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POLYNOMIAL SOLUTIONS FOR THE PELL'S EQUATION

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Let D be a given square-free natural number. In this paper some polynomial solutions are given for the Pell's equations $A^2 - DB^2 = \pm 1$ which are significant from computational point of view.

1. INTRODUCTION

Let D be a given square-free natural number. It is well known that the Pell's equation

$$A^2 - DB^2 = 1 \qquad \text{... (1)}$$

always has an infinite number of integral solutions (see e.g. Nagell[3]). All the solutions of (1) with positive A and B are obtained by the formula $A_r + B_r\sqrt{D} = (a + b\sqrt{D})^r$ where $r = 1, 2, 3, \ldots$, and $a + b\sqrt{D}$ is the fundamental solution of (1), i.e., $A_1 = a$ and $B_1 = b$.

The Pell's equation

$$A^2 - DB^2 = -1 \qquad \text{... (2)}$$

may not always have integral solutions. When this equation is solvable in integers for some given D, if $L + M\sqrt{D}$ be its fundamental solution, we have the relation

$$a + b\sqrt{D} = (L + M\sqrt{D})^2 \qquad \text{... (3)}$$

where $a + b\sqrt{D}$ is the fundamental solution of (1).

For a given value of D, in order to determine the fundamental solution of (1), or (2) (when it exists), one may use the method of continued fraction expansion of $\sqrt{D}$. The penultimate convergent of this expansion provides the fundamental solution. From computational point of view, it would be advantageous to group the values of D into several classes such that the values of D in a specific class are associated with a similar pattern of continued fraction expansion. It is found that polynomial expressions for D come in handy for this purpose. As a consequence, some polynomial solutions are given in this paper for the Pell's equations $A^2 - DB^2 = \pm 1$.

2. The Pell's Equation $A^2 - DB^2 = 1$

Theorem 1 — Suppose $D = 4t^2 + 12t + 5$, where t is any natural number. Then $a + b\sqrt{D}$ is the fundamental solutions of (1) where $a = 4t^3 + 18t^2 + 24t + 9$, $b = 2t^2 + 6t + 4$.

Proof : For real x, let the symbol $[x]$ denote the greatest integer $\leq x$. We have

$$(2t+2) < \sqrt{D} < (2t+3)$$

and so $[\sqrt{D}] = 2t + 2$. Now, expanding $\sqrt{D}$ into a continued fraction expansion, we obtain

$$\sqrt{D} = [\sqrt{D}] + \sqrt{4t^2 + 12t + 5} - (2t + 2)$$

$$= (2t + 2) + \cfrac{1}{\cfrac{\sqrt{4t^2 + 12t + 5} + (2t + 2)}{4t + 1}}$$

$$= (2t + 2) + \cfrac{1}{1 + \cfrac{1}{\cfrac{\sqrt{4t^2 + 12t + 5} + (2t - 1)}{4}}}$$

$$= (2t + 2) + \cfrac{1}{1 + \cfrac{1}{t + \cfrac{1}{\cfrac{\sqrt{4t^2 + 12t + 5} + (2t + 1)}{2t + 1}}}}$$

$$= \ldots = | 2t + 2, \overline{1, t, 2, t, 1, 4t + 4} | .$$

The penultimate convergent is

$$(2t + 2) + \frac{1}{1+}\,\frac{1}{t+}\,\frac{1}{2+}\,\frac{1}{t+}\,\frac{1}{1} = \frac{4t^3 + 18t^2 + 24t + 9}{2t^2 + 6t + 4}.$$

Thus we get the relation

$$(4t^3 + 18t^2 + 24t + 9)^2 - (4t^2 + 12t + 5)(2t^2 + 6t + 4)^2 = 1.$$

Theorem 2 — Suppose $D = m^2 t^4 + 2mnt^3 + n^2 t^2 + mt + n$ where m, n, t are any natural numbers with $mn \neq 0$. Then $a + b\sqrt{D}$ is the fundamental solution of (1) where $a = 2mt^3 + 2nt^2 + 1, b = 2t$.

Proof : $[\sqrt{D}] = mt^2 + nt$ and the continued fraction expansion of $\sqrt{D}$ is $| mt^2 + nt, \overline{2t, 2mt^2 + 2nt} |$. The penultimate convergent is $\dfrac{2mt^3 + 2nt^2 + 1}{2t}$.

Theorem 3 — Suppose $D = m^2t^4 + 2mnt^3 + n^2t^2 + 2mt + 2n$ where m, n, t are any natural numbers with $mn \neq 0$. Then $a + b\sqrt{D}$ is the fundamental solution of (1) where $a = mt^3 + nt^2 + 1$, $b = t$.

PROOF : $[\sqrt{D}] = mt^2 + nt$ and the continued fraction expansion of $\sqrt{D}$ is $| mt^2 + nt, \overline{t, 2mt^2 + 2nt} |$. The penultimate convergent is $\frac{mt^3 + nt^2 + 1}{t}$.

