Indian J. pure appl. Math., **25** (6) : 583-606, June 1994

THE GLOBAL CAUCHY PROBLEM FOR A HIGHER ORDER NONLINEAR SCHRÖDINGER EQUATION

XUESHANG FENG

The Seventh Research Division, Beijing University of Aeronautics and Astronautics, Beijing 100083, *China*

(*Received* 9 *March* 1993; *after revision* 5 *January* 1994; *accepted* 10 *January* 1994)

In this paper, the global existence of smooth solutions to the Cauchy problem for the second nonlinear Schrödinger equation in the Lax hierarchy of the nonlinear Schrödinger equation (NLS equation) is established by using Leray-Schauder fixed point theorem and delicate a priori estimates. In addition, the asymptotic properties of the solutions as $|x| \to +\infty$ are discussed

INTRODUCTION

The aim of this paper is to study the global existence of smooth solutions and their asymptotic behaviours of the Cauchy problem for the following so-called higher order NLS equation :

$$iu_t = u_{4x} + 4 \mid u \mid^2 u_{2x} + u^2 u^*_{2x} + 3u^* (u_x)^2 + 2 \mid u_x \mid^2 + u + \frac{3}{2} \mid u \mid^4 u \qquad \text{... (1)}$$

$$u(x, 0) = \varphi(x) \qquad \text{... (2)}$$

where subscripts stand for partial differentiation, $i = \sqrt{-1}$, $|u|$ is the norm of complex-valued function u, and u^* is the complex conjugate of u. Equation (1) is the second equation in the Lax hierarchy of the nonlinear Schrödinger equation

$$iu_t = \frac{\delta I_n}{\delta u^*}$$

in which the first two are (1) and the following standard NLS equation[23, 28, 29]

$$iu_t = -u_{2x} - |u|^2 u . \qquad \text{... (3)}$$

Here, I_n denotes the nth conserved quantity associated to NLS equation and $\frac{\delta I_n}{\delta u^*}$ is the functional derivative of I_n with respect to u^*[1, 24]. The initial value problem and initial-boundary value problem for equation (3) and its generalized forms have been widely studied in a lot of papers such as [2, 9-16, 25-27].

NLS equations have been of increasing interest due to their occurrences as mathematical models in several scopes of physics and their implications in the development of solitons and inverse scattering transform theory[1, 2, 8, 17, 20, 23, 28, 29].

NLS equations are completely integrable models[28], and exhibit solitary wave solutions in the Schwartz space $S(R)$ or with polynomial decay at infinity. Then it is of interest to solve the Cauchy problem in spaces with convenient weights. Ginibre-Velo[11], Tsutsumi[25], Hayashi, Nakamitsu and Tsutsumi[16] discussed the Cauchy problem for the second order NLS equation in weighted spaces $H^{s}\bigcap J_r^s$ with $s' = r = 1$, where $J_r^s(R^n) = [\omega^{r/2}(1-\Delta)^{s/2}]^{-1}L^2(R^n)$, $\omega(x) = \sqrt{1+|x|^2}$, $\Delta = \sum_{J=1}^{n} \partial_{x_1}^2$. Iorio discussed the Cauchy problem for the Benjamin-Ono equation (BO) in $H^{s'}\bigcap J_r^s$ with $s' = 0$, $r = 1, 2$ and $s' = 2$, $0 < r < 2$, and in comparable spaces by Hayashi *et al.*[16] and Guo and Tan *et al.*[9] for the second order NLS equation. Kato[19] discussed the Cauchy problem of the Korteweg-de Vries equation in $H^{2r}\bigcap J_r^s$. Feng[3-6] established the global existence and uniqueness of solutions in $J_r^s(R)$ to the Cauchy problem for KdV hierarchy, modified KdV hierarchy and higher order BO equation.

In the present work, we shall establish the global existence of smooth solutions in $H^s(R)$ and $J_r^s(R)$ to problem (1), (2).

The paper is organized as follows : Section 1 deals with preliminaries. Section 2 contains the a priori estimates of solutions to problem (1), (2). Sections 3 and 4 are devoted to solving problem (1), (2) in H^s and J_r^s, respectively.

1. Preliminaries and Statements

This part is to give the notation that will be used throughout this paper and to announce the results of this paper.

