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COMPLEMENT OF A FUZZY GRAPH

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The definition of complement of a fuzzy graph is modified and some properties of self complementary fuzzy graphs are studied. We show that automorphism groups of a fuzzy graph and its complement are identifical and a relative study of complement and some other operations on fuzzy graphs such as union, join and composition has been made.

Key Words : Fuzzy Graph; Fuzzy Subset; Mapping

1. INTRODUCTION

A mathematical frame work to describe the phenomena of uncertainty in real life situation has been suggested by Zadeh in 1965[19]. A fuzzy subset[8] of a set V is a mapping $\sigma: V \to [0, 1]$. For any $u \in V$, $\sigma(u)$ is called the degree of membership of u in σ. If σ is into $\{0, 1\}$, it is the characteristic function of a subset of V denoted by σ^* and is known as 'crisp'. A fuzzy relation on V is a fuzzy subset of $V \times V$. A fuzzy relation μ on V is a fuzzy relation on σ if $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all u, v in V. If μ and ν are fuzzy relations on V, then $\mu \circ \nu(u, w) = \text{Sup } \{\mu(u, v) \wedge \nu(v, w) : v \in V\}$ and $\mu^k(u, v) = \text{Sup } \{\mu(u, u_1) \wedge \mu(u_1, u_2) \wedge \ldots\ldots \wedge \mu(u_{k-1}, v) : u_1, u_2, \ldots, u_{k-1} \in V\}$, where '$\wedge$' stands for minimum.

Research on the theory of fuzzy sets has been witnessing an exponential growth; both within mathematics and in its applications. This ranges from traditional mathematical subjects like logic,

topology, algebra, analysis etc. to pattern recognition, information theory, artificial intelligence, operations research, neural networks and planning etc. [2], [8], [9], [10], [14]. Consequently, fuzzy set theory has emerged as a potential area of interdisciplinary research and fuzzy graph theory is of recent interest.

Rosenfeld[15] considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. During the same time Yeh and Bang[18] have also introduced various connectedness concepts in fuzzy graphs. A fuzzy graph is a pair of functions $G : (\sigma, \mu)$ where σ is a fuzzy subset of V and μ is a symmetric fuzzy relation on σ. We denote the underlying (crisp) graph of $G : (\sigma, \mu)$ by $G^* : (\sigma^*, \mu^*)$ where σ^* is referred to as the nonempty) set V of nodes and $\mu^* = E \subseteq V \times V$. Note that the crisp graph (V, E) is a special case of a fuzzy graph with each vertex and edge of (V, E) having degree of membership 1. A fuzzy graph $G : (\sigma, \mu)$ is a strong fuzzy graph if $\mu(u, v) = \sigma(u) \wedge \sigma(v) \quad \forall\ (u, v) \in \mu^*$ and is a complete fuzzy graph if $\mu(u, v) = \sigma(u) \wedge \sigma(v) \quad \forall\ u, v \in \sigma^*$. A path P of length n is a sequence of distinct nodes $u_0, u_1, ..., u_n$ such that $\mu(u_{i-1}, u_i) > 0, i = 1, 2, ..., n$ and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$ then P is called a cycle. Also, Sup $\{\mu^k(u, v) : k = 1, 2, ...\}$ gives the strength of connectedness between any two nodes u and v denoted by $\mu^\infty(u, v)$. A strongest $u - v$ path is that path which has strength $\mu^\infty(u, v)$. A node is a fuzzy cutnode of $G : (\sigma, \mu)$ if removal of it reduces the strength of connectedness between some other pair of nodes. Equivalently, w is a fuzzy cutnode if and only if there exist u, v distinct from w such that w is on every strongest $u - v$ path. A fuzzy graph is connected if $\mu^\infty(u, v) > 0$ for all u, v.

Consider the fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ with $\sigma_1^* = V_1$ and $\sigma_2^* = V_2$. An isomorphism[5] between two fuzzy graphs G_1 and G_2 is a bijective map $h : V_1 \to V_2$ that satisfies

$$\sigma_1(u) = \sigma_2 h(u) \quad \forall\ u \in V$$

and $$\mu_1(u, v) = \mu_2(h(u), h(v)) \quad \forall\ u, v \in V \text{ and we write } G_1 \approx G_2.$$

An automorphis of G is an isomorphism of G with itself.

After the pioneering work of Rosenfeld[15] and Yeh and Bang[18] in 1975, when some basic fuzzy graph theoretic concepts and applications have been indicated, several authors have been finding deeper results, and fuzzy analogues of many other graph theoretic concepts. This include fuzzy trees[7,16], fuzzy line graphs[11], operations on fuzzy graphs[12], automorphism of fuzzy graphs[3, 5], fuzzy interval graphs[6], cycles and cocyles of fuzzy graphs[13] and metric aspects in fuzzy graphs[17].

