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PART SIZES OF RANDOM INTEGER PARTITIONS*

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Put a uniform probablity distribution on the set of partitions of the integer n. Given a fixed integer d, let $Y_{i,n}$ be the number of part sizes that are congruent to i mod d. The random vector $\mathbf{Y}_n = (Y_{0,n}, Y_{1,n}, \ldots, Y_{d-1,n})$ is asymptotically normally distributed. A generalization of this statement is proved by combining the continuity theorem for moment generating functions and some methods of Meinardus (*Math. Zeit* **59** (1954), 388-98).

1. INTRODUCTION

Given an infinite set of positive integers $\mathcal{A}$, let $U(n, \mathcal{A})$ be the set of partitions of the integer n into parts are elements of $\mathcal{A}$. If we put the uniform probability measure on $U(n, \mathcal{A})$, then any real valued function on $U(n, \mathcal{A})$ can be regarded as a random variable. This paper concerns the joint distribution of the number of part sizes in various subsets of $\mathcal{A}$.

Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_d$ be disjoint sets of positive integers whose union is $\mathcal{A}$. For $\lambda \in U(n, \mathcal{A})$, let $Y_{i,n}(\lambda)$ be the number of part sizes that λ has in $\mathcal{A}_i$, counted without multiplicity. The random variables $Y_{i,n}$ $(i = 1, 2, \ldots, d)$ are obviously not independent. If the sets satisfy certain conditions (essentially due to Meinardus), the random vector $\mathbf{Y}_n = (Y_{1,n}, Y_{2,n}, \ldots, Y_{d,n})$ is asymptotically normally distributed. A curious feature is that not all the sets need to satisfy all the conditions. The precise statement is in Theorem 1 of the next section.

2. MAIN THEOREM

Our sets $\mathcal{A}_i$ will all satisfy the following two conditions of Meinardus.

(M1) : Let $D_i(s) = \sum_{a \in \mathcal{A}_i} \frac{1}{a^s}$. (Throughout this paper, s is a complex number

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with real and imaginary parts σ and t respectively; $s = \sigma + it$.) Then the Dirichlet series $D_i(s)$ converges in the half plane $\mathcal{Re}(s) > \alpha_i > 0$, and it can be analytically continued to $\mathcal{Re}(s) > -C_0$ for some $C_0 \in (0, 1)$. In $\mathcal{Re}(s) > -C_0, D_i(s)$ is analytic, except for a simple pole at $s = \alpha_i$ with residue R_i.

(**M2**) : There is an absolute constant C_1 such that $D_i(s) = O(|t|^{C_1})$ as $|t| \to \infty$, and uniformly for $\mathcal{Re}(s) > -C_0$.

At least one of our sets will satisfy the following :

(**M3**) : Define $g_i(\tau) = \sum_{a \in \mathcal{A}_i} e^{-a\tau}$, where $\tau = y + 2\pi i x$, and x and y are real with $y > 0$. Then for some $\varepsilon > 0$

$$g_i(y) - \mathcal{Re}(g_i(\tau)) \gg y^{-\varepsilon}$$

as $y \to 0^+$, provided that $y \to 0^+$ in such a way that $|\arg(\tau)| > \frac{\pi}{4}$ and $|x| \leq 1/2$.

In order to state our results, we need to define some constants. Let $\mathcal{A} = \bigcup_{i=1}^{d} \mathcal{A}_i$, and let $R = \sum_{i=1}^{d} R_i$. Given $\alpha > 0$, let $c_\infty := R\Gamma(\alpha)\zeta(\alpha+1)$, and let $\vec{m}_n := (m_{1,n}, m_{2,n}, \ldots m_{d,n})$, where

$$m_{i,n} := n^{\alpha/(\alpha+1)} \cdot (\alpha c_\infty)^{-\alpha/(\alpha+1)} \Gamma(\alpha) R_i.$$

