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GLOBAL SOLUTIONS OF INITIAL VALUE PROBLEMS FOR NONLINEAR INTEGRAL-DIFFERENTIAL EQUATIONS OF MIXED TYPE IN BANACH SPACES*

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In this paper, the global solutions of initial value problems for nonlinear integral-differential equation of mixed type in Banach spaces are investigated. The existence and uniqueness of solutions and their iteractive approximation are obtained by using Mönch fixed point theorem and comparison results. The results presented here essentially improve, generalize and unify many well-known results.

Key Words : Integral-Differential Equations of Mixed Type; Measure of Non-Compactness; Global Solutions

1. INTRODUCTION

We consider, in this paper, the existence of solutions, iterative approximation of the unique solution for the following initial value problems (IVP) of the nonlinear integral-differential equations of mixed type in ordered Banach spaces E :

$$u' = f(t, u, u, Tu, Su),\ t \in I,\ u(0) = x_0, \qquad \ldots (1)$$

where $I = [0, a]\ (a > 0),\ x_0 \in E, f \in C[I \times E \times E \times E \times E, E]$ and

$$(Tu)(t) = \int_0^t k(t, s)\, u(s)\, ds,\ (Su)(t) = \int_0^a h(t, s)\, u(s)\, ds,\ t \in I. \qquad \ldots (2)$$

In (2), $k \in C[D, R^+]$ and $h \in C[D_0, R^+]$, where $D = \{(t, s) \in R^2 : 0 \le s \le t \le a\}$, $D_0 = \{(t, s) \in R^2 : (t, s) \in I \times I\}$, R^+ denotes the set of non-negative real numbers and

R denotes the set of real numbers. In the special case where $f(t, u) = f_0(t, u) + f_1(t, u) + f_2(t, u)$ and it does not contain Tu and Su, the solutions of IVP (1) have been obtained by means of the mixed monotone iterative technique in[6] with stronger conditions. In another special case where

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f does not contain the third argument and operator Tu and Su,7 obtained the solutions of IVP(1) with stronger lower-upper solution condition as well as Lipschitz condition and compactness type condition. The same results have also been got in[11] without Lipschitz condition. In[2], f has only the volterra type integral operator Tu, the results of[2] extended those of[11]. The method used in[2, 11] is obtaining the local solutions at first and then extending them to the whole interval.

In this paper, by using some comparison results and Mönch fixed point theorem, we researched the more general mixed type case IVP(1) and obtained its global solutions and the unique solution as well as its iterative approximation. The results presented in this paper essentially improve, generalize and unify many known results. Our method is different in essence from those of[2,6,7,11].

2. PRELIMINARIES AND LEMMAS

Let E be a real Banach space with norm $\|\cdot\|$. A nonempty closed convex subset P of E is said to be a *cone* if $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap \{-P\} = \{\theta\}$, where θ denotes the zero element of E. The cone P defines a partial ordering in E by $x \leq y$ iff $y - x \in P$. Recall tha a cone P is said to be *normal* if there exists a positive constant λ such that $\theta \leq x \leq y$ implies $\|x\| \leq \lambda \|y\|$. The cone P is normal iff every order interval $[x, y] = \{u \in E : x \leq u \leq y\}$ is bounded in norm. For further details on cone theory, one can refer to[3].

Let $P_c = \{u \in C[I, E] : u(t) \geq \theta \text{ for all } t \in I\}$, where $C[I, E]$ denotes the Banach space of all continuous mapping $u : I \to E$ with norm $\|u\|_c = \max \{\|u(t)\| : t \in I\}$. It is easy to see that P_c is a cone in $C[I, E]$, and so it defines a partial order in $C[I, E]$ by $u \leq v$ iff $u(t) \leq v(t)$ for all $t \in I$. Obviously, the normality of P implies the normality of P_c and the normal constants of P_c and P are the same. Given a cone P, the dual of P is defined as $P^* = \{\varphi \in E^* : \varphi(u) \geq 0 \text{ for all } u \in P\}$, where E^* is the dual of E. Let α denote the Kuratowski measure of noncompactness in E or $C[I, E]$, the properties of which may be found in[4]. Let

$$k_0 = \max \{k(t, s) : (t, s) \in D\}, \quad h_0 = \max \{h(t, s) : (t, s) \in D_0\}.$$

For any $B \subset C[I, E]$ and $t \in I$, let

$$B(t) = \{u(t) : u \in B\}, \quad (TB)(t) = \{(Tu)(t) : u \in B\}, \quad (SB)(t) = \{(Su)(t) : u \in B\}.$$

The following comparison results play an important role in this paper.