In this paper the definition of complement of a fuzzy graph is modified and some properties of self colementary fuzzy graphs are studies. We show that the automorphism groups of a fuzzy graph and its complement are identical. Also we consider the operations of union, join, composition of fuzzy graphs and prove that complement of the union of two fuzzy graphs is the join of their complements and the complement of the join of two fuzzy graphs is the union of their complements. Finally we prove that complement of the composition of two strong fuzzy graphs is the composition of their complements.

Through out this paper, we assume that μ is reflexive and need not consider loops. Also, the underlying set V is assumed to be finite and σ can be chosen in any manner so as to satisfy the definition of a fuzzy graph in all the examples.

2. Complement of a Fuzzy Graph

Mordeson[12] has defined the complement of a fuzzy graph $G:(\sigma, \mu)$ as a fuzzy graph $G^c:(\sigma^c, \mu^c)$ where $\sigma^c = \sigma$ and $\mu^c(u, v) = 0$ if $\mu(u, v) > 0$ and $\mu^c(u, v) = \sigma(u) \wedge \sigma(v)$ otherwise. It follows from this definition that G^c is a fuzzy graph even if G is not and that $(G^c)^c = G$ if and only if G is a strong fuzzy graph. In Fig. 1, $(G^c)^c \neq G$.

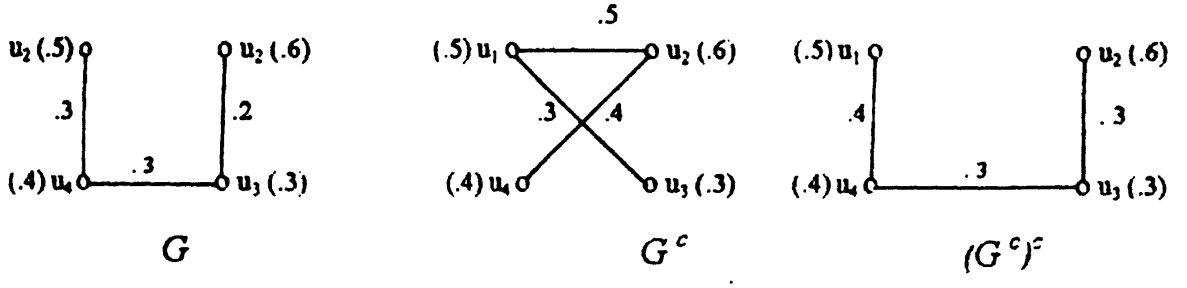


FIG. 1.

Also, automorphism group of G and G^c are not identical; for, consider the fuzzy graph G and G^c in Fig. 2. The automorphism group of G consists of two maps h_1 and h_2 where h_1 is the identity map and h_2 is given by the permutation $(v_1)(v_2\ v_4)(v_3)$. But the automorphism group of G^c consists of four maps h_1, h_2, h_3 and h_4 where h_1 and h_2 are automorphisms of G and h_3 and h_4 are given by $h_3 = (v_1\ v_3)(v_2)(v_4)$ and $h_4 = (v_1\ v_3)(v_2\ v_4)$.

FIG. 2.

These observations motivate us to modify the notion of complement of a fuzzy graph.

Definition 1 — The complement of a fuzzy graph $G:(\sigma, \mu)$ is a fuzzy graph $\overline{G}:(\overline{\sigma}, \overline{\mu})$ where $\overline{\sigma} \equiv \sigma$ and $\overline{\mu}(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v)$ for all u, v in V.

Example —

FIG. 3.

We have

$$\bar{\bar{\sigma}} = \bar{\sigma} = \sigma \text{ and } \bar{\bar{\mu}}(u, v) = \bar{\sigma}(u) \wedge \bar{\sigma}(v) - \bar{\mu}(u, v)$$

$$= \sigma(u) \wedge \sigma(v) = ((\sigma(u) \wedge \sigma(v)) - \mu(u, v))$$

$$= \mu(u, v) \quad \forall\ u, v.$$

Hence $\bar{\bar{G}} = G$.

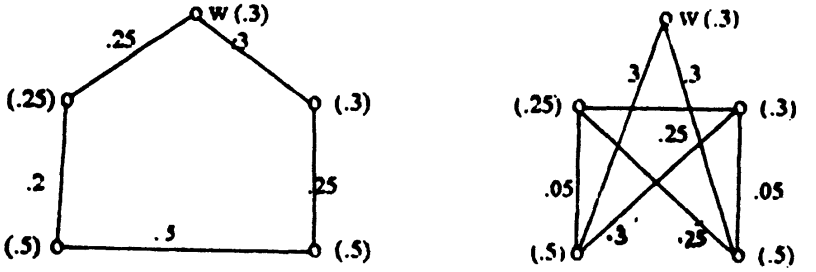


FIG. 4.