Define the $d \times d$ matrix $\widetilde{Q} = (\widetilde{Q}_{i,j})_{i,j}$ by

$$Q_{i,j} = \left(1 + \frac{1}{\alpha}\right)(\alpha c_\infty)^{\frac{1}{\alpha+1}} \left\{ -\frac{R_i R_j \alpha \Gamma(\alpha)^2}{(\alpha+1)^2 c_\infty^2} + \delta_{i,j} \frac{R_i \Gamma(\alpha)(1 - \frac{1}{2^\alpha})}{(\alpha+1) c_\infty} \right\}$$

where $\delta_{i,j}$ is one if $i = j$, and zero otherwise. Finally, let $\widetilde{Q}_{i,j} = \widetilde{Q}_{i,j}/(\det \widetilde{Q})^{1/d}$, and let $B_n : (\det \widetilde{Q})^{1/2d} n^{(\alpha/2\alpha+2)}$. The main theorem of this section is

Theorem 1 — Suppose that each of the sets $\mathcal{A}_i$ satisfies $M1$ and $M2$ with a common pole $\alpha_i = \alpha$ ($i = 1, 2, \ldots, d$). If at least one of the sets $\mathcal{A}_i$ satisfies $M3$, then the distribution of the random vector $\left(\frac{Y_n - \vec{m}_n}{B_n}\right)$ converges to a d-dimensional normal distribution, centered at the origin, with covariance matrix Q.

First we remark that much of the proof parallels Meinardus' estimates[1,4] for the cardinality of $\mathcal{U}(n, \mathcal{A})$.

PROOF : Let $t_1, t_2, \ldots, t_d$ be real numbers that are fixed (independent of n), but otherwise arbitrary. Let $\vec{t} = (t_1, t_2, \ldots, t_d)$. By the Cramer-Wold device[2], it suffices to prove that

$$E\left(\exp\left[\sum_{i=1}^{d} t_i\left(\frac{\mathbf{Y}_{i,n}-m_{i,n}}{B_n}\right)\right]\right) = \exp\left[\frac{1}{2}\ \vec{t}\ Q\ \vec{t}^{\ .T}\right] + o\ (1) \ldots (1)$$

Given $\vec{k} = (k_1, k_2, \ldots, k_d)$, define

$$p_{n,\vec{k}} := \text{Prob}\ (\mathbf{Y}_{i,n} = k_i \text{ for all } i) = \frac{\left|\left\{\lambda \in U(n, \mathcal{A}) : \mathbf{Y}_n = \vec{k}\right\}\right|}{|\ U(n, \mathcal{A})\ |}.$$

Then (see (4))

$$p_{n,\vec{k}} = \frac{1}{|\ U(n, \mathcal{A})\ |}\ [[\ w^n y_1^{k_1}\ y_2^{k_2} \ldots y_d^{k_d}\ ]]\ \prod_{i=1}^{d}\ \prod_{a \in \mathcal{A}_i}\left(1 + \frac{y_i\, w^a}{1 - w^a}\right).$$

Replacing y_i with $\exp\left(t_i\left(\frac{\mathbf{Y}_{i,n} - m_{i,n}}{B_n}\right)\right)$, we get

$$E\left(\exp\left[\sum_{i=1}^{d} t_i\left(\frac{Y_{i,n}-m_{i,n}}{B_n}\right)\right]\right) = \frac{\exp\left[-\sum_{i=1}^{d} t_i\left(\frac{m_{i,n}}{B_n}\right)\right]}{|\ U(n, \mathcal{A})\ |}\ H(n, \vec{t}),$$

where

$$H(n, \vec{t}) = [[\ w^n\ ]]\ \prod_{i=1}^{d}\ \prod_{a \in \mathcal{A}_i}\left(1 + \frac{e^{t_i/B_n}\, w^a}{1 - w^a}\right).$$

Integrating around a circle of radius less than one, we get,

$$H(n, \vec{t}) = \frac{1}{2\pi i}\int \prod_{i=1}\ \prod_{a \in \mathcal{A}_i}\left(1 + \frac{e^{t_i/B_n}\, w^a}{1 - w^a}\right)\cdot \frac{dw}{w^{n+1}}.$$

Next let $X_{i,n} = e^{t_i/B_n} - 1$, and observe that

$$\left(1 + \frac{e^{t_i/B_n}\, w^a}{1 - w^a}\right) = \left(\frac{1 + X_{i,n}\, w^a}{1 - w^a}\right).$$