Lemma 1 (Liu[8]) — Assume that $u \in C^1[I, E]$ satisfies

$$u'(t) \leq -Mu(t) - N \int_0^t k(t, s) u(s)\, ds, \; t \in I, \; u(0) \leq \theta,$$

where $M \geq 0$ and $N \geq 0$ are constants. Then $u(t) \leq \theta$ for $t \in I$ provided one of the following two conditions hold :

$$\text{(a) } M > 0, \; aNk_0 (e^{aM} - 1) \leq M; \qquad \ldots (3)$$

(b) $a(M + aNk_0) \leq 1.$ ... (4)

Lemma 2 (Liu[8]) — Assume that $m \in C[I, R^+]$ satisfies

$$m(t) \leq M_1 \int_0^t m(s)\,ds + M_2 t \int_0^t m(s)\,ds + M_3 t \int_0^a m(s)\,ds, t \in I,$$

where $M_i \geq 0$ $(i = 1, 2, 3)$ are constants. Then $m(t) \equiv 0$ for $t \in I$ provided one of the following two conditions hold :

(i) $M_1 + aM_2 > 0, aM_3 (e^{a(M_1 + aM_2)} - 1) < M_1 + aM_2;$ ... (5)

(ii) $a(2M_1 + aM_2 + aM_3) < 2.$ ... (6)

Lemma 3 (Liu[9]) — Let $B_1, B_2 \subset C[I, E]$ be two countable subset satisfying $\overline{B}_1 = \overline{co}\ (\{u_0\} \cup B_2)$ for some $u_0 \in C[I, E]$. Then $\overline{B_1(t)} = \overline{co}\ (\{u_0(t)\} \cup B_2(t))$ for any $t \in I$.

Lemma 4 (Heinz[5]) — Let B be a countable set of strongly measurable functions $u : I \to E$ such that there exists a $m \in L[I, R^+]$ such that $\| u(t) \| \leq m(t)$ for almost all $t \in I$ and $u \in B$. Then $\alpha(B(t)) \in L[I, R^+]$ and

$$\alpha\left(\left\{\int_I u(t)\,dt : u \in B\right\}\right) \leq 2 \int_I \alpha(B(t))\,dt.$$

Lemma 5 (Banas and Goebel[1]) — Let $B \subset C[I, E]$ be bounded and equicontinuous. Define $m(t) = \alpha(B(t)), t \in I$, then $m(t)$ is continuous on $t \in I$ and

$$\alpha\left(\int_I B(s)\,ds\right) \leq \int_I \alpha(B(s))\,ds.$$

Lemma 6 (Mönch H[10]) — Let X be a Banach space, $K \subset X$ closed and convex and $F : K \to K$ continuous with the further property that for some $x \in K$ we have $B \subset K$ countable $\overline{B} = \overline{co}\ (\{x\} \cup F(B))$ imply B is relatively compact. Then F has a fixed point in K.

Let us list the following assumptions for convenience :

(H_1) There exists $v_0, w_0 \in C^1[I, E]$ such that $v_0 \leq w_0$, for any $\sigma \in [v_0, w_0]$

$$= \{u \in C[I, E] : v_0 \leq u \leq w_0\},\ t \in I,$$

$$v_0' \leq f(t, \sigma, w_0, T\sigma, S\sigma) - M(v_0 - \sigma) - NT(v_0 - \sigma),\ v_0(0) \leq x_0,$$

$$w_0' \geq f(t, \sigma, v_0, T\sigma, S\sigma) - M(w_0 - \sigma) - NT(w_0 - \sigma),\ w_0(0) \geq x_0,$$

where $M \geq 0$ and $N \geq 0$ are constants satisfying (3) or (4).

(H_2) For any countable bounded equicontinuous set $B \subset [v_0, w_0]$ and $t \in I$,

$$\alpha(f(t, B(t), B(t), (TB)(t), (SB)(t))) \leq \lambda_1 \alpha(B(t)) + \lambda_2 \alpha((TB)(t)) + \lambda_3 \alpha((SB)(t)),$$

where $\lambda_i \geq 0$ ($i = 1, 2, 3$) are constants satisfying one of the following two conditions :

$$\text{(i)}\ a\lambda_3 h_0 (e^{a(\lambda_1 + M + a\lambda_2 k_0 + 2aNk_0)} - 1) < \lambda_1 + M + a\lambda_2 k_0 + 2aNk_0,$$

$$\lambda_1 + M + a\lambda_2 k_0 + 2aNk_0 > 0; \qquad \ldots (7)$$

$$\text{(ii)}\ a(2\lambda_1 + 2M + 2a\lambda_2 k_0 + 2aNk_0 + a\lambda_3 h_0) < 1. \qquad \ldots (8)$$

