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REMARKS ON *N*-POINT ORDER COMPACTIFICATIONS

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Let $(X^*, \tau^*, \leqslant^*)$ be an n-point T_2-ordered compactification of the $T_{3,5}$-ordered topological space $(X, \tau, \leqslant)$. Let $\mathscr{V}^{\uparrow}_{*}(a)$ be the filter on X^* having a base of $\leqslant^*$-increasing τ^*-open neighbourhoods of $a \in X^*$, with $\mathscr{V}^{\downarrow}_{*}(a)$ defined dually. Both the topology τ^* and the order $\leqslant^*$ are determined by the collection $C^* = \{\mathscr{V}^{\uparrow}_{*}(a), \mathscr{V}^{\downarrow}_{*}(a) : a \in X^*\}$. An intrinsic characterization on X for this collection is pointed out.

A partially ordered topological space $(X, \tau, \leqslant)$ is T_2 ordered if the graph of the order is closed in $X \times X$. By a space we shall mean a T_2-ordered topological space. An order compactification of $(X, \tau, \leqslant)$ is a topological compactification (X^*, τ^*) of (X, τ) together with a closed order $\leqslant^*$ that extends the order $\leqslant$. A space has an order compactification iff it is $T_{3,5}$-ordered (completely regularly ordered in Nachbin[3]). A subset A of a poest $(X, \leqslant)$ is increasing if $a \in A$ and $x \geqslant a$ imply $x \in A$. Decreasing sets are defined dually. For further information on ordered topological spaces. see Choe[1], Nachbin[3] and Richmond[4]. With the order $\mathscr{F} \leqslant \mathscr{G}$ iff $\mathscr{F} \subseteq \mathscr{G}$, the supermum $\mathscr{F} \vee \mathscr{G}$ of filters $\mathscr{F}$ and $\mathscr{G}$ exists iff ϕ $B \equiv \{F \cap G : F \in \mathscr{F}. \ G \in \mathscr{G}\}$. in which case B is a base for $\mathscr{C} \vee \mathscr{G}$.

Let $\{G_i\}_{i=1}^{n}$ be an n-star[2] corresponding to (X^*, τ^*), where $X^* = X \cup \{\omega_i\}_{i=1}^{n}$ and $(X^*, \tau^*, \leqslant^*)$ is an n-point order compactification of the $T_{3,5}$-ordered topological space $(X, \tau, \leqslant)$. Let $\mathscr{C} = \{\mathscr{V}\uparrow(a), \mathscr{V}\downarrow(a) : a \in X^*\}$ where $\mathscr{V}\uparrow(a)$ is the trace on X of $\mathscr{V}^{\uparrow}_{*}(a)$, with $\mathscr{V}\downarrow(a)$ defined dually. For $x \in X$, let $\mathscr{V}(x)$ be the neighbourhood filter at x. Let $K = X \setminus \cup_{i=1}^{n} G_i$, and let $\mathscr{V}(\omega_i)$ be the filter generated by $\{N \subseteq X : (K \cup G_i) \setminus N$'s compact$\}$.

Theorem 1—$\mathscr{C}$ is the unique collection of filters on $(X, \tau, \leqslant)$ satisfying the following conditions for any $a, b \in X^*$ and any $x, y \in X$.

(1) $\mathscr{V}\uparrow(a)$ (respectively, $\mathscr{V}\downarrow(a)$) has a filter base of τ-open $\leqslant$-increasing (respectively, $\leqslant$-decreasing) sets.

(2) $\mathscr{V}\uparrow(a) \vee \mathscr{V}\downarrow(a) = \mathscr{V}(a)$.

(3) $\mathscr{V}\uparrow(x) \vee \mathscr{V}\downarrow(b)$ exists $\Rightarrow \mathscr{V}\uparrow(a) \leqslant \mathscr{V}\downarrow(b) \leqslant \mathscr{V}(a)$,

(4) $\mathscr{V}\downarrow(a) \vee \mathscr{V}\downarrow(b)$ exists $\Rightarrow \mathscr{V}\uparrow(a) \leqslant \mathscr{V}(b)$ and $\mathscr{V}\uparrow(b) \leqslant \mathscr{V}(a)$.

Proof : It is easy to see that $\mathscr{C}$ satisfies these conditions. Suppose that $\{\mathscr{U}\uparrow(a), \mathscr{U}\downarrow(a) : a \in X^*\}$ is any other family of filters satisfying these conditions.