We shall be working on the one-dimensional space R equipped with the Lebesgue measure dx. Thus, as usual, $L^p(R)$, $1 \le p \le +\infty$ and $H^s(R)$, $s \in R$ will be the usual Lebesgue and Sobolev spaces with norms $|\cdot|_p$ and $||\cdot||_s$, respectively. If l is an interval and X a Banach space, with norm $||\cdot||_X$, $L^p(l;X) = \{u : l \to X$ such that $||u||_X \in L^p(l)\}$. $W_p^r(0,T;H^k(R))$ denotes the space of function $f(x, t)$ that has derivatives $D_t^s D_x^h f(t,x) \in L^p(0,T;L^2(R))$ with $0 \le s \le r$, $0 \le h \le k$. We denote by C, $C(\cdot,\cdot,\cdot)$ generic constants, not necessarily the same at each occurrence, which depend in an increasing way on the indicated quantities.

Let $Z = \{0, 1, 2, ...\}$, $\omega(x) = (1+x^2)^{1/2}$. Then for any $s, r \in R$, H_r^s is the completion of $S(R)$ under the norm $||u||_{r,s} = \left| \omega^r (1-D_x^2)^{\frac{s}{2}} u \right|_2$. Let $J_r^s = H_r^0 \cap H_0^s$ with the norm $|||u|||_{r,s} = (||u||_{r,0}^2 + ||u||_{0,s}^2)^{\frac{1}{2}}$. Under the inner product

$$(f, g)_{r,s} = (\omega^r (1-D_x^2)^{\frac{s}{2}} f,\ \omega^r (1-D_x^2)^{\frac{s}{2}} g).$$

H_r^s is a Hilbert space with dual $(H_r^s)' = H_{-s}^{-r}$; here $(\bullet\ ,\ \bullet)$ denotes the usual L^2 inner product, J_r^s is also a Hilbert space with the inner product $[u, v]_{r,s} = (u, v)_{r,0} + (u, v)_{0,s}$. Obviously, $H_0^s = H^s$ and $L^2 = H_0^0$.

The properties we shall need are the following[3, 25].

Theorem 1.1 — (a) $H_r^s \subseteq H_{r'}^{s'}$; $J_r^s \subseteq J_{r'}^{s'}$, $s \geq s'$; $r \geq r'$

(b) $[H_{r_1}^{s_1} ; H_{r_2}^{s_2}]_\theta = H_{(1-\theta)r_1 + \theta r_2}^{(1-\theta)s_1 + \theta s_2}$, $0 < \theta < 1, s_j, r_j \in R,\ j = 1, 2.$

where $[;]_\theta$ denotes the complex interpolation.

(c) $J_r^s \subseteq [H_r^0 ; H_0^s]_\theta = H_{(1-\theta)r}^{\theta s}$, $0 < \theta < 1,\ s,\ r \in R.$

(d) $\bigcap_{r,s\in R} H_r^s = \bigcap_{r,s\in Z} H_r^s = \bigcap_{r,s\in R} J_r^s = \bigcap_{r,s\in Z} J_r^s = S(R)$

(e) $F(H_r^s) = H_s^r$ with equivalent norms, where F is the Fourier transform.

(f) $(J_r^s)' = H_{-r}^0 + H_\theta^{-s}$

(g) Let $r,\ s > 0$. If $u \in J_r^{rs}$, then $u \in H_{r-1}^s$ and

$$||\ u\ ||_{r-1,s} \leq C(r, s)\ |||u|||_{r,rs}$$

(h) Let $r \in R$ and $s \geq 0$. Then for $h \in Z$ we have

$$|||D_x^h u|||_{r,s} \leq C\ |||\ u\ |||_{r',s'}$$

where $r' = (s+h)r/s$ and $s' = s+h$.

(i) If $s > \frac{1}{2}$, then for all $u, v \in J_r^s$ we have

$$|||\ uv\ |||_{r,s} \leq C(r, s)\ |||\ u\ |||_{r,s}\ |||\ v\ |||_{r,s}.$$

Remark : By the Theorem above and Sobolev embedding theorem we see that if $u \in J_r^{r(s+1)}$ with $s, r \in Z$ and $r \neq 0$. Then

$$\sup_{x\in R} |\ \omega(x)^{r-1} D_x^s u(x)\ | \leq C(r, s)||u||_{r-1,s+1} \leq C(r, s)\ |||u|||_{r,(s+1)r}$$

that is, $|\ D_x^s u(x)\ | = O(\ |x|^{-(r-1)}\ )$ as $|x| \to +\infty$.