Remark 1 : A node can be a fuzzy cut node of both G and $\bar{G}$ as distinct from crisp graph theory. In Fig. 4, w is a fuzzy cutnode of G and $\bar{G}$.

Definition 2 — A fuzzy graph G is self complementary if $G \approx \bar{G}$.

In the following theorems we present a necessary and then a sufficient condition for a fuzzy graph to be self complementary.

Theorem 1 — *Let* $G : (\sigma, \mu)$ *be a selfcomplementary fuzzy graph. Then*

$$\sum_{u \neq v} \mu(u, v) = \frac{1}{2} \sum_{u \neq v} (\sigma(u) \wedge \sigma(v)).$$

PROOF : Let $G : (\sigma, \mu)$ be a selfcomplementary fuzzy graph. Then there exists an isomorphism $h : V \to V$ such that

$$\bar{\sigma} h(u) = \sigma(u) \quad \forall\ u \in V$$

and $\overline{\mu}(h(u), h(v)) = \mu(u, v) \quad \forall\ u, v \in V.$

Now by definition of $\overline{G}$, we have,

$$\overline{\mu}(h(u), h(v)) = \overline{\sigma}(h(u)) \wedge \overline{\sigma}(h(v)) - \mu(h(u), h(v))$$

i.e., $$\mu(u, v) = \sigma(u) \wedge \sigma(v) - \mu(h(u), h(v))$$

i.e., $$\sum_{u \neq v} \mu(u, v) + \sum_{u \neq v} \mu(h(u), h(v)) = \sum_{u \neq v} \sigma(u) \wedge \sigma(v))$$

i.e., $$2 \sum_{u \neq v} \mu(u, v) = \sum_{u \neq v} \sigma(u) \wedge \sigma(v))$$

i.e., $$\sum_{u \neq v} \mu(u, v) = \frac{1}{2} \sum_{u \neq v} (\sigma(u) \wedge \sigma(v)).$$

Hence the theorem.

Remark 2 : If $G : (V, E)$ is a self complementary (crisp) graph, then from Theorem 1, it follows that $2m = \frac{n(n-1)}{2}$ where $m = |E|$ and $n = |V|$ which is equivalent to the result that every self complementary (crisp) graphs has $4k$ or $4k+1$ nodes for some k.

Remark 3 : The condition given in Theorem 1 is not sufficient. In the following example (Fig. 5), G is not isomorphism to $\overline{G}$ but,

$$\sum_{u \neq v} \mu(u, v) = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$$

and $$\frac{1}{2} \sum_{u \neq v} \sigma(u) \wedge \sigma(v) = \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2} + 1\right] = 1.$$

FIG. 5.

Theorem 2 — *Let* $G : (\sigma, \mu)$ *be a fuzzy graph. If* $\mu(u, v) = \frac{1}{2}(\sigma(u) \wedge \sigma(v)) \quad \forall\ u, v \in V$, then G is self complementary.

PROOF : Let $G : (\sigma, \mu)$ bea fuzzy graph such that $\mu(u, v) = \frac{1}{2}(\sigma(u) \wedge \sigma(v)) \quad \forall\ u, v \in V$. Then $G \approx \overline{G}$ under the identity map on V.

Remark 4 : The condition in Theorem 2 is not necessary. In the following example (Fig. 6) $G \approx \overline{G}$ where the isomorphism $h : V \to V$ is given by $h(u) = v, h(v) = x, h(w) = u, h(x) = w, h(y) = y$.

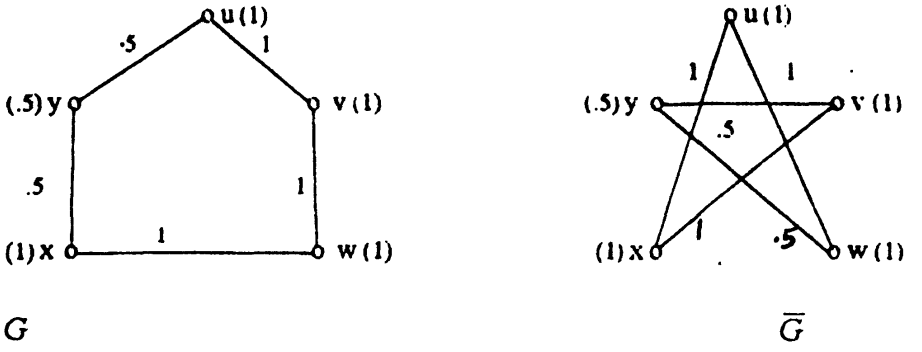


FIG. 6.

