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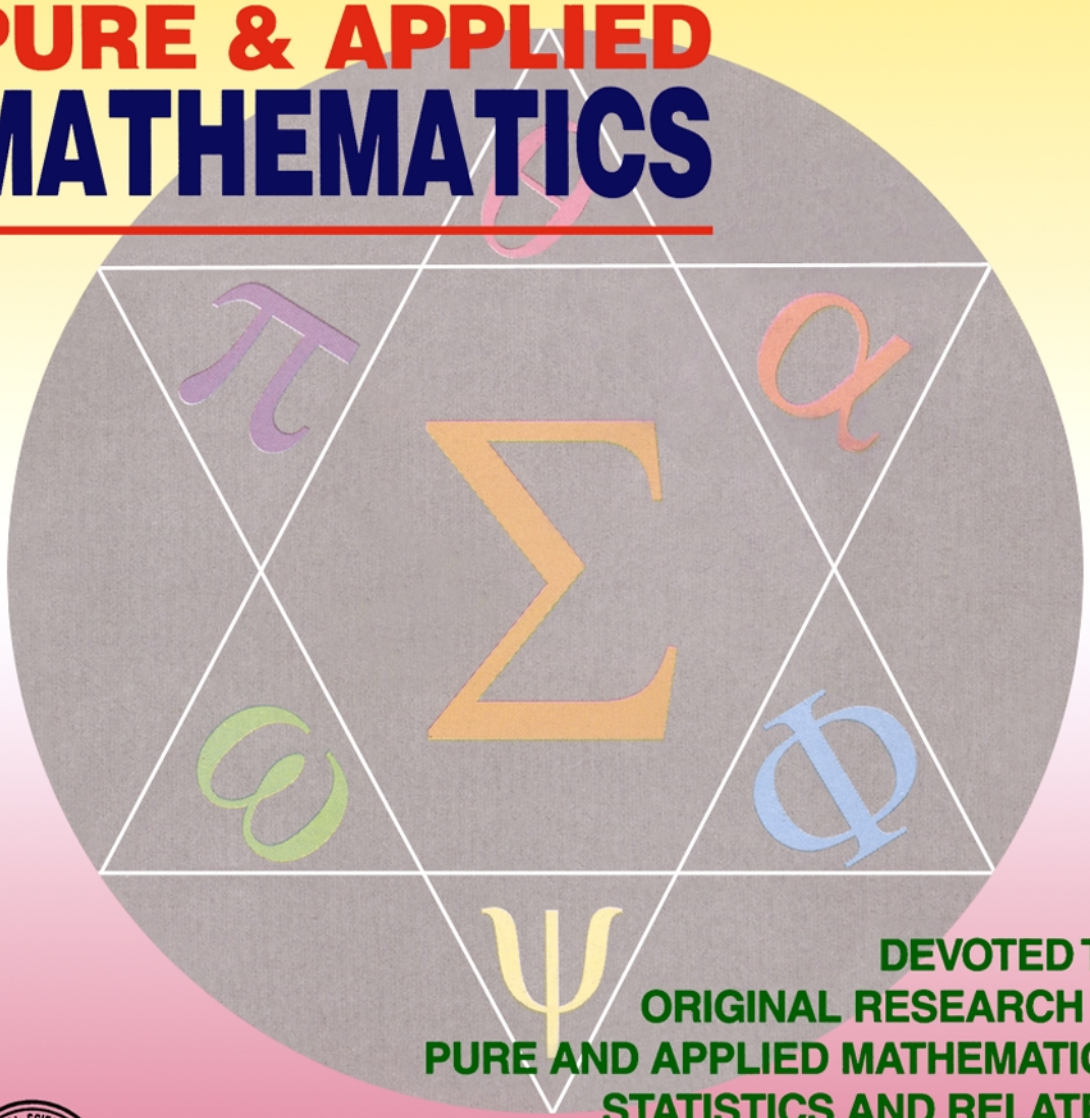

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MULTIPLE FIXED POINTS OF WEAKLY INWARD MAPPING*

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Using variational method and the theory of ordinary differential equations in Banach spaces, a three fixed points theorem of weakly inward mapping is proved.

Key Words : Weakly Inward; Variational Method; Ordinary Differential Equations; Fixed Point

1. INTRODUCTION

In 1965, Halpern put forward inward mappings in his doctoral thesis[1]. Since then many results have appeared in the literature concerning inward mapping in Halpern's sense[3, 6 &7].

In this paper, we use variational method and the theory of ordinary differential equations in Banach spaces to show that a weakly inward mapping has at least three fixed points under some suitable conditions.