The following Theorem follows from a straightforward calculation and its proof is omitted here.

Theorem 1.2 — Let $\alpha(x) \in C_0^\infty(R)$ such that $0 \le \alpha \le 1$, $\alpha = 1$ if $|x| \le 1$ and $\alpha = 0$ if $|x| \ge 2$. Let $\alpha_\varepsilon(x) = \alpha(\varepsilon x)$ for $0 < \varepsilon < 1$. Then $\varepsilon \to 0$.

$\alpha_\varepsilon(x) \to 1$ uniformly on any bounded set of R,

$D_x^j \alpha_\varepsilon(x) \to 0$ uniformly on R for $j \ne 0$.

Moreover, for any $j \in Z$ we have

$$|D_x^j \alpha_\varepsilon(x)| \le C(j)\, \varepsilon^h (\omega(x))^{-(j-h)}, \quad 0 \le h \le j$$

where the constant $C(j) > 0$ is independent of ε.

The following well-known Gagliardo-Nirenberg's inequality will be unexplainedly used many times throughout the paper.

Theorem 1.3 — Let q, r be any real numbers satisfying $1 \le q, r \le \infty$, and j, $m \in Z$ such that $j \le m$. Then

$$|D_x^j u|_p \le C(j, m, q, r, a)\ |D_x^m u|_r^a\ |u|_q^{1-a}$$

holds for $D_x^m u \in L^r$ and $u \in L^q$, where $\frac{1}{p} = j + a\left(\frac{1}{r} - m\right) + (1-a)\frac{1}{a}$ for all a in the interval $j/m \le a \le 1$.

For the proof see e.g. Friedman[7].

Theorem 1.4 — (Gronwall's inequality). Suppose that $g(t)$, $h(t) \ge 0$, satisfy the inequality

$$g(t) \le M_1 + M_2 \int_0^t g(s)\, h(s)\, ds, \text{ for any } 0 \le t \le T$$

where $M_1, M_2 \ge 0$ are two constants. Moreover, $h(t)$ satisfies $\int_0^T h(t)\, dt < \infty$. Then we have

$$g(t) \le M_1 \exp\left(M_2 \int_0^t h(s)\, ds\right), \quad t \in [0, T].$$

Our results are as follows :

Theorem A — Let T be any given positive constant. For any initial data $\varphi \in H^s$, $s \in Z$ and $s \ge 4$, then problem (1), (2) has a smooth solution u such that

$$u \in \bigcap_{k+4h \le s} W_\infty^h\ (0, T; H^k(R))$$

where $k, h \in Z$.

Theorem B — Let T be any given positive constant. For any initial data $\varphi \in J_r^s(R)$ with $s, r \in Z$ and $s \ge \max(3r, 4)$, then problem (1), (2) has a solution u such that

$$u \in \bigcap_{\substack{s'+4h \leq s \\ r's+4hr \leq rs}} W_\infty^h \left(0, T; J_{r'}^{s'}\right)$$

where $s', h \in Z$ and $r' \geq 0$ real.

Corollary C — For any $T > 0$, if $\varphi \in S(R)$. Then the solution u to problem (1), (2) belongs to $C^\infty([0, T]; S(R))$.

2. Global *a Priori* Estimates

In the study of global existence for the dispersive equations, the global a priori estimates play an important role. This part is to establish the global a priori estimates for the solution to (1), (2).

Lemma 2.1 — Let u be the solution of problem (1), (2) with the given initial data $\varphi \in H^s$, $s \in Z$ and $s \geq 4$, then we have the following identities

$$E_0(t) = \frac{1}{2}\int |u|^2 dx = E_0(0)$$

$$E_1(t) = \int \left(\frac{1}{2}|u_x|^2 - \frac{1}{4}|u|^4\right) dx = E_1(0)$$

$$E_2(t) = \int\left(\frac{1}{2}|u_{2x}|^2 - 2|u_x|^2|u|^2 - \frac{1}{2}\mathrm{Re}\,(u^2(u_x^*)^2) + \frac{1}{4}|u|^6\right) dx = E_2(0)$$

