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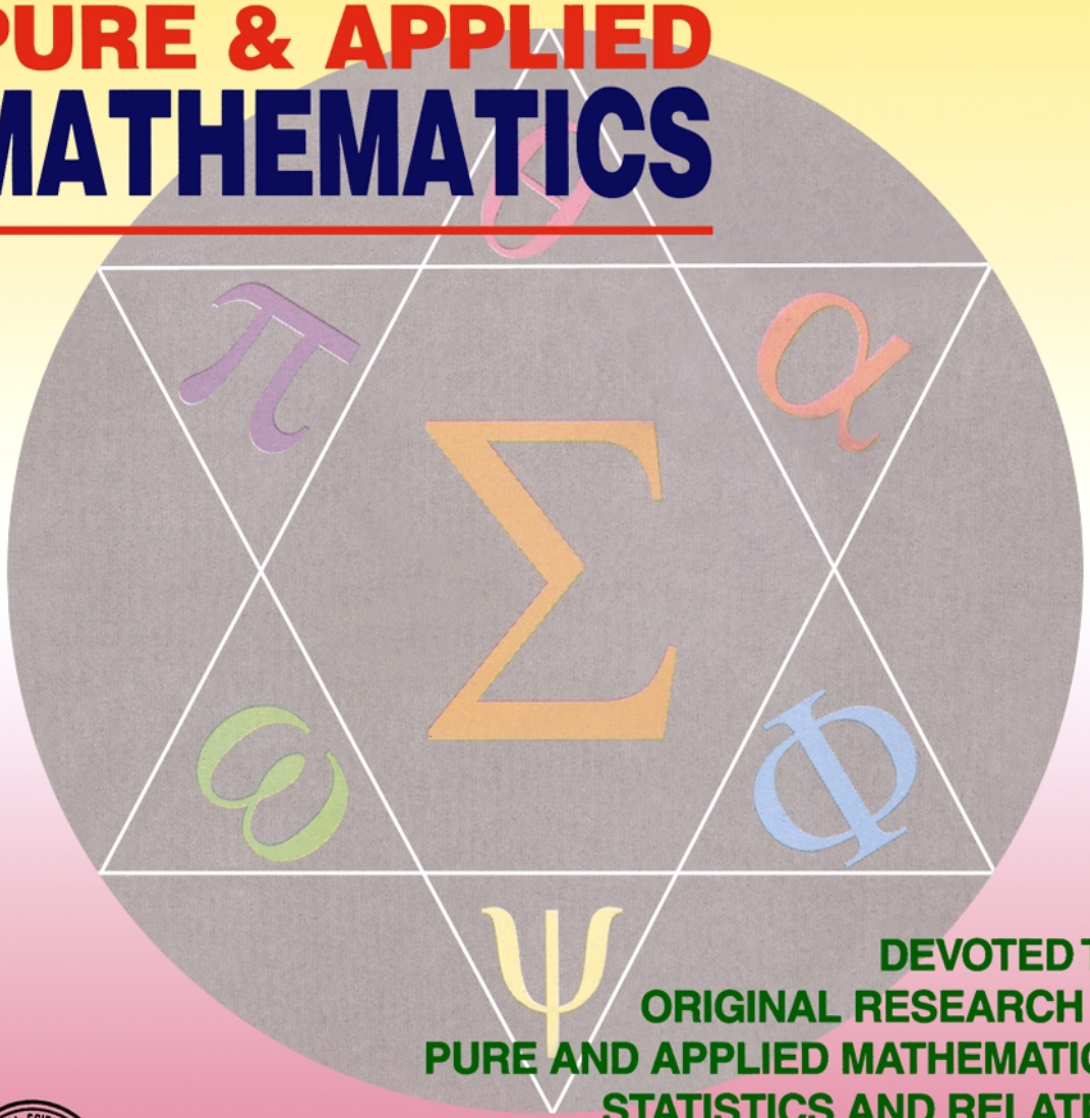

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ON EXACT SOLUTIONS FOR NONLINEAR DIFFUSION WITH SOURCE TERM

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Simple exact solutions are obtained for nonlinear diffusion and source term with power-law dependence forms. These solutions are derived by suitable transformations and standard techniques for an arbitrary number of dimensions $i \geq 0$ with particular emphases on $i = 0$, 1 and 2 corresponding to the slab, cylinder and sphere respectively. The results obtained include, as special cases, some earlier known results; they also apply to numerous equations which have hitherto not been studied. Some of the solutions obtained also exhibit uniqueness, multiplicity and the phenomenon of singularity, criticality, oscillation, extinction and blow up. We show some of these phenomena graphically.