3. The Pell's Equation $A^2 - DB^2 = -1$

Theorem 4 — If $D = 4t^2 + 4t + 5$ where t is any natural number, then $L + M\sqrt{D}$ is the fundamental solution of (2) where $L = 4t^3 + 6t^2 + 6t + 2$, $M = 2t^2 + 2t + 1$.

PROOF : $[\sqrt{D}] = 2t + 1$ and the continued fraction expansion of $\sqrt{D}$ is $| 2t + 1, \overline{t, 1, 1, t, 4t + 2} |$. The penultimate convergent is

$$(2t + 1) + \frac{1}{t+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{t} = \frac{4t^3 + 6t^2 + 6t + 2}{2t^2 + 2t + 1}.$$

Thus we get the relation

$$(4t^3 + 6t^2 + 6t + 2)^2 - (4t^2 + 4t + 5)(2t^2 + 2t + 1)^2 = -1.$$

Theorem 5 — If $D = 4t^4 + 4t + 2$, where t is any natural number ≥ 2, then $L + M\sqrt{D}$ is the fundamental solution of (2) where $L = 4t^4 - 4t^3 + 2t^2 + 2t - 1$, $M = 2t^2 - 2t + 1$.

PROOF : $[\sqrt{D}] = 2t^2$ and the continued fraction expansion of $\sqrt{D}$ is $| 2t^2, \overline{t - 1, 1, 1, t - 1, 4t^2} |$. The penultimate convergent is $\frac{4t^4 - 4t^3 + 2t^2 + 2t - 1}{2t^2 - 2t + 1}$.

Theorem 6 — If $D = 16t^4 + 8t^3 + 9t^2 + 6t + 2$ where t is any natural number, then $L + M\sqrt{D}$ is the fundamental solution of (2) where $L = 16t^4 + 4t^3 + 8t^2 + 3t + 1$, $m = 4t^2 + 1$.

PROOF : $[\sqrt{D}] = 4t^2 + t + 1$ and the continued fraction expansion of $\sqrt{D}$ is $| 4t^2 + t + 1, \overline{2t, 2t, 8t_2 + 2t + 2} |$. The penultimate convergent is $\frac{16t^4 + 4t^3 + 8t^2 + 3t + 1}{4t^2 + 1}$.

4. The General Pell's Equation

With the help of the following theorem, one may generate the solutions of the general Pell's equation

$$U^2 - DV^2 = N \qquad \text{... (4)}$$

where N is a non-zero integer.

Theorem 7 — Let u, v be integers such that $u^2 - Dv^2 = N$, where N is any given non-zero integer. Then

$$A_{(1)} = kv^2 \pm u$$

$$B_{(1)} = v,$$

$$D_{(1)} = k^2 v^2 \pm 2k\dot{u} + D$$

satisfy $A^2_{(1)} - D_{(1)} B^2_{(1)} = N$, where k is any natural number.

PROOF : Taking $A_{(1)}, B_{(1)}$ and $D_{(1)}$ as above, we have $A^2_{(1)} - D_{(1)} B^2_{(1)} = u^2 - Dv^2 = N$.

Corollary 1 — Let k, t be natural numbers. Then

$$A = kt^2 \pm 1$$

$$B = t$$

$$D = k^2 t^2 \pm 2k$$

satisfy (1).

5. THE DIOPHANTINE EQUATIONS $A^2 - DB^4 = \pm 1$

The solutions of the Diophantine equations

$$A^2 - DB^4 = \pm 1 \qquad \text{... (5)}$$

may be generated from the solutions of the Pell's equation

$$u^2 - 2v^2 = \pm 1 \qquad \text{... (6)}$$

as provided by the following.

Theorem 8 — Let u, v be integers such that

$$u^2 - 2v^k = \pm 1. \qquad \text{... (7)}$$

Then

$$A = u\,(v^k \mp 1)$$

$$B = v$$

$$D = u^2 \mp 4$$

satisfy

$$A^2 - DB^{2k} = \pm 1. \qquad \text{... (8)}$$

PROOF : With A, B and D as above, we obtain

$$A^2 - DB^{2k} = u^2 \mp 2u^2 v^k \pm 4v^{2k}$$

$$= u^2 \mp 2v^k (u^2 - 2v^k)$$

$$= u^2 \mp 2v^k (\pm 1) = u^2 - 2v^k = \pm 1.$$

When $k = 2$, (7) reduces to the Pell's equation (6) with $D = 2$ and hence it possesses an infinite number of solutions for both signs. Thus there are an infinite number of values of D for which the equation (5) has non-trivial integral solutions, as provided by Theorem 8. However, when $k \geq 3$, by Thue's theorem (Mordell[2]), (7) has at most a finite number of integral solutions. For e.g., in the case of $k = 4$, Ljunggren[1] proved that the only positive integer solutions of the Diophantine equation

$$u^2 - 2v^4 = -1$$

are $(u, v) = (1, 1)$ and $(239, 13)$. Using the latter solution of this equation in theorem 8, we obtain the relation

$$6826318^2 - 57125 \cdot 13^8 = -1$$

thus providing a solution of the Diophantine equation

$$A^2 - DB^8 = -1$$

with $D = 57125$.

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