(H_3) For any $x_i \in [v_0, w_0]$, $y_i \in [T v_0, Tw_0]$, $z_i \in [S v_0, Sw_0]$ ($i = 1, 2$) and $t \in I$,

$$\| f(t, x_1, x_1, y_1, z_1) - f(t, x_2, x_2, y_2, z_2) \|$$

$$\leq \lambda_1 \| x_1 - x_2 \| + \lambda_2 \| y_1 - y_2 \| + \lambda_3 \| z_1 - z_2 \|,$$

where $\lambda_i \geq 0$ ($i = 1, 2, 3$) are constants satisfies (7) or (8).

(H_4) $f(t, u, v, Tu, Su)$ is nonincreasing in v relative to P.

3. Main Theorems

We are now in a position to prove our main results concerning the solutions of IVP(1). At first we'll prove the following theorem on global solutions of IVP(1).

Theorem 1 — *Let E be a real Banach space and P be a normal cone in E. Assume that conditions (H_1), (H_2) and (H_4) hold, then IVP (1) has a solution $u^* \in [v_0, w_0]$.*

Proof : We will divide the rather long proof into five steps.

(I) For any $h_1, h_2 \in [v_0, w_0] \subset C[I, E]$, consider the following IVP of linear integral differential equation

$$u'(t) + Mu = g(t) - NTu,\ t \in I,\ u(0) = x_0, \qquad \ldots (9)$$

where $g(t) = f(t, h_1(t), h_2(t), (Th_1)(t), (Sh_1)(t)) + Mh_1(t) + N(Th_1)(t)$. We shall show that there exists a unique solution of LIVP(9). It is easy to check that $u \in C^1[I, E]$ is a solution of LIVP (9) if and only if $u \in C[I, E]$ is a fixed point of the following operator

$$(Fu)(t) \equiv x_0 e^{-Mt} + \int_0^t e^{M(s-t)} [g(s) - N(Tu)(s)]\, ds,\ t \in I. \qquad \ldots (10)$$

For any $v, w \in C[I, E]$ and $t \in I$, from (10), it is easy to show by induction that

$$\| (F^n v)(t) - (F^n w)(t) \| \leq \frac{(Nk_0)^n t^{2n}}{(2n)!} \| v - w \|_c, \quad n = 1, 2, 3, \ldots, t \in I.$$

Hence $$\| F^n v - F^n w \|_c \leq \frac{(Nk_0 a^2)^n}{(2n)!} \| v - w \|_c, n = 1, 2, 3, \ldots.$$

For n large enough such that $(Nk_0 a^2)^n/(2n)! < 1$ and by the extension of the contraction principle, the operator F has a unique fixed point u in $C[I, E]$, and this u is the unique solution of LIVP (9) in $C^1[I, E]$.

(II) Now for any $h_1, h_2 \in [v_0, w_0]$, we can define an operator $A : [v_0, w_0] \times [v_0, w_0] \to C[I, E]$ by $A[h_1, h_2] = u$, where u is the unique solution of LIVP (9) corresponding to the $h_1, h_2 \in [v_0, w_0]$. Then the operator A satisfies

$$A[h_1, h_2] = x_0 e^{-Mt} + \int_0^t e^{M(s-t)} [f(s, h_1(s), h_2(s), (Th_1)(s), (Sh_1)(s))$$

$$+ Mh_1(s) + N(Th_1)(s) - N(TA[h_1, h_{2]})(s)] ds. \quad \ldots (11)$$

Evidently, for any $h \in [v_0, w_0]$, h is a solution of IVP (1) Iff $h = A[h, h]$ i.e., h is a fixed point of A.