Key Words : Nonlinear Diffusion; Source Term; Power-Law Dependence

1. INTRODUCTION

Nonlinear diffusion equations play an important role in many areas of science and engineering. Partial differential equations like

$$\frac{\partial \phi}{\partial t} = \frac{1}{x^i} \frac{\partial}{\partial x} \left(x^i \phi^m \frac{\partial \phi}{\partial x} \right) + \lambda \phi^n, \qquad \ldots (1.1)$$

constitute the mathematical models of meaningful physical interest as would be seen in the following examples: eq. (1.1) with $\lambda = i = 0$ occur in the context of saturated regions in porous[17] media ($m = 1$), thin sheet of viscous liquid spreading under gravity[2] ($m = 3$), percolation of gas through porous media ($m \geq 1$)[13], models of diffusion processes[3] (integer values of m), dopant diffusion in silicon[9] ($m \geq 0$), diffusion in high-polymeric[15] systems ($m = -2$) as well as in models of 'fast' diffusion case ($m = -1$)[1&10].

On the other hand, eqs. (1.1) with $i = 0$ and $\lambda \neq 0$ have been employed to model materials undergoing heating by microwave[7] radiation ($n = m + 1$ and $n = 1$) and have also been previously studied by Kersner[8] and Hill[6] when $n = 1 - m$. In particular, Hill *et al.*[7] employed the separation of variables

$$\phi(x, t) = A(t)\, B(x), \qquad \ldots (1.2)$$

in their work where $A(t)$ and $B(x)$ denote functions of t and x respectively. Many of the simple exact solutions found in these papers exhibit the phenomena of extinction and blowup. Further examples of these types of phenomenon can be found in Kubicek and Hlavacek[11] who considered

the mass diffusion and nth order chemical reaction in an isothermal porous catalyst particle, taking into consideration the concentration dependence of the diffusion coefficient, when $i = 0$ and $m = -1$ and Wilhelmsson[18] who found similarity forms of explosive type solutions, when $i = 0$ and $n = m + 1$.

Frey *et al.*[5] have applied the Lie-group-based similarity method in a systematic way, finding symmetries and exact similarity solutions for a generalized diffusion reaction equations, which include some of those mentioned above. Further exact similarity solutions are developed by Philips[16] for (1.1) with $n = 1 - m$, $0 < m \leq 1$ and for an arbitrary number of dimensions $i \geq 0$.

For the 'fast' diffusion case ($m = -1$) and the special cases of $n = 1 - m$ the solutions obtained for (1.1) does not exhaust all possible exact solutions available. Hence, we provide a systematic route to determine some exact solutions for an arbitrary number of dimensions $i \geq 0$ with particular emphases on physically applicable values (i.e., $i = 0$, 1 and 2 corresponding to the slab, cylinder and sphere respectively). We also investigated the conditions for steady state solutions for the whole range of physical values of n when $m = -1$ as well as for the special case of $n = m + 1$. Our results involve the celebrated Frank-Kamenetskii solution which is not introduced as an approximation but arises as a consequence of the form (1.1) with $m = -1$ and appropriate substitution.

2. UNSTEADY STATE SOLUTION

2.1 *Closed-Form Solutions for* $n = 1-m$

We proceed by letting

$$\phi(x, t) = V^p. \qquad \ldots (2.1)$$

Putting this into (1.1) yields

$$V_t = V^{mp}(V_{xx} + ix^{-1}V_x) + (p\{1 + m\} - 1)V^{mp-1}(V_x)^2 + \frac{\lambda}{p}V^{(n-1)p+1}. \qquad \ldots (2.2)$$

Eq. (2.2) permits closed-form solutions for the special case of $mp = 1$ and $n = 1 - m$. Assuming this to be the case, we readily obtain

$$V_t = V(V_{xx} + ix^{-1}V_x) + p(V_x)^2 + \frac{\lambda}{p}. \qquad \ldots (2.3)$$