$\mathscr{U}\uparrow(a) \leqslant \mathscr{V}\uparrow(a)$: For any $U' \in \mathscr{U}\uparrow(a)$, there exists τ-open $\leqslant$-increasing $U \in \mathscr{U}\uparrow(a)$ such that (1) $U \subseteq U'$, and (2) $\omega \in X^*\setminus X$ and $\omega \ngeqslant a$ imply that there exists $N \in \mathscr{U}\downarrow(\omega)$ with $N \cap U = \phi$. Let $U^* = U \cup \{\omega \in X^*\setminus X : \omega \geqslant a\}$. Now U^* is τ^*-open since U was a neighbourhood of each of its points, and for $\omega \geqslant a$, the existence of $\mathscr{U}\uparrow(a) \vee \mathscr{U}\downarrow(\omega)$ implies $\mathscr{U}\uparrow(a) \leqslant \mathscr{U}(\omega)$, and thus U^* is neighbourhoood of ω. To see that U^* is $\leqslant^*$-increasing. suppose $b \in U^*$ and $b \leqslant^* c$. Clearly $c \in U^*$ if $b, c \in X$ or if $b, c \in X^*, X$. Suppose $b \in U^* \cap X$, $c = \omega \in X^*\setminus X$. If $\omega \notin U^*$. then there exists $N \in \mathscr{U}\downarrow$ such that $N \cap U = \phi$; but $b <^* \omega \Rightarrow \mathscr{U}\downarrow(\omega) \leqslant \mathscr{U}(b) \Rightarrow b \in N \cap U$, a contradiction. Finally, if $b = \omega \in U^*\setminus X$. $\omega <^* c$, and $c \in X$, then $a <^* c \Rightarrow \mathscr{U}\uparrow(a) \leqslant \mathscr{U}(c) \Rightarrow c \in U$. Now $U = U^* \cap X \subseteq U'$, where U' was an arbitrary element of $\mathscr{U}\uparrow(a)$ and U^* is the τ^*-open $\leqslant^*$-increasing neighbourhood of a described above. It follows that $\mathscr{U}\uparrow(a) \leqslant \mathscr{V}\uparrow(a)$.

$\mathscr{V}\uparrow(a) \leqslant \mathscr{U}\uparrow(a)$: First notice that $a \nleqslant^* c$ implies there exist $N \in \mathscr{U}\downarrow(c)$ and $U \in \mathscr{U}\uparrow(a)$ with $N \cap U = \phi$, and therefore $c \notin \mathrm{cl}_{\tau^*}(U)$. Thus, $\cap\{\mathrm{cl}_{\tau^*}(U) : U \in \mathscr{U}\uparrow(a)\} = i_*(a) \equiv \{x \in X^* : a \leqslant^* x\}$. Now suppose $\mathscr{V}\uparrow(a) \nleqslant \mathscr{U}\uparrow(a)$. Then there exists $\mathscr{V} \in \mathscr{V}\uparrow(a)$ such that for all $U \in \mathscr{U}\uparrow(a)$, $U\setminus V \neq \phi$. Let V^* be a τ^*-open $\leqslant^*$-increasing neighbourhood of a in X^* such that $V^* \cap X \subseteq V$. Without loss of generality, $\omega \in V^*\setminus X$ iff $a \leqslant^* \omega$. Because $U\setminus V \neq \phi \Rightarrow \mathrm{cl}_{\tau^*}(U)\setminus V^* \neq \phi$ ($\forall U \in \mathscr{U}\uparrow(a)$), we have for any finite collection $\{U_i \in \mathscr{U}\uparrow(a) : i = 1, \ldots, n\}$. $\phi \neq [\mathrm{cl}_{\tau^*}(\cap_{i=1}^{n} U_i)]\setminus V^* \subseteq [\cap_{i=1}^{n} \mathrm{cl}_{\tau^*} U_i]\setminus V^* = \cap_{i=1}^{n}[\mathrm{cl}_{\tau^*} U_i\setminus V^*]$. Thus $\mathscr{C} = \{\mathrm{cl}_{\tau^*} U : V^* : U \in \mathscr{U}\uparrow(a)\}$ is collection of τ-*-closed sets in X^* satisfying the finite intersection propetry, but $\cap \mathscr{C} = i_*(a)/V^* = \phi$, contrary to the compapactness of X^*

The dual arguments for $\mathscr{U}\downarrow(a)$ and $\mathscr{V}\downarrow(a)$ complete the proof.

By Magill[2] $\mathscr{V}(\omega_i)$ is the filter of punctured neighbourhoods of the theorem. $\mathscr{C}$ defines the topology T^*. The extension of item (3) to arbitrary points of X^* defines the order $\leqslant^*$. Thus, the collection $\mathscr{C}_*$ is determined by $\mathscr{C}$, and Theorem 1 gives a characterization of $\mathscr{C}$ intrinsic to X.