(III) We shall show that A is an operator from $[v_0 w_0] \times [v_0, w_0]$ to $[v_0, w_0]$. In fact, for any $h_1, h_2 \in [v_0, w_0]$, let $u = A[h_1, h_2]$. From the definition of A we know that u satisfies (9). All we need to do is to prove $v_0 \leq u \leq w_0$. For any $\varphi \in P^*$, let $p(t) = \varphi(u(t) - v_0(t))$, then $p(0) \geq 0$. For any $\sigma \in [v_0, w_0]$, by assumption (H_1)

$$p'(t) = \varphi(u'(t) - v_0'(t))$$

$$\geq \varphi[f(t, h_1(t), h_2(t), (Th_1)(t), (Sh_1)(t)) - M(u - h_1)(t) - NT(u - h_1)(t)$$

$$- f(t, \sigma(t), w_0(t), (T\sigma)(t), (S\sigma)(t)) + M(v_0 - \sigma)(t) - NT(v_0 - \sigma)(t)].$$

Let $\sigma = h_1$ in above inequality and note (H_4), then

$$p'(t) \geq -\varphi(M(u - v_0)(t)) - NT(\varphi((u - v_0(t)))$$

$$= -Mp(t) - N \int_0^t k(t, s) p(s) ds.$$

By Lemma 1 we know $p(t) \geq 0$, then $u \geq v_0$. By similar method we can obtain $u \leq w_0$. Then the operator A maps $[v_0, w_0] \times [v_0, w_0]$ into $[v_0, w_0]$.

(IV) We shall show that $A : [v_0, w_0] \times [v_0, w_0] \to [v_0, w_0]$ is continuous. For any two sequences $\{h_n\}$ $\{\bar{h}_n\} \subset [v_0, w_0]$ and $h, \bar{h} \in [v_0, w_0]$ with

$$\| h_n - h \|_c \to 0, \| \bar{h}_n - \bar{h} \|_c \to 0 \ (n \to \infty). \qquad \ldots (12)$$

By the compactness of interval $I = [0, a]$ and the continuity of f, it is easy to prove that

$$\| f(t, h_n(t), \bar{h}_n(t), (Th_n)(t), (Sh_n)(t))$$

$$- f(t, h(t), \bar{h}(t), (Th)(t), (Sh)(t)) \|_c \to 0 \ (n \to \infty). \qquad \ldots (13)$$

For any $t \in I$, (11) implies that

$$\| A[h_n, \bar{h}_n](t) - A[h, \bar{h}](t) \|$$

$$\leq \int_0^t \| f(s, h_n(s), \bar{h}_n(s), (Th_n)(s), (Sh_n)(s))$$

$$- f(t, h(s), \bar{h}(s), (Th)(s), (Sh)(s)) \| ds$$

$$+ aM \| h_n - h \|_c + aN \| T(h_n - h) \|_c$$

$$+ N \int_0^t e^{-M(t-s)} \left[\int_0^s k(s, \tau) \| A[h_n, \bar{h}_n](\tau) - A[h, \bar{h}](\tau) \| d\tau \right] ds$$

$$\leq a \| f(t, h_n(t), \bar{h}_n(t), (Th_n)(t), (Sh_n)(t)) - f(t, h(t), \bar{h}(t), (Th)(t), (Sh)(t)) \|_c$$

$$+ aM \| h_n - h \|_c + a^2 Nk_0 \| h_n - h \|_c$$

$$+ Nk_0 \| A[h_n, \bar{h}_n] - A[h, \bar{h}] \|_c \int_0^t e^{-M(t-s)} s ds. \qquad \ldots (14)$$

If $M > 0$ and $aNk_0 (e^{aM} - 1) \leq M$, then

$$aNk_0 (1 - e^{-aM}) = aNk_0 (e^{aM} - 1) e^{-aM} < aNk_0 (e^{aM} - 1) \leq M.$$

Hence, $M - aNk_0 (1 - e^{-aM}) > 0$. From (14), we have

$$\| A[h_n, \bar{h}_n](t) - A[h, \bar{h}](t) \|$$

$$\leq a \| f(t, h_n(t), \bar{h}_n(t), (Th_n(t), (Sh_n)(t)) - f(t, h(t), \bar{h}(t), Th(t), (Sh)(t) \|_c$$

$$+ (aM + a^2 Nk_0) \| h_n - h \|_c + aNk_0 \| A[h_n, \bar{h}_n] - A[h, \bar{h}] \|_c \int_0^t e^{-M(t-s)} ds$$

$$\leq a \| f(t, h_n(t), \bar{h}_n(t), (Th_n)(t), (Sh_n)(t))$$

$$- f(t, h(t), \bar{h}(t), (Th)(t), (Sh)(t) \|_c$$

$$+ (aM + a^2 N k_0) \| h_n - h \|_c$$

$$+ \frac{aNk_0}{M} (1 - e^{-aM}) \| A[h_n, \bar{h}_n] - A[h, \bar{h}] \|_c, \quad t \in I.$$