We seek solutions of (2.3) in the form

$$V(x, t) = C(t) + W(x, t). \qquad \ldots (2.4)$$

On substitution of (2.4) into (2.3), we obtain, after some straightforward rearrangement

$$C' - C(W_{xx} + ix^{-1}W_x) - \frac{\lambda}{p} = -W_t + W(W_{xx} + ix^{-1}W_x) + p(W_x)^2. \qquad \ldots (2.5)$$

Assuming it is possible to find a function $f(t)$ such that

$$f(t) = W_{xx} + ix^{-1}W_x, \qquad \ldots (2.6)$$

then one likely and reasonable possibility of eq. (2.5) is that both sides can vanish identically. Based on this requirement, we have

$$C' = f(t)\, C + \frac{\lambda}{p} \qquad \ldots (2.7)$$

and $$W_t = W(W_{xx} + ix^{-1} W_x) + p(W_x)^2.$$

It follows that eq. (2.6) is the necessary and sufficient condition for eq. (1.1) to have solutions of the form (2.4). Now eq. (2.7*b*) can be readily solved using the form $W(x, t) = A(t)\, B(x)$ to yield

$$\frac{A'}{A^2} = \frac{B(B'' + ix^{-1} B') + p(B')^2}{B} = \mu_1, \qquad \ldots (2.8)$$

where primes denote differentiation with respect to the indicated variable and μ_1 denotes separation constant. It is worth pointing out that the solutions to (2.8*b*) are found by luck and guesswork — often by trying functions of a particularly simple form. Thus, it is easy to show that

$$B = \frac{m \mu_1 x^2}{(2(2 + (1 + i) m)}, \quad A = \frac{1}{(\alpha - \mu_1 t)}, \qquad \ldots (2.9)$$

where α is an arbitrary constant of integration. It follows that

$$W(x, t) = \frac{m \mu_1 x^2}{2(2 + (1 + i) m)(\alpha - \mu_1 t)}. \qquad \ldots (2.10)$$

Putting this into (2.6), we find that

$$f(t) = \frac{m \mu_1 (1 + i)}{(2 + (1 + i) m)(\alpha - \mu_1 t)}. \qquad \ldots (2.11)$$

Hence, $C(t)$ is given by

$$C(t) = \frac{b}{(\alpha - \mu_1 t)^a} - \frac{m \lambda (\alpha - \mu_1 t)}{\mu_1 (a + 1)}, \qquad \ldots (2.12)$$

where $$a = \frac{m(1 + i)}{2 + (1 + i) m},$$

and b is an arbitrary constant of integration. Therefore with this $C(t)$ in mind

$$\phi(x, t) + \left(C(t) = \frac{m \mu_1 x^2}{2(2(1 + i) m)(\alpha - \mu_1 t)} \right)^{\frac{1}{m}}. \qquad \ldots (2.13)$$

Thus completing the solution. As a check on the solution, it gives the well known exact solution due to Kersner[8] for the case of $i = 0$ when $\alpha = 0$ and $\mu_1 = -1$ in ours. We note that even

for this very simple expression, the two subcases $m < 0$ and $m > 0$ give qualitatively different behaviour for all $i \geq 0$.

2.2 Exact Solutions Arising From n = 1

It is apparent that the case of $n = 1$ cannot be deduced from the previous section. To handle this situation of $n = 1$, eq. (2.2) with $mp = 1$ reduces to

$$V_t = V(V_{xx} + ix^{-1} V_x) + p(V_x)^2 + \frac{\lambda}{p} V, \qquad \text{... (2.14)}$$

which readily admits solutions of the form (2.4). It follows at once that

$$W_t = W(W_{xx} + ix^{-1} W_x) + p(W_x)^2 + \frac{\lambda}{p} W \qquad \text{... (2.15)}$$

and $$C' = f(t) C + \frac{\lambda}{p} C,$$

provided $f(t)$ satisfies (2.6). We now proceed to use separation of variables to obtain

$$\frac{A' + \frac{\lambda}{p} A}{A^2} = \frac{B(B'' + ix^{-1} B') + p(B')^2}{B} = \mu_2, \qquad \text{... (2.16)}$$