Definition[1-3] — Let E be a topological linear space and D a convex subset of E. If $x \in D$ we define the inward set, $I_D(x)$, of x with respect to D as follows :

$$I_D(x) = \{(1-\lambda)x + \lambda y \mid \lambda \geq 0, y \in D\}.$$

We say that a mapping $A : D \to E$ is inward in case $Ax \in I_D(x)$ for each $x \in D$. We say that A is weakly inward in case $Ax \in \overline{I}_D(x)$ for each $x \in D$, where $\overline{I}_D(x)$ denotes the closure of $I_D(x)$.

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Lemma 1[2] — Let K be a convex subset of a normed linear space X and $A : K \to X$. Then for each $x \in K$

$$\lim_{h \to 0+} h^{-1} d(x - hA(x), K) = 0$$

if and only if $I - A$ is weakly inward.

In this paper, we always assume that H is a Hilbert space and D is a closed convex subset of H, $f : H \to R$ is a functional defined by

$$f(x) = \frac{1}{2} \| x \|^2 - g(x),$$

where

$$g \in C^1(H, R), g'(x) = Ax, \text{ so } f'(x) = x - Ax.$$

2. MAIN RESULTS

Theorem 1 — *Assume that $A : D \to H$ is weakly inward and locally Lipschitzian. Suppose that Ω_1 and Ω_2 are convex open subsets of D, $\Omega_1 \cap \Omega_2 = \phi$, and $A : \overline{\Omega}_i \to H$ $(i = 1, 2)$ is also weakly inward. We also assume that A has no fixed point on $\partial\Omega_1 \cup \partial\Omega_2$ ($\partial\Omega_i$ $(i = 1, 2)$ means the boundary of Ω_i relative to D). If f satisfies the (PS) condition on D (any sequence $\{x_n\} \subset D$ such that $\sup_n |f(x_n)| < +\infty, f'(x_n) \to \theta$ contains a convergent subsequence), and*

$$\inf_{x \in \overline{\Omega}_1 \cup \overline{\Omega}_2} f(x) > -\infty. \qquad \text{... (1)}$$

Then A has at least three fixed points, $x_1 \in \Omega_1, x_2 \in \Omega_2, x_3 \in D \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$.

PROOF : Consider the initial value problem in H

$$\begin{cases} \dfrac{dx}{dt} = Ax - x, \\ x(0) = x_0 \in D. \end{cases} \qquad \text{... (2)}$$

Since A is locally lipschitzian, so by the theory of ordinary differential equations in Banach spaces[4], for each $x_0 \in \overline{\Omega}_1$, (2) has a unique solution $x(t, x_0)$, with right maximal interval of existence $[0, T(x_0))$ $(T(x_0) \leq +\infty)$, and by Lemma 1 and the criterion of invariant sets of flow in [5], we know that $\overline{\Omega}_1$ is an invariant set of flow of $Ax - x$, i.e.,

$$x(t, x_0) \in \overline{\Omega}_1, \quad \forall t \in [0, T(x_0)).$$

Let

$$c_1 = \inf_{x \in \overline{\Omega}_1} f(x) > -\infty.$$

Since

$$\frac{df(x(t, x_0))}{dt} = (f'(x, t, x_0)), x'(t, x_0))$$

$$= (x(t, x_0) - Ax(t, x_0), Ax(t, x_0) - x(t, x_0))$$

$$= -\| f'(x, t, x_0)) \|^2 \leq 0,$$

then $f(x(t, x_0))$ is decreasing in t, and

$$\| f'(x(t, x_0)) \|^2 = -\frac{df(x(t, x_0))}{dt},$$

so for

$$t \in [0, T(x_0)),$$

$$\int_0^t \| f'(x, t, x_0)) \|^2 dt$$

$$= -\int_0^t \frac{df(x(t, x_0))}{dt} dt - f(x_0) - fx(t, x_0)) \leq f(x_0) - c_1. \qquad \ldots (3)$$

For

$$0 \leq t_1 \leq t_2 < +\infty,$$

$$\| x(t_2, x_0) - x(t_1, x_0) \| \leq \int_0^t \| f'(x(t, x_0)) \| dt$$

$$\leq \left(\int_0^t \| f'(x, t, x_0)) \|^2 dt \right)^{\frac{1}{2}} (t_2 - t_1)^{1/2}$$

$$\leq (f(x_0) - c_1)^{1/2} (t_2 - t_1)^{1/2}.$$

Case (i) — If $T(x_0) < +\infty$, then there exists some $x_1 \in \overline{\Omega}_1$, such that

$$\lim_{t \to T(x_0)-0} \| x(t, x_0) - x_1 \| = 0.$$

We can prove that $Ax_1 - x_1 = \theta$, otherwise, the right existence interval of $x(t, x_0)$ would be $[0, T(x_0) + T(x_1))$ for some $T(x_1) > 0$, this contradicts the maximality of $[0, T(x_0))$. so $Ax_1 = x_1$.