Therefore, we have

$$\| A[h_n, \bar{h}_n] - A[h, \bar{h}] \|_c$$

$$\leq \frac{M}{M - aNk_0 (1 - e^{-aM})} [a \| f(t, h_n(t), \bar{h}_n(t), (Th_n)(t), (Sh_n)(t))$$

$$- f(t, h(t), \bar{h}(t), (Th)(t), (Sh)(t)) \|_c + (aM + a^2 Nk_0) \| h_n - h \|_c]. \quad \ldots (15)$$

If $a(M + aNk_0) \leq 1$, then by (14), for any $t \in I$, we have

$$\| A[h_n, \bar{h}_n](t) - A[h \bar{h}](t) \|$$

$$\leq a \| f(t, h_n(t), \bar{h}_n(t), (Th_n)(t), (Sh_n)(t)) - f(t, h(t), \bar{h}(t), (Th)(t), (Sh)(t)) \|_c$$

$$+ a(M + aNk_0) \| h_n - h \|_c + Nk_0 \| A[h_n, \bar{h}_n] - A[h, \bar{h}] \|_c \int_0^t sds$$

$$\leq a \| f(t, h_n(t), \bar{h}_n(t), (Th_n)(t), (Sh_n)(t)) - f(t, h(t), \bar{h}(t), (Th)(t), (Sh)(t)) \|_c$$

$$+ \| h_n - h \|_c + \frac{a^2 N k_0}{2} \| A[h_n, \bar{h}_n] - A[h, \bar{h}] \|_c$$

$$\leq a \| f(t, h_n(t), \bar{h}_n(t), (Th_n)(t), (Sh_n)(t)) - f(t, h(t), \bar{h}(t), (Th)(t), (Sh)(t)) \|_c$$

$$+ \| h_n - h \|_c + \frac{1}{2} \| A[h_n, \bar{h}_n] - A[h, \bar{h}] \|_c.$$

Hence

$$\| A[h_n, \bar{h}_n] - A[h, \bar{h}] \|_c \leq 2a \| f(t, h_n(t), \bar{h}_n(t), (Th_n)(t), (Sh_n)(t))$$

$$- f(t, h(t), \bar{h}(t), (Th)(t), (Sh)(t)) \|_c$$

$$+ 2 \| h_n - h \|_c. \qquad \ldots (16)$$

It follows from (12), (13), (15) and (16) that

$$\| A [h_n, \bar{h}_n] - A [h, \bar{h}] \|_c \to 0, n \to \infty.$$

Therefore, the operator $A : [v_0, w_0] \times [v_0, w_0] \to [v_0, w_0]$ is continuous.

(V) In the end we shall show that A has a fixed point $u^* \in [v_0, w_0]$. Define a operator $F : [v_0, w_0] \to [v_0, w_0]$ by $Fu = A [u, u]$. It is evident that $u \in [v_0, w_0]$ is a solution of the IVP (1) iff u is a fixed point of F in $[v_0, w_0]$. For any $u \in [v_0, w_0]$, by (III) we have $v_0 \le A (u, u) = Fu = A (u, u) \le w_0$. Hence, $F : [v_0, w_0] \to [v_0, w_0]$. Let $K = \overline{co} (F ([v_0, w_0]))$. Then F is a continuous operator from K into K by step (IV). It is easy to see that the normality of P implies the normality of P_c, and so, for any $h_1, h_2 \in [v_0, w_0]$, by means of the assumptions (H_1), we have

$$v_0' + M v_0 + NT v_0 \le f(t, h_1, w_0, Th_1, Sh_1) + Mh_1 + NTh_1$$

$$\le f(t, h_1, h_2, Th_1, Sh_1) + Mh_1 + NTh_1$$

$$\le f(t, h_1, v_0, Th_1, Sh_1) + Mh_1 + Th_1$$

$$\le w_0' + Mw_0 + NTw_0,$$

and consequently, the normality of P_c implies $\{f(t, h_1, h_2, Th_1, Sh_1) + Mh_1 + NTh_1 : h_1, h_2 \in [v_0, w_0]\}$ is a bounded set in $C[I, E]$. From these and (11) and the normality of P_c we can see that $K \subset [v_0, w_0]$ is uniformly bounded and equicontinuous on I. Let $B \subset K \subset C [I, E]$ be any countable subset satisfying $\overline{B} = \overline{co} (\{u\} \cup F (B))$ for some $u \in K$. It follows from Lemma 3 that $\overline{B(t)} = \overline{co} (\{u(t)\} \cup (FB) (t))$ for any $t \in I$. From the Lemma 5, we have

$$\alpha ((TB) (s)) = \alpha \left(\int_0^s k (s, \tau) B (\tau) d \tau \right) \le k_0 \int_0^s m (\tau) d \tau, t \in I,$$

$$\alpha ((SB) (s)) = \alpha \left(\int_0^s h (s, \tau) B (\tau) d \tau \right) \le h_0 \int_0^a m (\tau) d \tau, t \in I.$$

