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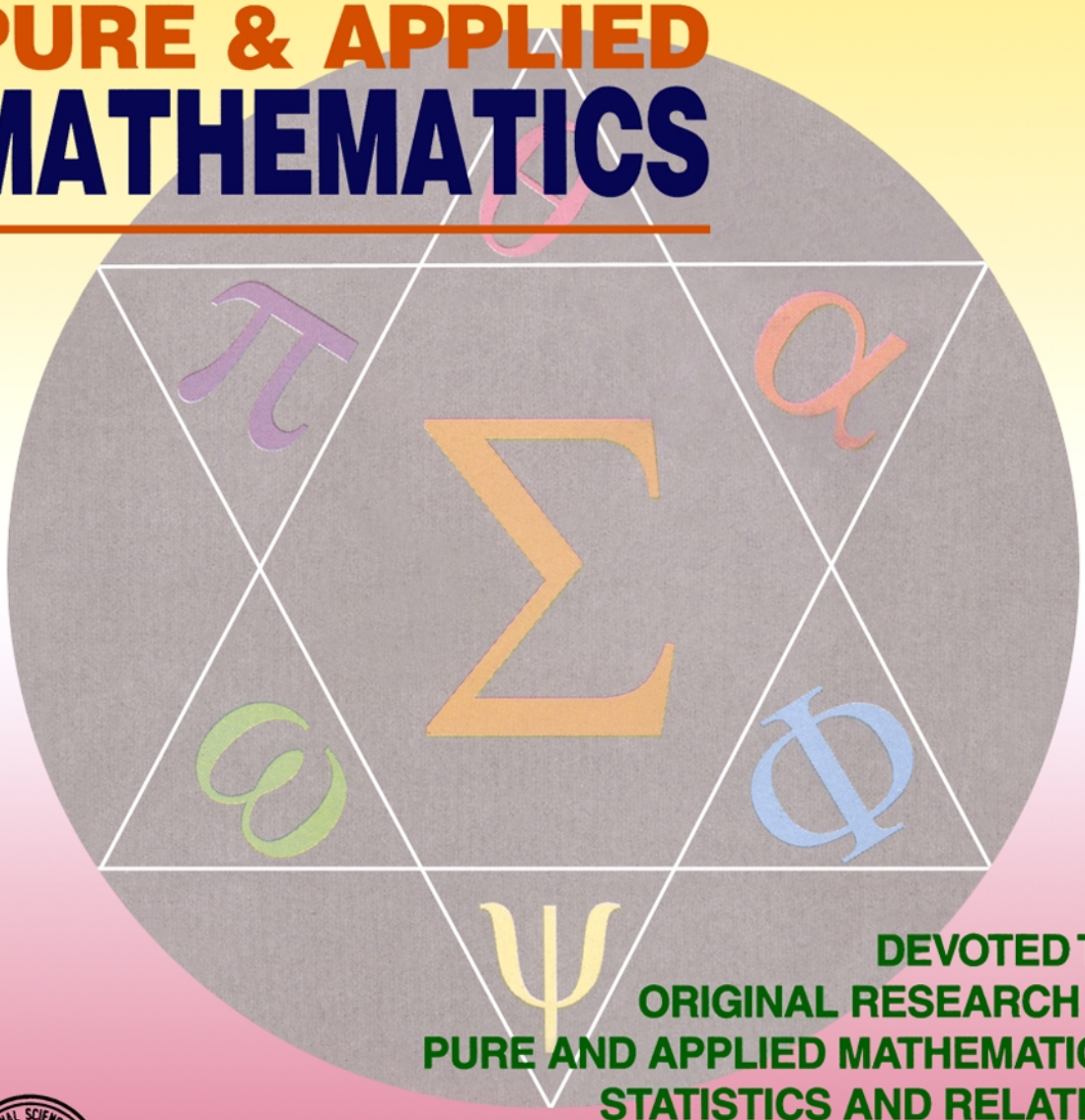

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SOME STRUCTURAL CONSIDERATIONS ON THE THEORY OF GRAVITATIONAL FIELD IN FINSLER SPACES — IV

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Some structural features underlying the Finslerian gravitational field are considered by emphasizing the concept of "nonlocalization" carried by the internal variables. In essence, it is shown that the concept of "nonlozalization" is geometrically absorbed into the covariant derivatives.

Key Words : Finslerian Gravitational Field; Nonlocalization; Covariant Derivatives

1. INTRODUCTION

In the theory of gravitational field in Finsler spaces (Ikeda[1], Ikeda[2]), the vector $\boldsymbol{y}$ $(=y^i;\ i = 1, 2, 3, 4)$ is attached to each point $\boldsymbol{x}$ $(=x^j;\ j = 1, 2, 3, 4)$ as the independent internal variable. In general, it may be said (Yukawa[3]) that the concept of "nonlocalization" is embodied by the internal variable, so that the Finslerian gravitational field is regarded as a "nonlocal" field "nonlocalized" by the internal vector $\boldsymbol{y}$.