Note that if h is an automorphism of G then h^{-1} is also an automorphism of G^5, for;

$$\sigma(h^{-1}(u)) = \sigma(u') = \sigma(h(u')) = \sigma(u)$$

and $$\mu(h^{-1}(u), h^{-1}(v)) = \mu(u', v') = \mu(h(u), h(v)) = \mu(u, v).$$

Similarly if h and g are automorphism of G then their composition $h \circ g$ is also an automorphism of G^3 and we have,

Theorem 3^3 — *The set of all automorphisms of a fuzzy graph G is a group under set-theoretic composition of maps as the binary operation.*

Notation — We denote by ΓG, the group of automorphism of the fuzzy graph G.

Theorem 4 — *Let* $G : (\sigma, \mu)$ *be a fuzzy graph. Then the automorphism groups of* G *and* $\overline{G}$ *are identical.*

PROOF : Note that if $h \in \Gamma G$, then $h \in \Gamma \overline{G}$, for;

$h : V \to V$ is bijective,

$$\overline{\sigma}(h(u)) = \sigma(h(u)) = \sigma(u) = \overline{\sigma}(u)$$

and $$\overline{\mu}(h(u), h(v)) = (\sigma h(u) \wedge \sigma h(v)) - \mu(h(u), h(v))$$

$$= (\sigma(u) \ \sigma(v)) - \mu(u, v)$$

$$= \overline{\mu}(u, v) \ \forall \ u, v \in V.$$

Hence the theorem.

In the following example (Fig. 7), $\Gamma G = \Gamma \overline{G} = \{I, h\}$, where I is the identity map and h is given by the permutation $(v_1, v_3)(v_2, v_4)$.

FIG. 7.

3. OPERATIONS ON FUZZY GRAPHS

The operations on (crisp) graphs such as union join, cartesian product and composition are extended to fuzzy graphs and some of their properties are studied in[12]. In the following discussions an arc between two nodes u and v is denoted by uv rather than (u, v), because in the cartesian product of two graphs, a node of the graph is infact, an ordered pair.

Definition 3[14] — Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ with $V_1 \cap V_2 = \phi$ and let $G^* = G_1^* \cup G_2^* = (V_1 \cup V_2, E_1 \cup E_2)$ be the union of G_1^* and G_2^*. Then the union of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G = G_1 \cup G_2 : (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ defined by

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u) \text{ if } u \in V_1 - V_2 \\ \sigma_2(u) \text{ if } u \in V_2 - V_1 \end{cases}$$

and

$$(\mu_1 \cup \mu_2)(u\,v) = \begin{cases} \mu_1(u\,v) \text{ if } u\,v \in E_1 - E_2 \\ \mu_2(u\,v) \text{ if } u\,v \in E_2 - E_1 \end{cases}$$

Definition 4[14] — Consider the join $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ of graphs where E' is the set of all arcs joining the nodes of V_1 and V_2 where we assume that $V_1 \cap V_2 = \phi$. Then the join of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G = G_1 + G_2 : (\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ defined by

$$(\sigma_1 + \sigma_2)(u) = (\sigma_1 \cup \sigma_2)(u) \text{ " } u \in V_1 \cup V_2 \text{ and}$$

$$(\mu_1 + \mu_2)(u\,v) = \begin{cases} (\mu_1 \cup \mu_2)(u\,v) \text{ if } u\,v \in E_1 \cup E_2 \text{ and} \\ \sigma_1(u) \wedge \sigma_2(v) \text{ if } u\,v \in E'. \end{cases}$$

Theorem 5 — *Let* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *be two fuzzy graphs. Then*

1) $\overline{G_1 + G_2} \approx \overline{G}_1 \cup \overline{G}_2$

and 2) $\overline{G_1 \cup G_2} \approx \overline{G}_1 + \overline{G}_2$

PROOF : We shall prove that the identity map is the required isomorphism.

Let $I : V_1 \cup V_2 \to V_1 \cup V_2$ be the identity map.

Required to prove $\overline{\sigma_1 + \sigma_2}\,(u) = \overline{\sigma}_1 \cup \overline{\sigma}_2\,(u)$

and $\overline{\mu_1 + \mu_2} = (u\ v) = \overline{\mu}_1 \cup \overline{\mu}_2\,(u\ v)$.