By applying the Lemma 4 to (11) and noting (H_2), we get

$$\alpha (B (t)) = \alpha \overline{(B (t))} = \alpha ((FB) (t))$$

$$\leq \alpha\Bigg(x_0 e^{-Mt} + \int_0^t e^{-M(t-s)} [f(s, B(s), B(s), (TB)(s), (SB)(s))$$

$$+ MB(s) + N(TB)(s) - N(T(FB))(s)]\, ds \Bigg)$$

$$\leq 2 \int_0^t [\lambda_1 \alpha(B(s)) + \lambda_2 \alpha((TB)(s)) + \lambda_3 \alpha((SB)(S)$$

$$+ M\alpha(B(s)) + n\alpha((TB)(s)) + N\alpha(T(FB))(s))]\, ds$$

$$\leq 2 \int_0^t \Bigg[\lambda_1 \alpha(B(s)) + \lambda_2 k_0 \int_0^s \alpha(B(\tau))\, d\tau + \lambda_3 h_0 \int_0^a \alpha(B(\tau))\, d\tau$$

$$+ M\alpha(B(s)) + Nk_0 \int_0^s \alpha(B(\tau))\, d\tau + Nk_0 \int_0^s \alpha((FB)(\tau))\, d\tau \Bigg] ds$$

$$\leq 2(\lambda_1 + M) \int_0^t \alpha(B(s))\, ds + 2(\lambda_2 + 2N) k_0 t \int_0^t \alpha(B(s))\, ds$$

$$+ 2\lambda_3 h_0 t \int_0^a \alpha(B(s))\, ds, t \in I. \qquad \ldots (17)$$

Hence, with the Lemma 2 and one of the two conditions (i) and (ii) of (H_2), $\alpha(B(t)) = 0$ for $t \in I$. Thus, by the Ascoli-Arzela theorem, B is relative compact in $X = C[I, E]$. It follows from Lemma 6 that $F(u) = A[u, u]$ has a fixed point $u^* \in K = \overline{co}(F([v_0, w_0])) \subset [v_0, w_0]$, which is a solution of the IVP (1). This completes the proof of Theorem 1.

By further discussing we can obtain the unique solution and its iterative approximation.

Theorem 2 — *Let E be a real Banach space and P be a normal cone in E, assume that conditions $(H_1), (H_3)$ and (H_4) hold, then IVP (1) has a unique solution $u^* \in [v_0, w_0]$. Moreover, there exists iterative sequences $\{u_n\} \subset [v_0, w_0]$, defined as*

$$u_n(t) = x_0 e^{-Mt} + \int_0^t e^{M(s-t)} [f(s, u_{n-1}(s), u_{n-1}(s), (Tu_{n-1})(s), (Su_{n-1}(s))$$

$$+ Mu_{n-1}(s) - N(T(u_n - u_{n-1}))(s)]\, ds, n = 1, 2, 3, \ldots, t \in I, \qquad \ldots (18)$$

which converges uniformly to u^* on $t \in I$ for any initial $u_0 \in [v_0, w_0]$.

PROOF : From (H_1) and the proof of Theorem 1 we can conclude that the operator $A : [v_0, w_0] \times [v_0, w_0] \to [v_0 w_0]$ defined by (11) is continuous and (18) can be written as

$$u_n = A[u_{n-1}, u_{n-1}], n = 1, 2, 3, \ldots. \qquad \ldots (19)$$

We also divide the following proof into three steps.

(I) We'll show that (H_3) can imply (H_2). In fact, for any $B \subset [v_0, w_0]$, it's easy to know that B is bounded because of the normality of P. For any $\varepsilon > 0$, from the definition of noncompactness we know that there exist three kinds of division methods saisfying

$$B = \bigcup_{i=1}^{m_1} B_i, TB = \bigcup_{j=1}^{m_2} TB_j, SB = \bigcup_{k=1}^{m_3} SB_k .$$

$$d(B_i) < \alpha(B) + \varepsilon, i = 1, 2, 3, ..., m_1,$$

$$d(TB_j) < \alpha(TB) + \varepsilon, j = 1, 2, 3, ..., m_2,$$

$$d(SB_k) < \alpha(SB) + \varepsilon, k = 1, 2, 3, ..., m_3.$$

Note that

$$f(t, B, B, TB, SB) = \bigcup_{i=1}^{m_1} \bigcup_{j=1}^{m_2} \bigcup_{k=1}^{m_3} f(t, B_i, B_i, TB_j, SB_k),$$