In other words, the Finslerian gravitational field is considered a "unified" field between the external $(\boldsymbol{x})$-field spanned by points $\{\boldsymbol{x}\}$ and the internal (y)-field spanned by vectors $\{\boldsymbol{y}\}$. Therefore, the Finslerian gravitational field is regarded as the "nonlocal" field obtained by "nonlocalizing" the Riemannian $(\boldsymbol{x})$-field. Namely, we may say, in general, that the "nonlocal" field is characterized by some internal variable, which causes the "deviation" from the Riemannian field (or the Riemann geometry) (Ikeda[4]).

Along these lines, in this paper, we shall take up some structural features underlying the Finslerian gravitational field by emphasizing the concept of "nonlocalization" caused by the internal variables. Namely, we shall develop some epistemological considerations on the concept of "nonlocalization".

2. ON THE "NONLOCALIZATION" — I

As mentioned in Section 1, the Finslerian gravitational field is regarded as the "nonlocal" field "nonlocalized" by the internal variable y $(= y^i;\ i = 1, 2, 3, 4)$. In this case, the first feature reflecting the concept of "nonlocalization" is represented by the following adapted frame (Ikeda[1], Miron and Anastasiei[5])

$$\begin{cases} X^A \equiv (dx^i,\ \delta y^i = dy^i + N^i_j\, dx^j), \\ X_A \equiv \left(\dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - N^j_i \dfrac{\partial}{\partial y_j},\ \dfrac{\partial}{\partial y^i} \right), \end{cases} \qquad \text{... (2.1)}$$

where N^i_j denotes the nonlinear connection, which arises from the attachment of the internal vector y (or the internal (y)-field) to each point x of the external (x)-field.

Physically, δy shows the intrinsic behaviour of y itself (Ikeda[2], Ikeda[6]): For example, if the gauge transformation of y is given by $\bar{y}^i = K^i_j(x)\, y^j$, then δy is formulated as

$$\begin{cases} \delta y^i = dy^i - \dfrac{\partial K^i_k}{\partial x^j} y^k\, dx^j \\ \equiv dy^i + N^i_j\, dx^j \end{cases} \qquad \text{... (2.2)}$$

where $K^i_j(x) = K^i_j(0) + \dfrac{\partial K^i_j}{\partial x^k} dx^k$, and $dy^i \equiv \bar{y}^i - K^i_j(o)\, y^j$ and $N^i_j \equiv -\dfrac{\partial K^i_k}{\partial x^i} y^k$.

On the basis of the adapted frame (2.1), the "nonlocalized" Finslerian connection structure is prescribed by

$$\begin{cases} \nabla_{x_c} X_B = \Gamma^A_{BC} X_A; \\ \Gamma^A_{BC} \equiv (L^i_{jk},\ C^i_{jk}). \end{cases} \qquad \text{... (2.3)}$$

Then, the following covariant derivatives can be introduced, for an arbitrary vector $V\,(= V^i(x, y))$,

$$\begin{cases} V^i_{lk} = \dfrac{\delta V^i}{\delta x^k} + L^i_{jk} V^i, \\ V^i\,{}_{l_k} = \dfrac{\partial V^i}{\partial y^k} + C^i_{jk} V^j \end{cases} \qquad \text{... (2.4)}$$

Of course, the connection coefficient C^i_{jk} embodies the concept of "nonlocalization". That is to say, if $C^i_{jk} = 0$, then the resulting field becomes Riemannian. This fact insists that the concept of "nonlocalization" is geometrically absorbed into the covariant derivative (V^i/k).

From (2.4), if the condition $y^i_{lk} = 0$ is imposed, then the nonlinear connection N^i_j is determined by $N^i_j = L^i_{kj} y^k$.

By the way, the formulations mentioned above can also be applied to a scalar y^0, where the vector y^i is reduced to the scalar y^0. In this case, N^i_j is reduced to N^0_j (i.e., $\delta y^0 = dy^0 + N^0_j\, dx^j$) and

$\Gamma^{A}_{BC} \equiv (L^{i}_{jk}, L^{0}_{0k}, C^{i}_{j0}, C^{0}_{00})$.