Case *(ii)* If $T(x_0) = +\infty$, then let $t \to +\infty$ in (3), we have

$$\int_0^{+\infty} \| f'(x(t, x_0)) \|^2 \, dt \le f(x_0) - c_1 < +\infty,$$

so there exist $t_n \in [0, +\infty)$, $t_n \to +\infty$, such that

$$f'(x(t_n, x_0)) \to \theta,$$

by the (PS) condition, there is a convergence subsequence of $\{x(t_n, x_0)\}$, which we also denote by $\{x(t_n, x_0)\}$. Let $x_1 = \lim_{n \to \infty} x(t_n, x_0)$. It is obvious that $f'(x_1) = \theta$, i.e., $Ax_1 = x_1$. In either case, we have $Ax_1 = x_1$. Noticing that A has no fixed point on $\partial \Omega_1$, we have $x_1 \in \Omega_1$. In the same way, we can prove that A has a fixed point $x_2 \in \Omega_2$.

In the following we try to prove that there is $x_3 \in D \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$, such that $Ax_3 = x_3$.

Denote $U_1^* = \{x_0 \mid x_0 \in D$ and there is $0 \le t' < T(x_0)$ such that $x(t', x_0) \in \Omega_1\}$. We can prove that $\overline{\Omega}_1 \subseteq U_1^* \subset D$. In fact, it is easy to know that $\Omega_1 \subset U_1^* \subset D$, if there exists $\tilde{x}_0 \in \overline{\Omega}_1$, but $\tilde{x}_0 \notin U_1^*$, so $\tilde{x}_0 \in \partial \Omega_1$. Since $\overline{\Omega}_1$ is an invariant set of flow of $Ax - x$, so $x(t, \tilde{x}_0) \in \overline{\Omega}_1$, $\forall t \in [0, T(\tilde{x}_0))$. Hence, $x(t, \tilde{x}_0) \in \partial \Omega_1$, $\forall t \in [0, T(\tilde{x}_0))$. Then as the proof of existence of x_1, A has a fixed point on $\partial \Omega_1$. which contradicts with A has no fixed point on $\partial \Omega_1$. So $\overline{\Omega}_1 \subset U_1^*$. By the continuous dependence of ordinary differential equations on initial data, we know that U_1^* is an open subset of D. Similarly, $U_2^* = \{x_0 \mid x_0 \in D$ and there is $0 \le t' < T(x_0)$ such that $x(t', x_0) \in \Omega_2\}$ is an open subset of D. From $\Omega_1 \cap \Omega_2 = \phi$, it is easy to prove that $U_1^* \cap U_2^* = \phi$. By the connectness of D, we have $U_1^* \ne D$. So $\partial U_1^* \ne 0$ (∂U_1^* means the boundary of U_1^* relative to D). If $x_0 \in \partial U_1^*$, then by the theory of ordinary differential equations in Banach spaces, problem (2) has a unique solution $x(t, x_0)$, with right maximal interval of existence $[0, T(x_0))$.

Claim I — $x(t, x_0) \in \partial U_1^*$, $\forall t \in [0, T(x_0))$.

If there is $0 \le t' < T(x_0)$, such that $x(t', x_0) \notin \partial U_1^*$, then $x(t', x_0) \in U_1^* \cup (D \setminus \overline{U_1^*})$. Case (i) if $x(t', x_0) \in U_1^*$, then $x_0 \in U_1^*$, which contradicts $x_0 \in \partial U_1^*$. Case (ii) if $x(t', x_0) \in D \setminus \overline{U_1^*}$. We consider problem (2), since D is convex and $A : D \to H$ is weakly inward, we know that for each $x' \in D, x(t, x') \in D$, $\forall t \in [0, T(x'))$. Since $D \setminus \overline{U_1^*}$ is an open subset of D and $x_0 \in D$, by the continuous dependence of ordinary differential equations on initial data, there is a neighbourhood U of x_0 in D, such that for any $x^* \in U, x(t', x^*) \in D \setminus \overline{U_1^*}$. Select $x^* \in U \cap U_1^*$, then $x(t', x^*) \in U_1^*$, which bring a contradiction. In either case, we come to a contradiction. Claim I is proved.

Claim II — $\inf\limits_{\partial U_1^*} f(x) \ge \inf\limits_{\partial \Omega_1} f(x) > -\infty$.