Lemma 2—If $(Y, \tau, \leqslant)$ is a compoct T_2-ordered lattice, with $\mathscr{V}\uparrow(y)$ representing filter generated by the τ-open $\leqslant$increasing neighbourhoods of y, then $\mathscr{V}\uparrow(x) \vee \mathscr{V}\uparrow(y) = V\uparrow(x \vee y)$.

Proof : Clearly $\mathscr{V}\uparrow(x) \vee \mathscr{V}\uparrow(y) \leqslant \mathscr{V}\uparrow(x \vee y)$. Conversely, let N be an open increosing element of $\mathscr{V}\uparrow(x \vee y)$. For $c \in Y\setminus N$, c is not an upper bound of

x and y, so either $x \not\leqslant c$ or $\not\leqslant c$. Thus, there exists an open decreasing neighbourhood N_c of c disjoint from some open increasing neighbourhood M_c of. x or y. Now $\{N_c : c \in Y \setminus N\}$ is an open cover of the compact $Y \setminus N$. If $\{N_c : c \in F\}$ is a finite subcover, then $\bigcap_{c \in F} M_c \subseteq N$ and $\bigcap_{c \in F} M_c \in \mathscr{V}^{\uparrow}(x) \vee \mathscr{V}^{\uparrow}(x) \vee \mathscr{V}^{\uparrow}(y)$.

Theorem 3—Let X^* be an n-point order compactification X, and $\mathscr{C} = \{\mathscr{V}^{\uparrow}(a), \mathscr{V}^{\downarrow}(a) : a \in X^*\}$ be the associated collection of trace filters on X. Then X^* is a lattice iff both $\mathscr{C}^{\uparrow} \equiv \{\mathscr{V}^{\downarrow}(a) : a \in X^*\}$ and $\mathscr{C}^{\downarrow} \equiv \{\mathscr{V}^{\downarrow}(a) : a \in X^*\}$ from upper semi-lattices.

Proof : That $\mathscr{C}^{\uparrow}$ and $\mathscr{C}^{\downarrow}$ from upper semi-lattices, when X is lattice follows from Lemma 2 and its dual Conversely, if $C^{\uparrow}$ is an upper semi-lattice then for any $a,b \in X^*$. $V^{\uparrow}(a) \vee \mathscr{V}^{\uparrow}(b)$ exists and is equal to an element $\mathscr{V}^{\uparrow}(c) \in C^{\uparrow}$ for some $c \in X^*$. Now $\mathscr{V}^{\uparrow}(a) \leqslant \mathscr{V}^{\uparrow}(c) \leqslant \mathscr{V}(c)$ implies $\mathscr{V}^{\uparrow}(a) \vee \mathscr{V}^{\downarrow}(c)$ exists, whence $a \leqslant^* c$. Similarly, $b \leqslant^* c$, so that c is an upper bounded of a and b. If d is another upper bound of a and b, then the existence of $\mathscr{V}^{\uparrow}(a) \vee \mathscr{V}^{\downarrow}(d)$ implies $\mathscr{V}^{\uparrow}(a) \leqslant \mathscr{V}(d)$ and $\mathscr{V}^{\uparrow}(b) \leqslant \mathscr{V}(d)$. Thus. $\mathscr{V}^{\uparrow}(c) = \mathscr{V}^{\uparrow}(a) \vee \mathscr{V}^{\uparrow}(b) \leqslant \mathscr{V}(d)$ so that $\mathscr{V}^{\uparrow}(c) \vee \mathscr{V}^{\downarrow}(d)$ exists, that is, $c \leqslant^* d$.

A similar argument shows that the existence of $a \wedge b$ follows from $\mathscr{C}^{\downarrow}$ being an upper semi-lattice.

Let X be a bounded $T_{3,5}$-ordered poset and X^* be any T_2-order compactification of X. Then X^* must be bounded. Furthermore, any pair $a, b \in X^*$ must have a minimal upper bound since the upper bounds of a and b are given by $i(a) \cap i(b)$, which is closed since X^* is T_1-ordered, and a theorem of Wallace (Theorem 1 of Ward[5]) guarantees that a non-empty compact T_2-ordered space must have a minimal element. Thus, X^* fails to be a lattice iff there exist $a, b \in X^*$ such that a and b have two minimal upper bounds, or two maximal lower bounds.

References

1. T H. Choe, *An. Acad. Brasil. Ciênc.* **51** (1) 1979), 53-63.
2. K. D. Magill (Jr). *Am. Math. Monthly,* **72** (1965), 1075-81.
3. L. Nachbin, *Topology and Order*, Van Nostrand, New York Math Studies, 4. Princeton, NJ, (1965).
4. T. A. Richmond, *Math. Proc. Camb Phil. Soc.* **102** (1987), 466-73.
5. L. E. Ward (Jr.), *Proc. Am. Math. Soc.* **5** (1954), 144-61.