$$E_3 = \int \left\{-\frac{1}{2}|u_{3x}|^2 + 3|u|^2|u_{2x}|^2 + \frac{1}{2}\mathrm{Re}\,(u^2(u_{2x}^*)^2) - 3\mathrm{Re}\,|u|^2(uu_{2x}^*)\right.$$

$$+ \frac{3}{4}\mathrm{Re}\left(|u|^4(u^* u_{2x})\right) - \frac{11}{4}|u_x|^4 - 3|u|^4|u_x|^2$$

$$\left.-\frac{1}{2}|u|^2\left(\left(|u|^2\right)_x\right)^2 + \frac{5}{16}|u|^8\right\}dx = E_3(0).$$

This lemma can be proved directly by using the formula of conserved quantities associated to NLS equation (see Ablowitz and Segur[1]).

Proposition 2.2 — Under the condition of Lemma 2.1, we have

$$\sup_{0 \leq t < \infty} \|u(t)\|_3 \leq C(\|\varphi\|_3) \qquad \text{... (2.1)}$$

where $C(\|\varphi\|_3)$ depends only on the size of φ.

This proposition can be proved by using Lemma 2.1, Theorem 1.3 and Young's inequality.

Proposition 2.3 — Let $T > 0$ be arbitrarily given. Under the conditions of Lemma 2.1, we have

$$\sup_{0 \leq t \leq T} \|u(t)\|_s \leq C. \qquad \text{... (2.2)}$$

PROOF : In order to show (2.2) we consider the following derivative

$$\frac{d}{dt}\left\{\int\left[\frac{1}{2}\,|u_{sx}|^2+\operatorname{Re}C_s\,(u^2\,(u^*_{(s-1)x})^2)+D_s\,|u|^2\,|\,u^*_{(s-1)x}\,|^2\right]dx\right\}$$

where C_s and D_s are constants to be determined later. For the first two terms in the above derivative we have

$$\frac{d}{dt}\left[\int\frac{1}{2}\,|u_{sx}|^2\,dx+\operatorname{Re}C_s\int u^2\,(u^*_{(s-1)x})^2\,dx\right]$$

$$= \operatorname{Re}\int u^*_{sx}\,u_{sxt}\,dx+2C_s\operatorname{Re}\int uu_t\,(D^{s-1}u^*)^2\,dx$$

$$+\ 2C_s\operatorname{Re}\int u^2\,D^{s-1}u^*\,D^{s-1}u^*_t\,dx$$

$$= \operatorname{Re}\ \Big[-i\int D^s u^*\,D^s\,(u_{4x}+4\,|u|^2\,u_{2x}+u^2\,u^*_{2x}$$

$$+\ 3u^*\,(u_x)^2+2\,|u_x|^2\,u+\frac{3}{2}\,|u|^4\,u)\,dx\,\Big]$$

$$+\ 2C_s\operatorname{Re}\Big[-i\int u(u_{4x}+4\,|u|^2\,u_{2x}+u^2\,u^*_{2x}$$

$$+\ 3u^*(u_x)^2+2\,|u_x|^2\,u+\frac{3}{2}\,|u|^4\,u)(D^{s-1}u^*)^2\,dx\,\Big]$$

$$+\ 2C_s\operatorname{Re}\Big[\,i\int u^2\,D^{s-1}u^*\,D^{s-1}\,(u^*_{4x}+4\,|u|^2\,u^*_{2x}+(u^*)^2\,u_{2x}$$

$$+\ 3u^*\,(u_x)^2+2\,|u_x|^2\,u^*+\frac{3}{2}\,|u|^4\,u^*)\,dx\,\Big]$$

$$= \operatorname{Im}\int D^s u^*\,D^s\left[4\,|u|^2\,u_{2x}+u^2u^*_{2x}+3u^*(u_x)^2+2\,|u_x|^2\,u+\frac{3}{2}\,|u|^4\,u\right]dx$$

$$+\ 2C_s\operatorname{Im}\int uu_{4x}\,(D^{s-1}u^*)^2\,dx+2C_s\operatorname{Im}\int u(4\,|u|^2\,u_{2x}+u^2\,u^*_{2x}+3u^*.(u_x)^2$$