$\overline{\sigma_1 + \sigma_2}\,(u) = (\sigma_1 + \sigma_2)\,(u)$ by definition of compliment

$$= \begin{cases} \sigma_1\,(u) & \text{if } u \in V_1 \\ \sigma_2\,(u) & \text{if } u \in V_2 \end{cases}$$

$$= \begin{cases} \overline{\sigma}_1\,(u) & \text{if } u \in V_1 \\ \overline{\sigma}_2\,(u) & \text{if } u \in V_2 \end{cases}$$

$$= \overline{\sigma}_1 \cup \overline{\sigma}_2\,(u).$$

$$\overline{\mu_1 + \mu_2}\,(u\ v) = (\sigma_1 + \sigma_2)\,u \wedge (\sigma_1 + \sigma_2)\,v - (\mu_1 + \mu_2)\,uv$$

$$= \begin{cases} (\sigma_1 \cup \sigma_2)u \wedge (\sigma_1 \cup \sigma_2)\,v - (\mu_1 \cup \mu_2)\,u\,v & \text{if } uv \in E_1 \cup E_2 \\ (\sigma_1 \cup \sigma_2)\,u \wedge (\sigma_1 \cup \sigma_2)\,v - \sigma_1\,(u) \wedge \sigma_2\,(v), & uv \in E' \end{cases}$$

$$= \begin{cases} \sigma_1\,(u) \wedge \sigma_1\,(v) - \mu_1\,(u\ v), & \text{if } uv \in E_1 \\ \sigma_2\,(u) \wedge \sigma_2\,(v) - \mu_2\,(uv) & \text{if } uv \in E_2 \\ \sigma_1\,(u) \wedge \sigma_2\,(v) - \sigma_1\,(u) \wedge \sigma_2\,(v) & \text{if } uv \in E' \text{ where } u \in V_1,\ v \in V_2 \end{cases}$$

$$= \begin{cases} \overline{\mu}_1\,(u\mu), & uv \in E_1 \\ \overline{\mu}_2\,(uv), & uv \in E_2 \\ 0 \qquad , & uv \in E' \end{cases}$$

$$= \overline{\mu}_1 \cup \overline{\mu}_2\,(uv).$$

2) $\overline{G_1 \cup G_2} \approx \overline{G}_1 + \overline{G}_2$.

Let $I : V_1 \cup V_2 \to V_1 \cup V_2$ be the identity map.

To prove $\overline{\sigma_1 \cup \sigma_2}\,(u) = (\overline{\sigma}_1 + \overline{\sigma}_2)\,(uv)$

and $\overline{\mu_1 \cup \mu_2}\,(u\,v) = (\overline{\mu}_1 + \overline{\mu}_2)\,(uv).$

$$\overline{\sigma_1 \cup \sigma_2}\,(u) = (\sigma_1 \cup \sigma_2)\,(u)$$

$$= \begin{cases} \sigma_1\,(u) \text{ if } u \in V_1 \\ \sigma_2\,(u), \text{ if } u \in V_2 \end{cases}$$

$$= \begin{cases} \overline{\sigma}_1\,(u) \text{ if } u \in V_1 \\ \overline{\sigma}_2\,(u), \text{ if } u \in V_2 \end{cases}$$

$$= (\overline{\sigma}_1 \cup \overline{\sigma}_2)\,(u)$$

$$= (\overline{\sigma}_1 + \overline{\sigma}_2)\,(u).$$

$$\overline{\mu_1 \cup \mu_2}\,(uv) = (\sigma_1 \cup \sigma_2)\,u \wedge (\sigma_1 \cup \sigma_2)\,v - (\mu_1 \cup \mu_2)\,(u\,v)$$

$$= \begin{cases} \sigma_1\,(u) \wedge \sigma_1\,(v) - \mu_1\,(u\,v), \text{ if } uv \in E_1 \\ \sigma_2\,(u) \wedge \sigma_2\,(v) - \mu_2\,(uv), \text{ if } uv \in E_2 \\ \sigma_1\,(u) \wedge \sigma_2\,(v) - 0, \text{ when } u \in V_1, v \in V_2 \end{cases}$$

$$= \begin{cases} \overline{\mu}_1\,(uv), & \text{if } uv \in E_1 \\ \mu_2\,(uv), & \text{if } uv \in E_2 \\ \sigma_1\,(u) \wedge \sigma_2\,(v), & \text{if } u \in V_1, v \in V_2 \end{cases}$$

$$= \begin{cases} (\overline{\mu}_1 \cup \overline{\mu}_2)\,uv & \text{if } uv \in E_1 \text{ or } E_2 \\ \sigma_1\,(u) \wedge \sigma_2\,(v) & \text{if } uv \in E^1 \end{cases}$$

