S_f} \omega(|\alpha f|)\, dA + \int\int_{D \setminus S_f} \omega(|\alpha f|)\, dA$$

$$< \int\int_{S_f} \omega\left(\frac{\varepsilon}{L} \cdot L\right) dA + \varepsilon$$

$$= \omega(\varepsilon) + \varepsilon < \eta$$

for all $f \in E$. This implies that E is topologically bounded.

Conversely, let E be a topologically bounded subset of $A(\omega)$. Then for all $\eta > 0$ there exists $\alpha = \alpha(\eta), |\alpha| \leq 1$, such that

$$\int\int_D \omega(|\alpha f|)\, dA = \|\alpha f\|_A < \eta \qquad \ldots (2.2)$$

for all $f \in E$. Fix a number $\eta_0 > 0$ and corresponding $\alpha_0 = \alpha(\eta_0)$. If we set

$M = \left(\left[\frac{1}{\alpha_0}\right] + 1\right)\omega(1) + \eta_0$, where $[x]$ stands for the integral part of x, then

$$\int\int_D \omega(|f|)\, dA = \int\int_D \omega\left(|\alpha_0 f| \cdot \frac{1}{|\alpha_0|}\right) dA$$

$$\leq \omega\left(\frac{1}{|\alpha_0|}\right) + \int\int_D \omega|\alpha_0 f|\, dA$$

$$\leq \left(\left[\frac{1}{|\alpha_0|}\right] + 1\right)\omega(1) + \|\alpha_0 f\|_A$$

$$< \left(\left[\frac{1}{|\alpha_0|}\right] + 1\right)\omega(1) + \eta_0$$

for every $f \in A(\omega)$. This proves the condition (i). Next, for any number $\varepsilon > 0$, we take η so that $\eta < \varepsilon/2$. Choose $\alpha = \alpha(\eta)$ so that (2.2) holds. If we choose $\delta > 0$ so that

$$\left(\left[\frac{1}{|\alpha|}\right]+1\right)\omega(1)\,\delta<\frac{\varepsilon}{2},$$

then we have for any $S\subset D$ with $dA(S)<\delta$,

$$\iint_S \omega(|f|)\,dA=\iint_S \omega\left(|\alpha f|\cdot\frac{1}{|\alpha|}\right)dA$$

$$<\|\alpha f\|_A+\left(\left[\frac{1}{|\alpha|}\right]+1\right)\omega(1)\,dA(S)$$

$$<\eta+\left(\left[\frac{1}{|\alpha|}\right]+1\right)\omega(1)\,\delta$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}.$$

This proves (*ii*).

As an application of the above theorem, we show that the spaces $A(\omega)$ have the montel property.

Theorem 2.2 — *The space $A(\omega)$ has the Montel property.*

PROOF : Let $E\subset A(\omega)$ be topologically bounded. Then we have to show that $\overline{E}$, the closure of E, is compact in $A(\omega)$. Let $\{f_n\}$ be a sequence in $\overline{E}$,. Since $A(\omega)$ is complete, $\overline{E}$ is closed and so complete, it is sufficient to show that $\{f_n\}$ has a subsequence which is Cauchy in $A(\omega)$. Since E is topologically bounded, by Theorem 1.13 (*f*) of Rudin[5], $\overline{E}$ is topologically bounded and so metric bounded. Therefore, by Proposition 1.1, for any compact set $K\subset D$,

$$|f_n(z)|\le\omega^{-1}(M.C_K),$$

where $C_K=\operatorname{Sup}\left\{\frac{1}{(1-|z|)^2}:z\in K\right\}$ and $M\ge 0$ such that $\|f_n\|_A\le M$ for all n. This implies that $\{f_n\}$ is uniformly bounded on each compact subset of D and hence, by Theorem 14.6 of Rudin[6], $\{f_n\}$ is a normal family, which means that $\{f_n\}$ contains a subsequence $\left\{f_{n_k}\right\}$ such that $f_{n_k}\to g$ uniformly on compact subsets of D for some $g\in A(\omega)$. Thus, if $\varepsilon>0$ is given, then for any compact set $K\subset D$ there exists a positive integer n_1 such that

$$\left|\left(f_{n_{k_1}}-f_{n_{k_2}}\right)(z)\right|<\omega^{-1}\left(\frac{\varepsilon}{3}\right) \qquad \ldots (2.3)$$

for all $n_{k_1}, n_{k_2}>n_1$ and $z\in K$. Also, by Lemma 2.2, there exists $\delta=\delta(\varepsilon)>0$ such that for any $S\subset D$, $dA(S)<\delta$ implies that

$$\int\int_S \omega\left(\left|f_{n_k}\right|\right)dA < \frac{\varepsilon}{3} \qquad \ldots(2.4)$$

for all n_k. Further, if we set

$E_k = \left\{ z \in D : |z| > 1 - \frac{1}{k} \right\}$, then $E_1 \supset E_2 \supset E_3 \supset \ldots.$, and $dA\left(\bigcap_{k=1}^{\infty} E_k\right) = 0$. Hence we can choose n_2 such that $dA\,(E_{n_2}) < \delta$. Let

$$K = \left\{ z \in D : |z| \leq 1 - \frac{1}{n_2} \right\}.$$

Then K is a compact subset of D and if $n_0 = \max\{n_1, n_2\}$, then for $n_{k_1}, n_{k_2} > n_0$, we have

$$\left|\left|f_{n_{k_1}} - f_{n_{k_2}}\right|\right|_A = \int\int_D \omega\left(\left|f_{n_{k_1}} - f_{n_{k_2}}\right|\right)dA$$

$$= \int\int_{E_{n_2}} \omega\left(\left|f_{n_{k_1}} - f_{n_{k_2}}\right|\right)dA + \int\int_K \omega\left(\left|f_{n_{k_1}} - f_{n_{k_2}}\right|\right)dA$$

$$\leq \int\int_{E_{n_2}} \omega\left(\left|f_{n_{k_1}}\right|\right)dA + \int\int_{E_{n_2}} \omega\left(\left|f_{n_{k_2}}\right|\right)dA + \frac{\varepsilon}{3}\int\int_K dA$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This proves that $\left\{f_{n_k}\right\}$ is Cauchy in $A\,(\omega)$. Hence the proof is complete.

Carollary 2.3 — Every continuous linear operator on $A\,(\omega)$ is topologically compact.

PROOF : Let T be a continuous linear operator on $A\,(\omega)$ and E be a topologically bounded subset of $A\,(\omega)$. The continuity of T implies that $T(E)$ is also topologically bounded. Hence by Theorem 2.2, $\overline{T(E)}$is compact. This proves that T is topologically compact.

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