Our solution for B carries over, but equation for A arising from (2.16a) can be solved as a standard Bernoulli equation with resulting solution

$$A = \left(d \exp(-m \lambda t) - \frac{\mu_2}{\lambda m} \right)^{-1}, \qquad \text{... (2.17)}$$

where d is an arbitrary constant of integration. Hence

$$W(x, t) = \frac{m \mu_2 x^2}{2(2 + (1 + i) m) \left(d \exp(-m \lambda t) - \frac{\mu_2}{m \lambda} \right)}. \qquad \text{... (2.18)}$$

It follows that

$$f(t) = \frac{(1 + i) m \mu_2}{(2 + (1 + i) m) \left(d \exp(-m \lambda t) - \frac{\mu_2}{m \lambda} \right)}, \qquad \text{... (2.19)}$$

and $$C(t) = \frac{b_0 \exp(m \lambda t)}{\left[d - \frac{\mu_2}{m \lambda} \exp(m \lambda t) \right]^a}, \qquad \text{... (2.20)}$$

where b_0 is an arbitrary constant of integration. Thus with this $C(t)$ in mind

$$\phi(x,t)=\left[C(t)+\frac{m\mu_2 x^2}{2(2+(1+i)m)\left(d\exp(-m\lambda t)-\frac{\mu_2}{m\lambda}\right)}\right]^{\frac{1}{m}}. \qquad \text{... (2.21)}$$

We note that the special case of $i=b_0=0$ and $\lambda<0$ corresponding to [7, eq. (2.11), p. 288] which is a good illustration of the phenomena such as extinction, singularity and waiting-time behaviour. Hence, our solution (2.21) is an extension of earlier works.

3. Steady State Solution

3.1 Closed-Form Solutions for all n and for m = − 1

Finally, in this section, we consider some properties for large time behaviour when stationary solution ϕ_∞ exists in the form

$$\frac{1}{x^i}\frac{d}{dx}\left(\frac{x^i}{\phi_\infty}\frac{d\phi_\infty}{dx}\right)+\lambda\phi_\infty^n=0. \qquad \text{... (3.1)}$$

For the slab we may integrate once to yield

$$\left(\frac{d\phi_\infty}{dx}\right)^2=A\phi_\infty^2-\frac{2\lambda}{n}\phi_\infty^{n+2}, \qquad \text{... (3.2)}$$

where A is an arbitrary constant of integration. So that $\phi_\infty(x)$ may be obtained from the integral

$$\int^{\phi_\infty(x)}\frac{d\phi}{\phi\left(A-\frac{2\lambda}{n}\phi^n\right)^{\frac{1}{2}}}=\pm(x-x_0). \qquad \text{... (3.3)}$$

Let $$\phi^n=\frac{nA}{2\lambda\cosh^2\Phi}, \qquad \text{... (3.4)}$$

then we readily evaluate the integral to obtain

$$\phi_\infty^n=\frac{A}{\cosh^2\left(\frac{\lambda nA}{2}\right)^{\frac{1}{2}}(x-x_0)}, \text{ if } n\neq 0, \qquad \text{... (3.5)}$$

and $$\phi_\infty=\exp\left(\frac{-\lambda x^2}{2}+A_1x+B_1\right), \text{ if } n=0, \qquad \text{... (3.6)}$$

where A, A_1, B_1 and x_0 are arbitrary constants of integration.

For the cylinder, it is easy to show that if $n = 0$

$$\phi_\infty = A_0 x \exp\left(\frac{-\lambda x^2}{4} + A_2\right), \quad \ldots (3.7)$$

and

$$\phi_\infty = A_3 \exp\left(\frac{-\lambda x^2}{4} + A_2\right), \quad \ldots (3.8)$$

are solutions where A_0, A_2 and A_3 are arbitrary constants of integration. However for $n \neq 0$, we apply the substitution

$$\phi_\infty^n = \exp(\upsilon), \; x = \exp(t) \text{ and } \theta = \upsilon + 2t, \quad \ldots (3.9)$$

to reduce (3.1) to the celebrated Frank-Kamenetskii equation

$$\theta' + n\lambda \exp(\theta) = 0, \quad \ldots (3.10)$$

whose solution is contained in[4]. Hence,

$$\phi_\infty^n = \frac{A}{x^2 \cosh^2\left(\sqrt{\frac{n\lambda A}{2}} \log_e x - x_0\right)}, \; n \neq 0. \quad \ldots (3.11)$$