$$= (\overline{\mu}_1 + \overline{\mu}_2)\,(uv).$$

Definition 5^{14} — Let $G^* = G_1^* \times G_2^* = (V, E'')$ be the cartesian product of two graphs where $V = V_1 \times V_2$ and $E'' = \{(u, u_2)\,(u, v_2) : u \in V_1, u_2\,v_2 \in E_2\} \cup \{(u_1, w)\,(v_1, w) : w \in V_2, u_1\,v_1 \in E_1\}$. Then the cartesian product $G = G_1 \times G_2 : (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ is a fuzzy graph defined by

$$(\sigma_1 \times \sigma_2)\,(u_1, u_2) = \sigma_1\,(u_1) \wedge \sigma_2\,(u_2) \quad \forall\ (u_1, u_2) \in V$$

and $$\mu_1 \times \mu_2\,((u, u_2)\,(u\,v_2)) = \sigma_1\,(u) \wedge \mu_2\,(u_2\,v_2) \quad \forall\ u \in V_1,\ \forall\ u_2\,v_2 \in E_2,$$

$$\mu_1 \times \mu_2\,((u_1, w)\,(v_1, w)) = \sigma_2\,(w) \wedge \mu_1\,(u_1\,v_1) \quad \forall\ w \in V_2,\ \forall\ u_1\,v_1 \in E_1.$$

Definition 6[14] — Let $G^* = G_1^* \circ G_2^* = (V_1 \times V_2, E)$ be the composition of two graphs, where $E \neq \{(u, u_2)(u, v_2) : u \in V_1, u_2 v_2 \in E_2\} \cup \{(u_1, w)(v_1, w) : w \in V_2, u_1 v_1 \in E_1\} \cup \{(u_1, u_2)(v_1, v_2) : u_1 v_1 \in E_1, u_2 \neq v_2\}$. Then the composition of fuzzy graphs $G = G_1 \circ G_2 : (\sigma_1 \circ \sigma_2, \mu_1 \circ \mu_2)$ is a fuzzy graph defined by

$$(\sigma_1 \circ \sigma_2)(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2), \quad \forall\ (u_1, u_2) \in V_1 \times V_2$$

and

$$(\mu_1 \circ \mu_2)((u, u_2)(u, v_2)) = \sigma_1(u) \wedge \mu_2(u_2 v_2), \quad \forall\ u \in V_1, \ \forall\ u_2 v_2 \in E_2;$$

$$(\mu_1 \circ \mu_2)((u_1, w)(v_1, w)) = \sigma_2(w) \wedge \mu_1(u_1 v_1) \quad \forall\ w \in V_2, \ \forall\ u_1 v_1 \in E_1;$$

$$(\mu_1 \circ \mu_2)((u_1, u_2)(v_1, v_2)) = \sigma_2(u_2) \wedge \sigma_2(v_2) \wedge \mu_1(u_1 v_1), \quad \forall$$

$$(u_1, u_2)(v_1, \mu_2) \in E - E',$$

where

$$E'' = \{(u, u_2)(u, v_2) : u \in V_1, \ \forall\ u_2 v_2 \in E_2\}$$

$$\cup \{(u_1, w)(v_1, w) : w \in V_2, u_1 v_1 \in E_1\}.$$

Remark 5 : Note that if G is a strong fuzzy graph, then $\overline{G}$ is also strong, for, let $u v \in \mu^*$, then

$$\overline{\mu}(u v) = \sigma(u) \wedge \sigma(v) - \mu(u v) = \sigma(u) \wedge \sigma(v) - \sigma(u) \wedge \sigma(v) = 0$$

and if $u v \notin \mu^*$ then

$$\overline{\mu}(u v) = \sigma(u) \wedge \sigma(v) - \mu(u v) = \sigma(u) \wedge \sigma(v) - 0 = \sigma(u) \wedge \sigma(v).$$

Theorem 6 — *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two strong fuzzy graphs. Then $G_1 \circ G_2$ is a strong fuzzy graph and $\overline{G_1 \circ G_2} \simeq \overline{G}_1 \circ \overline{G}_2$.*

PROOF : Let $G_1 \circ G_2 = G : (\sigma, \mu)$ where $\sigma = \sigma_1 \circ \sigma_2$, $\mu = \mu_1 \circ \mu_2$ and $G^* = (V, E)$ where $V = V_1 \times V_2$, $E = \{(u, u_2)(u, v_2) : u \in V_1, u_2 v_2 \in E_2\} \cup \{(u_1, w)(v_1, w) : w \in V_2, u_1 v_1 \in E_1\} \cup \{(u_1, u_2)(v_1, v_2) : u_1 v_1 \in E_1, u_2 \neq v_2\}$.