$$+\ 2\,|u_x|^2\,u+\frac{3}{2}\,|u|^4\,u)\,(D^{s-1}u^*)^2\,dx-2C_s\operatorname{Im}\int u^2\,D^{s-1}u^*\,D^{s-1}u^*_{4x}\,dx$$

$$-\ 2C_s\operatorname{Im}\int u^2\,D^{s-1}u^*\,D^{s-1}\,[4\,|u|^2\,u^*_{2x}+(u^*)^2\,u_{2x}\,]\,dx$$

$$-\ 2C_s\operatorname{Im}\int u^2\,D^{s-1}u^*\,D^{s-1}\left[3u(u^*_x)^2+2\,|u_x|^2\,u^*+\frac{3}{2}\,|u|^4\,u^*\right]dx$$

$$= \operatorname{Im}\int D^s u^*\,D^s\,(4\,|u|^2\,u_{2x}+u^2\,u^*_{2x})\,dx+\operatorname{Im}\int D^s u^*\,D^s\,(3u^*\,(u_x)^2$$

$$+\ 2\,|u_x|^2\,u)\,dx+2C_s\operatorname{Im}\int uu_{4x}\,(D^{s-1}u^*)^2\,dx$$

$$-\ 2C_s\operatorname{Im}\int u^2\,D^{s-1}u^*\,D^{s-1}u^*_{4x}\,dx$$

(*equation continued on p.* 589)

$$- 2C_s \operatorname{Im}\int u^2 D^{s-1} u^* D^{s-1} [4|u|^2 u^*_{2x}\, dx + (u^*)^2 u_{2x}]\, dx$$

$$+ \text{ Remaining term}$$

$$= \sum_{j=1}^{s} L_j + \text{ Remaining term.} \qquad \text{... (2.3)}$$

For the Remaining term we have

$$|\text{ Remaining term }| \le C(1 + |u_{sx}|_2^2). \qquad \text{... (2.4)}$$

In what follows we estimate each term L_j. Using integration by parts, Theorem 1.3 and Young's inequality we have

$$L_1 = \operatorname{Im}\int D^s u^* D^s \left(4|u|^2 u_{2x} + u^2 u^*_{2x}\right) dx$$

$$\le s\operatorname{Im}\int D^s u^* (4|u|^2)_x D^{s+1} u dx + 4\operatorname{Im}\int |u|^2 D^s u^* D^{s+2} u dx$$

$$+ s\operatorname{Im}\int D^s u^* (u^2)_x D^{s+1} u^* dx + \operatorname{Im}\int u^2 D^s u^* D^{s+2} u^* dx + C\left(1 + |u_{sx}|_2^2\right)$$

$$= (s-1)\operatorname{Im}\int D^s u^* (4|u|^2)_x D^{s+1} u dx - \operatorname{Im}\int u^2 (D^{s+1} u^*)^2 dx$$

$$+ \left(\frac{1-s}{2}\right)\operatorname{Im}\int (u^2)_{2x} (D^s u^*)^2 dx + C\left(1 + |u_{sx}|_2^2\right)$$

$$\le (s-1)\operatorname{Im}\int D^s u^* (4|u|^2)_x D^{s+1} u dx$$

$$- \operatorname{Im}\int u^2 (D^{s+1} u^*)^2 dx + C\left(1 + |u_{sx}|_2^2\right) \qquad \text{... (2.5)}$$

$$L_2 = \operatorname{Im}\int D^s u^* D^s (3u^* u_x^2 + 2|u_x|^2 u)\, dx$$

$$\le 3\operatorname{Im}\int u^* D^s u^* D^s u_x^2 dx + 2\operatorname{Im}\int u D^s u^* D^s |u_x|^2 dx + C(1 + |u_{sx}|_2^2)$$

$$\le 6\operatorname{Im}\int u^* u_x D^s u^* D^{s+1} u dx + 2\operatorname{Im}\int u u^*_x D^s u^* D^{s+1} u dx + C(1 + |u_{sx}|_2^2).$$