For each $x_0 \in \partial U_1^*$, then $x_0 \notin \overline{\Omega}_1$, so there is a sequence $\{x_n\}_1^\infty$ with $\{x_n\}_1^\infty \subset U_1^*$ and $\{x_n\}_1^\infty \cap \overline{\Omega}_1 = \phi$, such that

$$\lim_{n \to \infty} x_n = x_0.$$

Since $x_n \in U_1^*$ and $x_n \notin \overline{\Omega}_1$, then by the definition of U_1^*, there are $t_n > 0$, such that $x(t_n, x_n) \in \partial \Omega_1$, then it follows from $f(x(t))$ is decreasing in t that $f(x_n) \ge f(x(t_n, x_n)) \ge \inf\limits_{x \in \partial \Omega_1} f(x)$. Hence, $f(x_0) = \lim\limits_{n \to \infty} f(x_n) \ge \inf\limits_{x \in \partial \Omega_1} f(x)$. Therefore, $\inf\limits_{\partial U_1^*} f(x) \ge \inf\limits_{\partial \Omega_1} f(x) > -\infty$. Claim II is proved.

Claim III — There exists $x_3 \in D \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$, such that $Ax_3 = x_3$.

From Claim I, we know that for each $x_0 \in \partial U_1^*$,

$$x(t, x_0) \in \partial U_1^*, \quad \forall t \in [0, T(x_0)),$$

then as the proof of existence of x_1, we can prove that there is $x_3 \in \partial U_1^*$, such that $Ax_3 = x_3$. Since $\Omega_i \cap \partial U_1^* = \phi$ (i = 1, 2), and A has no fixed point on $\partial \Omega_1 \cup \partial \Omega_2$. So $x_3 \in D \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$. Theorem 1 is proved.

Remark 1 : If $D = H$ in Theorem 1, we have the same conclusions as Theorem 1.

Corollary 1 — Assume that $A : D \to D$ is locally lipschitzian, Ω_1 and Ω_2 are convex open subsets of D, $\Omega_1 \cap \Omega_2 = \phi$, $A(\overline{\Omega}_1) \subset \overline{\Omega}_1$, $A(\overline{\Omega}_2) \subset \overline{\Omega}_2$. We also assume that A has no fixed point on $\partial\Omega_1 \cup \partial\Omega_2$. If f satisfies the (PS) condition on D and

$$\inf_{x \in \overline{\Omega}_1 \cup \overline{\Omega}_2} f(x) > -\infty.$$

Then A has at least three fixed points.

Remark 2 : From the proof of Theorem 1, if we weaken (1), we have the following conclusions :

(*i*) If $\inf_{x \in \partial\Omega_1} f(x) > -\infty$ (or $\inf_{x \in \partial\Omega_2} f(x) > -\infty$), then A has at least one fixed point $x \in D\ (\overline{\Omega}_1 \cup \overline{\Omega}_2)$; and

(*ii*) if $\inf_{x \in \overline{\Omega}_1} f(x) > -\infty$ (or $\inf_{x \in \overline{\Omega}_2} f(x) > -\infty$), then A has at least two fixed points $x_1 \in \Omega_1$ (or $x_1 \in \Omega_2$), $x_2 \in D\ (\overline{\Omega}_1 \cup \overline{\Omega}_2)$.

Theorem 2 — *Assume that $A : D \to H$ is weakly inward and locally lipschitzian, Ω_1 is a convex open subset of D, and $A : \overline{\Omega}_1 \to H$ is weakly inward. We also assume that A has no fixed point on $\partial\Omega_1$. If f satisfies the (PS) condition on D and*

$$\inf_{x \in \overline{\Omega}_1} f(x) > -\infty,$$

and there exists $x_0 \in D \setminus \overline{\Omega}_1$, such that $f(x_0) < \inf_{x \in \partial\Omega_1} f(x)$. Then A has at least two fixed points, $x_1 \in \Omega_1, x_2 \in D \setminus \overline{\Omega}_1$.

PROOF : From the proof of Theorem 1, it is easy to prove the existence of x_1, and the existence of x_2 can be derived from $\partial U_1^* \neq \phi$. In fact, from the proof of Theorem 1, we know U_1^* is an open subset of D and $\overline{\Omega}_1 \subset U_1^*$. For any $x^* \in U_1^* \setminus \overline{\Omega}_1$, by the definition of U_1^* and $f(x(t))$ is decreasing in t, we have $f(x^*) \geq \inf_{x \in \partial\Omega_1} f(x) > f(x_0)$, so $x_0 \notin U_1^*$. Therefore, $\partial U_1^* \neq \phi$. Theorem 2 is proved.

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