Now,

1. $\mu(u, u_2)(u, v_2) = \sigma_1(u) \wedge \mu_2(u_2, v_2)$

$= \sigma_1(u) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2)$, G_2 being strong

$= (\sigma_1(u) \wedge \sigma_2(u_2)) \wedge (\sigma_1(u) \wedge \sigma_2(v_2))$

$= (\sigma_1 \circ \sigma_2)(u, u_2) \wedge (\sigma_1 \circ \sigma_2)(u, v_2)$

$$= \sigma(u, u_2) \wedge \sigma(u, v_2)$$

2. $\mu(u_1, w)(v_1, w) = \sigma_2(w) \wedge \mu_1(u_1, v_1)$

$$= \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(w), G_1 \text{ being strong}$$

$$= (\sigma_1(u_1) \wedge \sigma_2(w)) \wedge (\sigma_1(v_1) \wedge \sigma_2(w))$$

$$= (\sigma_1 \circ \sigma_2)(u_1, w) \wedge (\sigma_1 \circ \sigma_2)(v_1, w)$$

$$= \sigma(u_1, w) \wedge \sigma(v_1, w)$$

and 3. $\mu(u_1, u_2)(v_1, v_2) = \mu_1(u_1, v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2)$

$$= \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2), G_1 \text{ being strong}$$

$$= (\sigma_1(u_1) \wedge \sigma_2(u_2)) \wedge (\sigma_1(v_1) \wedge \sigma_2(v_2))$$

$$= (\sigma_1 \circ \sigma_2)(u_1, u_2) \wedge (\sigma_1 \circ \sigma_2)(v_1, v_2)$$

$$= \sigma(u_1, u_2) \wedge \sigma(v_1, v_2).$$

Thus from 1, 2 and 3 it follows that G is a strong fuzzy graph.

Next, to prove $\overline{G_1 \circ G_2} = \overline{G}_1 \circ \overline{G}_2$.

Let $\overline{G} : (\sigma, \overline{\mu}) = \overline{G_1 \circ G_2}$ $\overline{\mu} = \overline{\mu_1 \circ \mu_2}$, $\overline{G}^* = (V, \overline{E})$;

$\overline{G} : (\sigma, \overline{\mu}) = G_1^* = (V_1, \overline{E}_1)$;

$\overline{G}_2 : (\sigma_2, \overline{\mu}) = G_2^* = (V_2, \overline{E}_2)$

and $\overline{G}_1 \circ \overline{G}_2 : (\sigma_1 \circ \sigma_2, \overline{\mu}_1 \circ \overline{\mu}_2)$.

Now, the various types of arcs (say) e, joining the nodes of V are the following and it suffices to prove that $\overline{\mu_1 \circ \mu_2} = \overline{\mu}_1 \circ \overline{\mu}_2$ in each case.

Case I — $e = (u, u_2)(u, v_2)$, $u_2 v_2 \in E_2$

Then $e \in E$ and G being strong $\overline{\mu}(e) = 0$. Also, $\overline{\mu}_1 \circ \overline{\mu}_2(e) = 0$, since $u_2 v_2 \approx \overline{E}_2$.

Case II — $e = (u, u_2)(u, v_2)$, $u_2 \neq v_2$ and $u_2 v_2 \in E_2$

Here $e \in E$, so $\mu(e) = 0$.

Now,

$$\overline{\mu}(e) = \sigma(u, u_2) \wedge \sigma(u, v_2)$$

$$= (\sigma_1(u) \wedge \sigma_2(u_2)) \wedge (\sigma_1(u) \wedge \sigma_2(v_2)) \text{ and since}$$

$u_2\ v_2 \in \overline{E}_2$, we have,

$$\overline{\mu}_1 \circ \overline{\mu}_2 (e) = \sigma_1 (u) \wedge \overline{\mu}_2 (u_2\ v_2)$$

$$= \sigma_1 (u) \wedge \sigma_2 (u_2) \wedge \sigma_2 (v_2), \overline{G}_2 \text{ being strong}$$

$$= \overline{\mu} (e).$$

Case III — $e = (u_1, w)\ (v_1, w), u_1\ v_1 \in E_1$

Then $e \in E$, so $\overline{\mu} (e) = 0$ as in Case 1.

Also, since $u_1\ v_1 \notin \overline{E}_1$, we have $\overline{\mu}_1 \circ \overline{\mu}_2 (e) = 0$.