If we let $w = e^{-\tau}$, and let

$$F_n(\tau) := \prod_{i=1}^{d}\ \prod_{a \in \mathcal{A}_i}\left(\frac{1 + X_{i,n}\, e^{-a\tau}}{1 - e^{-a\tau}}\right)$$

then

$$H(n, \vec{t}) = \frac{1}{2\pi\, i}\int F_n(\tau)\ e^{n\tau}\, d\tau = \int_{-1/2}^{1/2} F_n(y + 2\,\pi\, xi)\ e^{n(y + 2\,\pi\, xi)}\, dx. \ \ldots (2)$$

In order to estimate this integral, we need to define some auxilliary functions. Let $\Delta_{i,n}(s) = \sum_{l \geq 1} \frac{(-1)^{l-1} X_{i,n}^{l}}{l^{s}}$, and let $Z_{i,n}(s) = \zeta(s) + \Delta_{i,n}(s)$. For all sufficiently large n, we have $X_{i,n} < 1/2$ $(1 \leq i \leq d)$. Therefore $\Delta_{i,n}(s)$ is an analytic function on $|\mathcal{R}e(s)| \leq C$. (Throughout this paper, C will denote a generic, unspecified positive constant; it need not have the same value in different places). It is also easy to see that

$$\Delta_{i,n}(s) = o(1) \qquad \text{... (3)}$$

uniformly as $n \to \infty$. Thus $Z_{i,n}(s)$ is analytic in the whole complex plane, except for a simple pole at $s = 1$ with residue 1. Finally, let $D_{\mathcal{A}}(s) = \sum_{a \in \mathcal{A}} \frac{1}{a^{s}}$.

With these definitions at our disposal, we can proceed to the following cruical lemma.

Lemma 1 — If $y \to 0^{+}$ in such a way that $|\arg \tau| \leq \frac{\pi}{4}$ then

$$F_{n}(\tau) = \exp\left[\left(\sum_{i=1}^{d} Z_{i,n}(\alpha_{i}+1) A_{i} \Gamma(\alpha) \tau^{-\alpha} + \Delta_{i,n}(1) D_{i}(0)\right) - D_{\mathcal{A}}(0) \log \tau + D_{\mathcal{A}}'(0) + O(y^{C})\right].$$

PROOF : Recall that $\mathcal{R}e(\tau) > 0$, and observe that $X_{i,n} \to 0$ as $n \to \infty$. Therefore, for all large n, we have

$$\log F_{n}(\tau) = \sum_{i=1}^{d} \sum_{a \in \mathcal{A}_{i}} \sum_{l=1}^{\infty} \left(\left(\frac{1 + (-1)^{l-1} X_{i,n}^{l}}{l}\right) e^{-a \tau l}\right).$$

Using the identity $e^{-w} = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} w^{-s} \Gamma(s)\, ds$, (with C large enough to justify the use of Fubini's theorem), we get

$$\log F_{n}(\tau) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \sum_{i=1}^{d} \sum_{l=1}^{\infty} \left(\frac{1 + (-1)^{l-1} X_{i,n}^{l}}{l^{s+1}}\right) \left(\sum_{a \in \mathcal{A}_{i}} \frac{1}{a^{s}}\right) \tau^{-s} \Gamma(s)\, ds$$

$$= \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \sum_{i=1}^{d} Z_{i,n}(s+1) D_{i}(s) \tau^{-s} \Gamma(s)\, ds.$$

We plan to shift the integration contour to the left. Combining (3) with well known properties of the Riemann zeta function, we see that, as $|t| \to \infty$, we have

$$Z_{i,n}(s+1) = \zeta(s) + O(1) = O(|t|^{C}).$$

It is well known that, for any $\varepsilon > 0$, we have $\Gamma(s) = O(e^{-\left(\frac{\pi}{2}-\varepsilon\right)|t|})$ in the vertical strip $-C_0 \le \sigma \le C$. Finally, because $|\arg(\tau)| \le \frac{\pi}{4}$, we have $|\tau^{-s}| \le |\tau|^{-\sigma} \exp(\pi|t|/4)$. We can therefore shift the line of integration to the left using the residue theorem :

$$\log F_n(\tau) = (\text{Res. at } s = \alpha) + (\text{Res. at } s = 0)$$

$$+ \frac{1}{2\pi i} \sum_{i=1}^{d} \int_{-C-i\infty}^{-C+i\infty} Z_{i,n}(s+1) D_{\mathcal{A}}(s) \tau^{-s} \Gamma(s)\, ds$$