Concerning the "nonlocalized" Finslerian metrical structure G of the Finslerian gravitational field, it should be given by

$$G \equiv g_{ij}(x, y)\, dx^{i} \otimes dx^{j} + g_{ij}(x, y)\, \delta y^{i} \otimes \delta y^{j}. \qquad \ldots (2.5)$$

One concrete form of the Finslerian metric tensor $g_{ij}(x, y)$ can be proposed as follows (Ikeda[1], Ikeda[7]): If the external (x)-field is governed by the Riemannian metric tensor $\gamma_{ij}(x)$, then one example of $g_{ij}(x, y)$ is given by

$$g_{ij}(x, y) = \gamma_{ij}(x) + h_{ij}(x, y), \qquad \ldots (2.6)$$

where the Riemannian metric tensor $h_{ij}(y)$ of the internal (y)-field is embedded on the external (x)-field by some kind of embedding operator $e^{i}_{j}(x)$ in the form $h_{ij}(x, y) = e^{k}_{i}\, e^{l}_{j}\, h_{kl}(y)$. (Of course, some other kinds of $g_{ij}(x, y)$ can be considered in various ways). Furthermore, if (2.6) is rewritten as

$$g_{ij}(x, y) = \gamma_{ij}(x)\ \exp\ 2\,\sigma(x, y), \qquad \ldots (2.7)$$

then the conformal scalar $\sigma(x, y)$ is stipulated as

$$\exp 2\,\sigma(x, y) = 1 + \frac{(h_{ij}(x, y)\, y^{i}\, y^{j})}{(\gamma_{ij}(x)\, y^{i}\, y^{j})}. \qquad \ldots (2.8)$$

3. ON THE "NONLOCALIZATION" — II

In this Section, we shall consider two generalizations of the concept of "nonlocalization" mentioned in Section 2.

First, if another vector $z\,(=z^{i};\ i = 1, 2, 3, 4)$ is further adopted as the independent internal variable at one more microscopic level than the Finslerian y-level, then the generalized "nonlocalized" adapted frame must be introduced as follows, in comparison with the second-order vector bundle-like structure (Miron and Anastasiei[5], Miron[8]):

$$\begin{cases} X^{A} \equiv (dx^{i},\ \delta y^{i} = dy^{i} + N^{i}_{j}\, dx^{ij},\ \delta z^{ij} = dz' + M^{i}_{j}\, dx^{j} + Q^{i}_{j}\, dy^{j}), \\ X_{A} \equiv \left(\dfrac{\delta}{\delta x^{i}} = \dfrac{\partial}{\partial x^{i}} - N^{j}_{i} \dfrac{\partial}{\partial y^{j}} - M^{j}_{i} \dfrac{\partial}{\partial z^{j}},\ \dfrac{\delta}{\delta y^{i}} = \dfrac{\partial}{\partial y^{i}} - Q^{j}_{i} \dfrac{\partial}{\partial z^{j}},\ \dfrac{\partial}{\partial z^{i}} \right) \end{cases} \qquad \ldots (3.1)$$

where three kinds of nonlinear connections N^{i}_{j}, M^{i}_{j} and Q^{i}_{j} must be introduced. In this case, the connection structure is represented by (see (2.3) and (2.4))

$$\left\{\begin{array}{l} \Gamma^A_{BC} \equiv (L^i_{jk},\ C^i_{jk},\ E^i_{jk}); \\ V^i/k = \dfrac{\delta V^i}{\delta x^k} + L^i_{jk}\, V^j, \\ V^i/k = \dfrac{\delta V^i}{\delta y^k} + C^i_{jk}\, V^j, \\ V^i/k = \dfrac{\partial V^i}{\partial z^k} + E^i_{jk}\, V^j. \end{array}\right. \qquad \ldots (3.2)$$

The concept of "nonlocalization" is embodied by the connection coefficients C^i_{jk} and E^i_{jk}, of course. And the metrical structure (2.5)is generalized as follows :

$$G \equiv g_{ij}(x, y, z)\, dx^i \otimes dx^j + g_{ij}(x, y, z)\, \delta y^j \otimes \delta y^j + g_{ij}(x, y, z)\, \delta z^i \otimes \delta z^j. \qquad \ldots (3.3)$$

Secondly, if any non-vectorial internal variables $y^{(\alpha)}(= y^{(\alpha)\,i};\ \alpha = 1, 2, 3, \ldots, m)$ are adopted at more microscopic levels, then the spatial structure of this most generally "nonlocalized" gravitational field becomes higher-order space-like. Then, the most generally "nonlocalized" adapted frame is set as follows, in comparison with the base connection in the theory of higher-order spaces (Kawaguchi[9], Kawaguchi[10]):