consequently, it follows from (H_3) that

$$d(f(t, B_i, B_i, TB_j, SB_k)) \le \lambda_1 d(B_i) + \lambda_2 d(TB_j) + \lambda_3 d(SB_k)$$

$$\le \lambda_1 \alpha(B) + \lambda_2 \alpha(TB) + \lambda_3 \alpha(SB)$$

$$+ (\lambda_1 + \lambda_2 + \lambda_3) \varepsilon.$$

From discussing above we know that

$$\alpha(f(t, B, B, TB, SB)) \le \lambda_1 \alpha(B) + \lambda_2 \alpha(TB) + \lambda_3 \alpha(SB).$$

Hence, (H_2) holds.

(II) We'll prove $\{u_n\}$ converges uniformly to a solution $u^* \in [v_0, w_0]$ of IVP (1). In reality, it's easy to see that the normality of P implies the normality of P_c, moreover, $\{u_n\} \subset [v_0, w_0]$, so $\{u_n(t)\}$ is uniformly bounded. From (H_3) we know

$$\{f(t, u_{n-1}(t), u_{n-1}(t), Tu_{n-1}(t), Su_{n-1}(t))\}$$

is uniformly bounded on I, hence

$$\{f(t, u_{n-1}(t), u_{n-1}(t), Tu_{n-1}(t), Su_{n-1}(t))$$

$$+ Mu_{n-1}(t) + NT(u_{n-1})(t)\}$$

is uniformly bounded on I. Then the equicontinuity of sequence $\{u_n\}$ follows from (18). Let $m(t) = \alpha(\{u_n(t) : n = 1, 2, ...\})$. Note the conclusion (I) above, then by the method which was used in proving (17), we get

$$m(t) \le 2(\lambda_1 + M) \int_0^t m(s)\, ds + 2(\lambda_2 + 2N) k_0 t$$

$$\int_0^{t_\varphi} m(s)\, ds + 2\lambda_3 h_0 t \int_0^a m(s)\, ds, t \in I.$$

Hence, by the Lemma 2, $m(t) = 0, t \in I$. Thus by the Ascoli-Arzela theorem $\{u_n\}$ is relatively compact in $C[I, E]$, so there exists subsequence of $\{u_n\}$ which converges uniformly to $u^* \in [v_0, w_0]$. If we can prove

$$\| u_n(t) - u_{n-1}(t) \| \to (n \to \infty), t \in I, \quad \ldots (20)$$

then it is clear from the definition of $\{u_n\}$ that the limit of any subsequence is U^*. It then follows that the full sequence $\{u_n(t)\}$ converges uniformly to u^*in I. Moreover, note (19) as well as that A is continuous, it's clear that u^* is a solution of IVP(1). So let

$$m(t) = \lim_{n \to \infty} \sup \| u_n(t) - u_{n-1}(t) \|, t \in I. \quad \ldots (21)$$

Note that

$$\{f(t, u_{n-1}(t), u_{n-1}(t), Tu_{n-1}(t), Su_{n-1}(t))$$

$$+ Mu_{n-1}(t) + NT(u_{n-1})(t)\}$$

is uniformly bounded on I, that is, there exists $F_0 > 0$ such that for any n

$$\| f(t, u_{n-1}(t), u_{n-1}(t), Tu_{n-1}(t), Su_{n-1}(t)) + Mu_{n-1}(t) + NT(u_{n-1}(t) \|$$

$$\le F_0, t \in I.$$

Then for any $\varepsilon > 0$, for $t_1, t_2 \in I$, we have

$$\| u_n(t_1) - u_{n-1}(t_1) \| \le \| u_n(t_2) - u_{n-1}(t_2) \| + \| u_n(t_2) - u_n(t_1) \|$$

$$+ \| u_{n-1}(t_1) - u_{n-1}(t_2) \|$$

$$\le m(t_2) + 2F_0 | t_2 - t_1 | + \varepsilon,$$

for large n. Therefore for given $\varepsilon > 0$, we know $m(t_1) \le m(t_2) + 2F_0 | t_2 - t_1 | + \varepsilon$. Since t_1, t_2 can be

interchanged and $\varepsilon > 0$ is arbitrary, we get $|m(t_1) - m(t_2)| \le 2F_0 |t_1 - t_2|$, it follows that $m(t) \in C[I, R^+]$. From (H_3) and (18) we know