$$= (\text{Residue at } s = \alpha) + (\text{Residue at } s = 0) + O(y^{C}).$$

To compute these residues, observe that, as $s \to 0$,

$$Z_{i,n}(s+1) = \frac{1}{s} + \gamma + \Delta_{i,n}(1) + O(s).$$

We also have $\tau^{-s} = 1 - s\log\tau + O(s^2)$ and $\Gamma(s) = \frac{1}{s} - \gamma + O(s)$. Hence

$$(\text{Residue at } s = \alpha) + (\text{Residue at } s = 0)$$

$$= \sum_{i=1}^{d} \left(Z_{i,n}(\alpha+1) R_i \Gamma(\alpha) \tau^{-\alpha} - D_i(0)\log\tau + D_i'(0) + \Delta_{i,n}(1) D_i(0) \right).$$

Now let $c_n = \sum_{i=1}^{d} Z_{i,n}(\alpha+1)\Gamma(\alpha) R_i$, set $\beta := 1 + \frac{5\alpha}{12}$, and let $\delta_n = \sum_{i=1}^{d} \Delta_{i,n}(1) D_i(0)$. Also let

$$y = y_n = \left(\frac{\alpha c_n}{n}\right)^{1/(\alpha+1)} \qquad \text{.... (4)}$$

Then from (2) we get

$$H(n, \vec{t}) = \int_{-y^{\beta}}^{y^{\beta}} F_n(y + 2\pi i x)\, e^{ny + 2\pi i n x}\, dx$$

$$+ \int_{y^{\beta} \le |x| \le \frac{1}{2}} \stackrel{\text{def}}{=} T_1 + T_2.$$

By Lemma 1, we have

$$T_1 = e^{ny + D_{\mathcal{A}}(0) + \delta_n} \int_{-y^\beta}^{y^\beta} \exp\left[c_n \tau^{-\alpha} - D_{\mathcal{A}}(0) \log \tau + 2\pi nix + O(y^C) \right] dx$$

$$= (1 + O(y^C)) \exp\left[ny + c_n y^{-\alpha} - D_{\mathcal{A}}(0) \log y + D'_{\mathcal{A}}(0) + \delta_n \right] \cdot I_1$$

where

$$I_1 = \int_{|x| \le y^\beta} \exp\left[c_n y^{-\alpha} \left(\left(1 + \frac{2\pi ix}{y}\right)^{-\alpha} - 1\right) + 2\pi inx - D_{\mathcal{A}}(0) \log\left(1 + \frac{2\pi ix}{y}\right) \right] dx.$$

Using the method of Laplace, one can verify that

$$I_1 = (2\pi c_n y^{-\alpha-2} \alpha(\alpha+1))^{-1/2} (1 + o(1)).$$

Using arguments similar to those of Meinardus (see appendix), one can verify that $T_2 = o(T_1)$. We therefore have

$$E\left(\exp\left[\sum_{i=1}^{d} t_i \left(\frac{Y_{i,n} - m_{i,n}}{B_n} \right) \right] \right) \qquad ..... (5)$$

$$= \frac{\exp\left[ny + c_n y^{-\alpha} - \sum_{i=1}^{d} \frac{t_i m_{i,n}}{B_n} \right]}{|U(n, \mathcal{A})|} \cdot \frac{\exp\left[-D_{\mathcal{A}}(0) \log y + D_{\mathcal{A}}(0)\right]}{(2\pi c_n y^{-\alpha-2} \alpha(\alpha+1))^{1/2}} (1 + o(1)).$$

Meinardus' theorem asserts that

$$|U(n, \mathcal{A})| = \exp\left[(\alpha c_\infty)^{1/\alpha+1} \left(1 + \frac{1}{\alpha}\right) n^{\alpha/\alpha+1} \right] \cdot n^\kappa \cdot K_0 (1 + o(1))$$

where $\kappa = \dfrac{D_{\mathcal{A}}(0) - 1 - \alpha/2}{1+\alpha}$, and $K_0 = \dfrac{e^{D_{\mathcal{A}}(0)} (\alpha c_\infty)^{(1 - 2D_{\mathcal{A}}(0))/(2+2\alpha)}}{\sqrt{2\pi(1+\alpha)}}$. Using (4), (5), the fact that $c_n \to c_\infty$, and the fact that $\delta_n \to 0$ (see (3)), we get