$$\left\{\begin{array}{l} X^A \equiv \left(dx^i,\ \delta^{y(\alpha)\,i} = M^{(\alpha_i)}_j \left(dy^{(\alpha)}{}_j + \sum_{\beta+1}^{\alpha-1} \Lambda^{(\alpha_i)}_{(\beta)\,k}\, dy^{(\beta)\,k} + \Lambda^{(\alpha)}_{0)\,k\,i}\, dx^k \right) \right), \\ X_A \equiv \left(\dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} + \sum_{\alpha+1}^{m} \prod_{(0)\,i}^{(\alpha k)} \dfrac{\partial}{\partial y^{(\alpha)\,k}},\ \dfrac{\delta}{\delta y^{(\alpha)\,k}} = \dfrac{\partial}{\partial y^{(\alpha)\,k}} + \sum_{\beta=\alpha+1}^{m} \prod_{(\alpha)\,k}^{(\beta)\,j} \dfrac{\partial}{\partial y^{(\beta)}{}_j} \right) \end{array}\right. \qquad \ldots (3.4)$$

where we have put

$$\left\{\begin{array}{l} M^{(\alpha)\,i}_k\, dy^{(\alpha)\,k} = \delta y^{(\alpha)\,i} + \sum_{\beta=1}^{\alpha-1} \prod_{(\beta)\,k}^{(\alpha)\,i} \delta y^{(\beta)\,k} + \prod_{(0)\,k}^{(\alpha)i} dx^k \\ \left(\begin{array}{l} M^{(0)\,i}_k\, dy^{(0)\,k} \equiv \delta y^{(0)\,i} + \prod_{(0)\,k}^{(0)\,i} dx^k, \\ \\ M^{(1)i}_k\, dy^{(1)k} = \delta y^{(1)\,i} + \prod_{(0)\,k}^{(1)\,i} dx^k;\ dy^{(0)} = \delta y^{(0)} = dx \end{array} \right) \end{array}\right. \qquad \ldots (3.5)$$

In this case, the connection structure is given by

$$\Gamma^A_{BC} \equiv \left(L^i_{jk},\ L^{(\alpha)\,i}_{(\beta)_j\,k},\ C^i_{j\,(\gamma)\,k},\ C^{(\alpha)\,i}_{(\beta)_j\,(\gamma)\,k} \right), \qquad \ldots (3.6)$$

and the metrical structure is given by

$$G \equiv g_{ij}(x, y^{(\alpha)})\, dx^i \otimes dx^j + g_{(\alpha)\, i\, (\beta)_j}(x, y(\alpha))\, \delta y^{(\alpha)_i} \otimes \delta y^{(\beta) j} \qquad \ldots (3.7)$$

The concept of "nonlocalization" is typically embodied by the connection coefficients $C^{i}_{j\,(r)\,k}$ and $C^{(\alpha)\,i}_{(\beta)\,j\,(r)\,k}$.

In order to consider more physically, if we liken the non-vector $y^{(\alpha)}$ to the spinor ψ, at one bounce, then we can set the spinorially "nonlocalized" adapted frame as follows (Ikeda[1], Sardanashivily[11]) :

$$\begin{cases} X^A \equiv (dx^i,\ \delta\psi = d\psi + H\psi\, d\psi + U_i\, \psi\, dx^i \equiv P\, d\psi + R_i\, dx^i), \\ X_A \equiv \left(\dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - R_i \dfrac{\delta}{\delta \psi},\ \dfrac{\delta}{\delta \psi} = (p^{-1}) \dfrac{\partial}{\partial \psi} \right) \end{cases} \qquad \ldots (3.8)$$

where $P = I + H\psi$ and $R_i \equiv U_i\, \psi$. Then, if we take up a spinorial "nonlocal" field such as $\overline{\Phi}(x, \psi)$, then we can put the covariant derivatives of $\overline{\Phi}(x, \psi)$ as follows :

$$\begin{cases} D\overline{\Phi} = d\overline{\Phi} - H_i\, \overline{\Phi}\, dx^i - K\, \overline{\Phi}\, d\psi \\ \quad = (\overline{\Phi}/i)\, dx^i + (\overline{\Phi}/)\, \delta\psi, \\ \overline{\Phi}/i = \dfrac{\delta \overline{\Phi}}{\delta x^i} - H_i\, \overline{\Phi}, \\ \overline{\Phi}/ = \dfrac{\delta \overline{\Phi}}{\delta \psi} - \overline{K}\, \overline{\Phi} \end{cases} \qquad \ldots (3.9)$$

where H_i and K are the spin gauge fields, and $\overline{H}_i \equiv H_i - R_i \overline{K}$ and $\overline{K} \equiv (p^{-1})\, k$. Thereofore, the concept of "nonlocalization" is carried by the spin gauge field $\overline{K}$ and the covariant derivative $(\overline{\Phi}/)$.

4. CONCLUSION

Thus, it is understood from the above that the concept of "nonlocalization" caused by the internal variables such as y, $y^{(\alpha)}$ or ψ is, in essence, geometrically absorbed into the connection structure or the covariant derivatives. And it is also emphasized that these formluations to obtain the covariant derivatives can be applied to any kind of internal variable.

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