It is worth pointing out that ϕ_∞ must be finite at $x = 0$ for $n \neq 0$. Now as $x \to 0$

$$\phi_\infty^n \sim \frac{4A}{x^2 \exp(2x_0) \exp\left(-2\sqrt{\frac{n\lambda A}{2}} \log_e x\right)}, \; n \neq 0. \quad \ldots (3.12)$$

The finiteness of ϕ_∞ at the origin implies

$$A = \frac{2}{n\lambda}, \; n \neq 0. \quad \ldots (3.13)$$

So that as $x \to 0$

$$\phi_\infty^n \sim \frac{8}{n\lambda \exp(2x_0)}, \; n \neq 0. \quad \ldots (3.14)$$

Hence, the solution becomes

$$\phi_\infty^n = \frac{2}{n\lambda x^2 \cosh^2(\log_e x - x_0)}, \; n \neq 0, \quad \ldots (3.15)$$

where x_0 is the only arbitrary constant of integration to be determined. Note that ϕ_∞ as contained in (3.15) is finite at $x = 0$, since its value is contained in (3.14) and therefore compatible with reality. Fig. 1 shows the various possibilities which can occur for these steady state solutions.

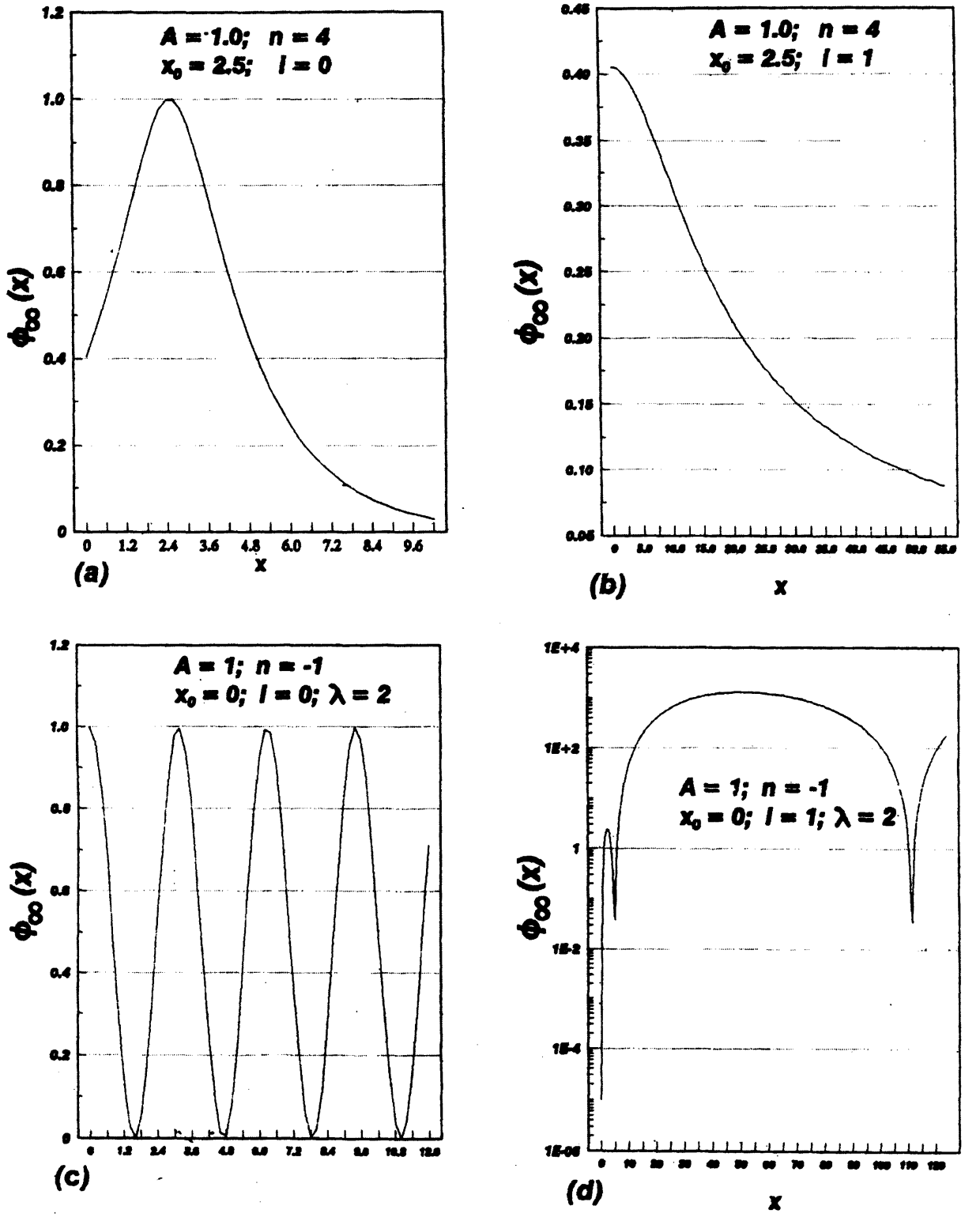

FIG. 1. Typical plots of steady state solutions.

It is important to note that (3.15) does not have a solution for all values of λ under some physically reasonable boundary conditions. For example, with the boundary condition

$$\phi_{\infty}(1) = 1, \qquad \ldots (3.16)$$

we obtain from (3.15) that

$$\cosh x_0 = \left(\frac{2}{n\lambda}\right)^{\frac{1}{2}}, n \neq 0, \qquad \ldots (3.17)$$

and this implies that a steady solution exists only for $0 \leq \lambda \leq \lambda_{cr}$, where

$$\lambda_{cr} = \frac{2}{n}, n \neq 0. \qquad \ldots (3.18)$$

3.2 Exact Solutions for $n = m + 1$

Briefly, if we apply the substitution

$$\theta = \phi_{\infty}^{m+1}$$

on (1.1), we have

$$\frac{1}{x^i}\frac{d}{dx}\left(x^i \frac{d\theta}{dx}\right) + (m+1)\lambda\, \theta^{\frac{n}{m+1}} = 0, \qquad \ldots (3.19)$$

from which it is apparent that the special case $n = m + 1$ give rise to linear equations

$$\frac{1}{x^i}\frac{d}{dx}\left(x^i \frac{d\theta}{dx}\right) + (m+1)\lambda\,\theta = 0. \qquad \ldots (3.20)$$