Case IV — $e = (u_1, w)\ (v_1, w), u_1\ v_1 \notin E_1$

Here $e \notin E$, hence $\mu (e) = 0$ and

$$\overline{\mu} (e) = \sigma (u_1, w) \wedge \sigma (v_1, w)$$

$$= \sigma_1 (u_1) \wedge \sigma_1 (v_1) \wedge \sigma_2 (w) \text{ and since } u_1\ v_1 \in \overline{E}_1$$

we have, $$\overline{\mu}_1 \circ \overline{\mu}_2 (e) = \sigma_2 (w) \wedge \overline{\mu}_1 (u_1\ v_1)$$

$$= \sigma_2 (w) \wedge \sigma_1 (u_1) \wedge \sigma_1 (v_1), \overline{G}_1 \text{ being strong}$$

$$= \overline{\mu} (e).$$

Case V — $e = (u_1, u_2)\ (v_1, v_2), u_1\ v_1 \in E_1$ and $u_2 \neq v_2$

Here $e \in E$, so $\overline{\mu} (e) = 0$ as in Case 1.

Also, since $u_1\ v_1 \in \overline{E}_1$, we have $\overline{\mu}_1 \circ \overline{\mu}_2 (e) = 0$.

Case VI — $e = (u_1, u_2)\ (v_1, v_2), u_1\ v_1 \in E_1$ and $u_2 \neq v_2$

Then $e \notin E$, hence $\mu (e) = 0$

Thus $$\overline{\mu} (e) = \sigma (u_1, u_2) \wedge \sigma (v_1, v_2)$$

$$= \sigma_1 (u_1) \wedge \sigma_1 (v_1) \wedge \sigma_2 (u_2) \wedge \sigma_2 (u_2)$$

and since $u_1\ \upsilon_1 \in \overline{E}_1$, we have,

$$\overline{\mu}_1 \circ \overline{\mu}_2 = \overline{\mu}_1 (u_1\ v_1) \wedge \sigma_2 (u_2) \wedge \sigma_2 (v_2)$$

$$= \sigma_1 (u_1) \wedge \sigma_1 (v_1) \wedge \sigma_2 (u_2) \wedge \sigma_2 (v_2), \overline{G}_1 \text{ being strong}$$

$$= \overline{\mu} (e).$$

Case VII — $e = (u_1, u_2)\ (v_1, v_2), u_1\ v_1 \notin E_1, u_2\ v_2 \notin E_2$

Here $e \notin E$, hence $\mu (e) = 0$

Thus $\overline{\mu}(e) = \sigma(u_1, u_2) \wedge \sigma(v_1, v_2)$

$= \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(v_2)$

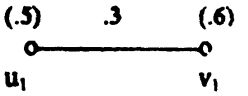
Now, $u_1 v_1 \in \overline{E}_1$ and if $u_2 = v_2 = w$, then we have Case IV.

Next, if $u_1, v_1 \in \overline{E}_1$ and $u_2 \neq v_2$, then we have Case VI.

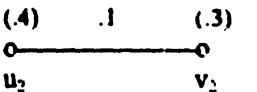
Thus from Cases I to VI, it follows that $\overline{G_1 \circ G_2} \approx \overline{G}_1 \circ \overline{G}_2$.

Remark 6 : In general $\overline{G_1 \circ G_2} \neq \overline{G}_1 \circ \overline{G}_2$. Consider the following example in which G_1 and G_2 are not strong.

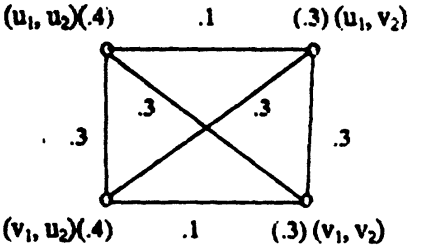
Thus $\overline{G_1 \circ G_2} \neq \overline{G}_1 \circ \overline{G}_2$.



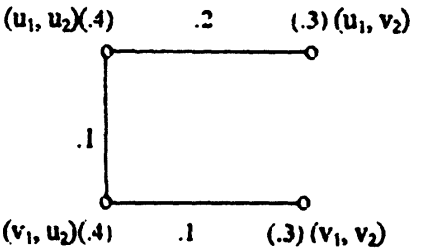
G_1



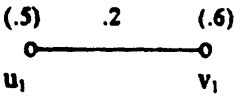
G_2

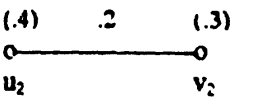


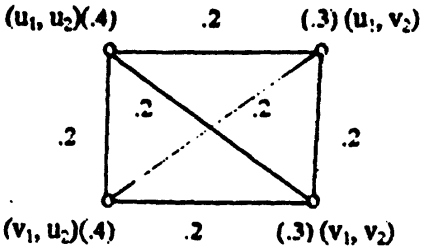
$G_1 \circ G_2$



$\overline{G_1 \circ G_2}$







$\overline{G}_1 \circ \overline{G}_2$

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