$$E\left(\exp\left[\sum_{i=1}^{d} t_i \left(\frac{Y_{i,n} - m_{i,n}}{B_n} \right) \right] \right)$$

$$= \exp\left[ny + c_n y^{-\alpha} - \left(1 + \frac{1}{\alpha}\right) (\alpha c_\infty)^{1/\alpha+1} n^{\alpha/\alpha+1} - \sum_{i=1}^{d} \frac{t_i m_{i,n}}{B_n} \right] (1 + o(1)).$$

We must show that the exponent approaches $\frac{1}{2} \vec{t} Q \vec{t}^{\,T}$. Using (4), we get

$$ny + c_n y^{-\alpha} - \left(1 + \frac{1}{\alpha}\right)(\alpha\, c_\infty)^{1/\alpha+1}\, n^{\alpha/\alpha+1}$$

$$= \left(1 + \frac{1}{\alpha}\right)(\alpha\, c_\infty)^{1/\alpha+1}\, n^{\alpha/\alpha+1} \left(-1 + \left(\frac{c_n}{c_\infty}\right)^{1/\alpha+1}\right) \qquad \ldots (6)$$

To simplify this, observe that

$$\frac{c_n}{c_\infty} = 1 + \sum_{i+1}^{d} \frac{t_i}{B_n} \cdot \frac{\Gamma(\alpha) R_i}{c_\infty} + \sum_{i=1}^{d} \frac{t_i^2}{B_n^2} \left(\left(\frac{1}{2} - \frac{1}{2^{\alpha+1}}\right) \frac{\Gamma(\alpha) R_i}{c_\infty}\right) + O_t\left(\frac{1}{B_n^3}\right),$$

and therefore

$$-1 + \left(\frac{c_n}{c_\infty}\right)^{\frac{1}{\alpha+1}} = \sum_{i=1}^{d} \frac{t_i}{B_n} \frac{\Gamma(\alpha) R_i}{(\alpha+1)\, c_\infty} + \sum_{i=1}^{d} \frac{t_i^2}{2B_n^2} \left(\left(1 - \frac{1}{2^\alpha}\right) \frac{\Gamma(\alpha) R_i}{(\alpha+1)\, c_\infty}\right) + \sum_{1 \le i,j \le d} \frac{t_i t_j}{B_n^2} \left(\frac{-\alpha \Gamma(\alpha)^2 R_i R_j}{2(\alpha+1)^2 c_\infty^2}\right) + O_t\left(\frac{1}{B_n^3}\right).$$

Putting this into (6), we get

$$ny + c_n y^{-\alpha} - \left(1 + \frac{1}{\alpha}\right)(\alpha\, c_\infty)^{1/\alpha+1}\, n^{\alpha/\alpha+1} = \sum_{i=1}^{d} \frac{t_i m_{i,n}}{B_n} + \frac{1}{2} \vec{t}\, Q\, \vec{t}^{\,T} + O_t\left(\frac{1}{B_n^3}\right)$$

and consequently

$$E\left(\exp\left[\sum_{i=1}^{d} t_i \left(\frac{Y_{i,n} - m_{i,n}}{B_n}\right)\right]\right) = \exp\left[\frac{1}{2} \vec{t}\, Q\, \vec{t}^{\,T} + O_t\left(\frac{1}{B_n^3}\right)\right].$$

3. Application

We close with a simple example that motivates Theorem 1. Let $\mathcal{A}_1$ be the set of even positive integers, and let $\mathcal{A}_2$ be the set of odd positive integers. Then