This linear equation have solutions of the form

$$\theta = \phi_{\infty}^{m+1} = \begin{cases} A \sin kx + B \cos kx, & i = 0 \\ AJ_0(kx) + BY_0(kx), & i = 1 \\ x^{-1}(A \sin kx + B \cos kx), & i = 2 \end{cases} \qquad \ldots (3.21)$$

where A and B denote arbitrary constants and $k = [m+1]\lambda]^{1/2}$.

A careful examination of each of the above special steady solutions indicates that there are no steady state solutions such that $\theta \geq 0$ at all times. We note, however, that θ tends to zero as x tends to infinity in the case of (3.21 c) provided $m > -1$. Furthermore, if we impose arbitrary boundary conditions (with physical applications) on (3.21), we readily deduce unique solutions.

4. REMARKS

(a) To handle the special case of $n = 1 + m$, the analysis is similar to that of section 2, and evidently we have from (1.1), (2.1) and (2.4) that

$$W_t = W(W_{xx} + ix^{-1} W_x) + p(W_x)^2 + \frac{\lambda}{p} W^2, \qquad \ldots (i)$$

$$C' = f(t)\, C + \frac{\lambda C^2}{p},$$

provided that

$$f(t) = W_{xx} + ix^{-1} W_x + \frac{2\lambda W}{p}. \qquad \ldots (ii)$$

These equations admit simple exact solutions for selected special values of m and i. For example, in the case of $i = 0$, equation (*ia*) has the solution

$$W(x,\ t) = D_1 \exp(\sqrt{4\lambda}\, x), \qquad \ldots (iii)$$

from which it is clear that $f(t) \equiv 0$ and $m = -2$. Therefore

$$C(t) = (-D_4 + 2\lambda t)^{-1}, \qquad \ldots (iv)$$

and hence

$$\phi(x, t) = [(-D_4 + 2\lambda t)^{-1} + D_1 \exp(\sqrt{4\lambda}\, x)]^{-\frac{1}{2}}. \qquad \ldots (v)$$

It is easy to see that this solution exhibits extinction in a finite time

$$t = \frac{D_4}{2\lambda}.$$

(*b*) It is worth pointing out that the need to obtain these new exact solutions of (1.1) is not unconnected with the fact that simple exact solution is not trivial to come by and that any example is always important and valuable in itself.

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