$$\| u_{n+1}(t) - u_n(t) \| \le \int_0^t [(\lambda_1 + M) \| u_n(s) - u_{n-1}(s) \|$$

$$+ (\lambda_2 + N) \| T(u_n - u_{n-1})(s) \|$$

$$+ \lambda_3 \| S(u_n - u_{n-1})(s) \| + N \| T(u_{n+1} - u_n)(s) \|] ds, t \in I, \quad \text{... (22)}$$

so we have from (22) and Fatou Lemma

$$m(t) = \lim_{n \to \infty} \sup \| u_{n+1}(t) - u_n(t) \|$$

$$\le \int_0^t [(\lambda_1 + M) \lim_{n \to \infty} \sup \| u_n(s) - u_{n-1}(s) \|$$

$$+ (\lambda_2 + N) \lim_{n \to \infty} \sup \left\| \int_0^s k(s, \tau)(u_n(\tau) - u_{n-1}(\tau)) d\tau \right\|$$

$$+ \lambda_3 \lim_{n \to \infty} \sup \left\| \int_0^a h(s, \tau)(u_n(\tau) - u_{n-1}(\tau)) d\tau \right\|$$

$$+ N \lim_{n \to \infty} \sup \left\| \int_0^s k(s, \tau)(u_{n+1}(\tau) - u_n(\tau)) d\tau \right\|] ds$$

$$\le (\lambda_1 + M) \int_0^t m(s) ds + (\lambda_2 + N) k_0 \int_0^t \int_0^s m(\tau) d\tau ds$$

$$+ \lambda_3 h_0 \int_0^t \int_0^a m(\tau) d\tau ds + N k_0 \int_0^t \int_0^s m(\tau) d\tau ds$$

$$\le (\lambda_1 + M) \int_0^t m(s) ds + (\lambda_2 + 2N) k_0 t \int_0^t m(s) ds$$

$$+ \lambda_3 h_0 t \int_0^a m(s) ds, t \in I. \quad \text{... (23)}$$

By applying Lemma 2 to (23), we get $m(t) \equiv 0, t \in I$, consequently (20) holds.

All the above (I)-(III) imply that $\{u_n\}$ converges uniformly to a solution $u^* \in [v_0, w_0]$ of IVP (1).

(III) In the end we shall show the uniqueness of solution. Assume that IVP (1) has two solutions u and $v \in [v_0, w_0]$, then u and v both satisfy (18). Let $m(t) = \| u(t) - v(t) \|$, then $m(t) = 0, t \in I$ by applying the method used in proving (19)-(23), i.e., $u = v$. This completes the uniqueness proof.

Remark 1 : Theorem 2.1 (b), the main results of [6], is the special case of Theorem 1 of this paper when $f = f_0(t, u) + f_1(t, u) + f_2(t, u)$ and it does not contain Tu and Su defined by (2). Moreover[6], implies that $f(t, u, v)$ is nondecreasing in u and nonincreasing in v. We only require $f(t, u, v)$ is nonincreasing in v in this paper. It is worthwhile to point out that we do not demand the key assumption (A_1) of[6] in this paper.

Remark 2 : If f does not contain the third argument In IVP (1), then the condition (H_4) of this paper is surplus, i.e., monotone assumption of f is unnecessary. If f does not contain operator Su either, then we can let $\lambda_3 = 0$. Thus,

⟨ 1 ⟩ On one hand, (ii) of (H_2) holds for any $\lambda_1 \geq 0, \lambda_2 \geq 0$. Now, if M, N in (H_1) satisfy (3) in this paper, then Theorem 3.1 in [2] is very easy. But when M, N of (H_1) satisfy (4), the problem becomes difficult to deal with for [2]. So Theorem 1 of this paper contained and extended Theorem 3.1 of [2].

⟨ 2 ⟩ On the other hand, in this paper 7, holds for any $\lambda_1 \geq 0, \lambda_2 \geq 0$, so Theorem 2.1 of [2] is the special case of Theorem 2 of this paper when f does not contain Su. Thus Theorem 2 of this paper contained and extended Theorem 2.1 of[2].

Remark 3 : In this paper if f does not contain the third argument, nor does it contain Tu and Su, then Theorem 1 and Theorem 2 improved and extended Theorem 2 of[11] and Theorem 2.1 of [7] respectively. At the same time we do not use the compactness type conditions used in[7].

Remark 4 : This paper weakens the monotone assumptions of f compared with[8].

Remark 5 : Our method is different in essence from those of[2, 6, 7, 11].

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