$D_1(s) = \frac{1}{2^s}\zeta(s)$, $D_2(s) = \left(1 - \frac{1}{2^s}\right)\zeta(s)$, $\alpha = 1$, and $R_1 = R_2 = \frac{1}{2}$. We set $m_{1,n} = m_{2,n} = \frac{\sqrt{6n}}{2\pi}$, and set $B_n = \left(\frac{3}{8\pi^2} - \frac{9}{4\pi^4}\right)^{1/4} n^{1/4}$. Then Theorem 1 implies that the random vector $\left(\frac{Y_{1,n} - m_{1,n}}{B_n}, \frac{Y_{2,n} - m_{2,n}}{B_n}\right)$ converges to a normally distributed vector with means $\vec{0} = (0, 0)$ and covariance matrix

$$Q = \left(\frac{3}{8\pi^2} - \frac{9}{4\pi^4}\right)^{-1/2} \begin{pmatrix} \frac{\sqrt{6}}{4\pi} - \frac{3\sqrt{6}}{4\pi^3} & \frac{-3\sqrt{6}}{4\pi^3} \\ \frac{-3\sqrt{6}}{4\pi^3} & \frac{\sqrt{6}}{4\pi} - \frac{3\sqrt{6}}{4\pi^3} \end{pmatrix}.$$

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APPENDIX

We must show that $T_2 = o(T_1)$. We have

$$|T_2| \le e^{ny} \int_{y^\beta \le |x| \le \frac{1}{2}} |F_n(y + 2\pi xi)|\, dx = T_{2,1} + T_{2,2},$$

where

$$T_{2,1} = e^{ny} \int_{y^\beta \le |x| \le \frac{y}{2\pi}} \qquad T_{2,2} = e^{ny} \int_{\frac{y}{2\pi} \le |x| \le \frac{1}{2}}$$

To estimate $T_{2,1}$, we argue as before :

$$\log F_n(y + 2\pi xi) = c_n \tau^{-\alpha} + O\left(\log \frac{1}{y}\right)$$

and therefore

$$\log \mid F_n(y + 2\pi xi) \mid \leq c_n \mid \tau \mid^{-\alpha} + O\left(\log \frac{1}{y}\right)$$

$$= c_n y^{-\alpha}\left(1 + \frac{4\pi^2 x^2}{y^2}\}\right)^{-\alpha/2} + O\left(\log \frac{1}{y}\right)$$

$$\leq c_n y^{-\alpha} - c_n y^{-\alpha} c \frac{x^2}{y^2} + O\left(\log \frac{1}{y}\right) \leq c_n y^{-\alpha} - Cy^{-\alpha + 2(\beta - 1)}.$$

Thus

$$\mid T_{2,1} \mid \leq (e^{ny\, e^{c_n y^{-\alpha}}})\, e^{-y^{-\epsilon}} = .o(T_1).$$

To estimate $T_{2,2}$, we bound the logarithm of the integrand :

$$\log \mid F_n(y + 2\pi xi) \mid$$

$$= \sum_{i=1}^{d} \sum_{a \in \mathcal{A}_1} \sum_{l \geq 1} \frac{(1 + (-1)^{l-1} X_{i,n}^l}{l} e^{-aly} \cos(2\pi xal).$$

Isolating the terms with $l = 1$, and then estimating the remaining terms, we get

$$\sum_{i=1}^{d} \sum_{a \in \mathcal{A}_1} \sum_{l \geq 2} \frac{(1 + (-1)^{l-1} X_{i,n}^l}{l} e^{-aly} \cos(2\pi xal)$$

$$+ \sum_{i=1}^{d} \sum_{a \in \mathcal{A}_i} (1 + X_{i,n})\, e^{-ay} \cos(2\pi xa)$$

$$\leq \sum_{i=1}^{d} \sum_{a \in \mathcal{A}_1} \sum_{l \geq 2} \frac{(1 + (-1)^{l-1} X_{i,n}^l)}{l} e^{-aly}$$

$$+ \sum_{i=1}^{d} \sum_{a \in \mathcal{A}_i} (1 + X_{i,n})\, e^{-ay} \cos(2\pi xa)$$

$$= \log F_n(y) - \sum_{i=1}^{d} (1 + X_{i,n} (g_i(y) - \mathcal{Re}(g_i(\tau))).$$

By assumption, we have $\mathcal{Re}(g_i(\tau)) - g_i(y) < -Cy^{-\epsilon}$ for $\frac{y}{2\pi} \leq \mid x \mid \leq \frac{1}{2}$.

We therefore have

$$\log \mid F_n(y + 2\pi xi) \mid \leq \log F_n(y) - Cy^{-\epsilon}$$

and consequently

$$\mid T_{2,2} \mid \leq (e^{ny} F_n(y))\, e^{-C_y^{-\epsilon}} = o(T_1).$$