$$= \operatorname{Im}\int (4|u|^2)_x D^s u^* D^{s+1} u dx + 2\operatorname{Im}$$

$$\int (u^* u_x - u u^*_x) D^s u^* D^{s+1} u dx + C(1 + |u_{sx}|_2^2)$$

$$= \operatorname{Im}\int (4|u|^2)_x D^s u^* D^{s+1} u dx + 2\operatorname{Im}$$

$$\int (u^*_x u)_x |D^s u|^2 dx + C(1 + |u_{sx}|_2^2)$$

$$\le \operatorname{Im}\int (4|u|^2)_x D^s u^* D^{s+1} u\, dx + C(1 + |u_{sx}|_2^2) \qquad \text{... (2.6)}$$

$$L_3 = 2C_3 \operatorname{Im}\int uu_{4x}(D^{s-1}u^*)^2\,dx$$

$$= -2C_s \operatorname{Im}\left[\int u_x u_{3x}(D^{s-1}u^*)^2\,dx + 2\int uu_{3x}D^{s-1}u^* D^s u^*\,dx\right]$$

$$\le C(1 + |u_{sx}|_2^2) \qquad \text{... (2.7)}$$

$$L_4 = -2C_s \operatorname{Im}\int u^2 D^{s-1}u^* D^{s+3}u^*\,dx$$

$$= -2C_s \operatorname{Im}\Big[\int u^2 (D^{s+1}u^*)^2\,dx + 2\int (u^2)_x D^s u^* D^{s+1}u^*\,dx$$

$$+\int (u^2)_{2x} D^{s-1}u^* D^{s+1}u^*\,dx\Big]$$

$$= -2C_s \operatorname{Im}\Big[\int u^2 (D^{s-1}u^*)^2\,dx - 2\int (u^2)_{2x}(D^s u^*)^2\,dx$$

$$-\int (u^2)_{3x} D^{s-1}u^* D^s u^*\,dx\Big]$$

$$\le -2C_s \operatorname{Im}\int u^2 (D^{s+1}u^*)^2\,dx + C(1 + |u_{sx}|_2^2). \qquad \text{... (2.8)}$$

For L_5 we can easily obtain

$$L_5 \le C\left(1 + |u_{sx}|_2^2\right) \qquad \text{... (2.9)}$$

Now consider

$$\frac{d}{dt} D_s \int |u|^2 |D^{s-1}u|^2\,dx = 2D_s \operatorname{Re}\int |D^{s-1}u|^2 u^* u_t\,dx$$

$$+ 2D_s \operatorname{Re}\int |u|^2 D^{s-1}u^* D^{s-1}u_t\,dx$$

$$= 2D_s \operatorname{Re}\Big\{-i\int |D^{s-1}u|^2 u^*\Big[u_{4x} + 4|u|^2 u_{2x} + u^2 u^*_{2x} + 3u^*(u_x)^2$$

$$+ 2|u_x|^2 u + \frac{3}{2}|u|^4 u\Big]dx\Big\}$$

$$+ 2D_s \operatorname{Re}\Big\{-i\int |u|^2 D^{s-1}u^* D^{s-1}\Big[u_{4x} + 4|u|^2 u_{2x} + u^2 u^*_{2x} + 3u^*(u_x)^2$$

$$+ 2|u_x|^2 u + \frac{3}{2}|u|^4 u\Big]dx\Big\}$$

$$\le 2D_s \operatorname{Im}\int |D^{s-1}u|^2 u^* u_{4x}\,dx + 2D_s \operatorname{Im}\int |u|^2 D^{s-1}u^* D^{s+3}u\,dx$$

$$+ C\left(1 + |u_{sx}|_2^2\right)$$

$$\le -2D_s \operatorname{Im}\int (|D^{s-1}u|^2 u^*)_x u_{3x}\,dx + 2D_s \operatorname{Im}\int |u|^2 D^{s-1}u^* D^{s+3} u dx$$
$$+ C\left(1 + |u_{sx}|_2^2\right)$$

$$\le D_s \operatorname{Im}\int (4|u|^2)_x D^s u^* D^{s+1} u dx + C\left(1 + |u_{sx}|_2^2\right). \quad \text{... (2.10)}$$

Considering (2.3)-(2.10) and taking $C_s = -\frac{1}{2}$, $D_s = -s$, we have

$$\frac{d}{dt}\left[\frac{1}{2}\int |u_{sx}|^2 dx - \frac{1}{2}\operatorname{Re}\int u^2 (D^{s-1}u^*)^2 dx - s\int |u|^2\,|D^{s-1}u|^2 dx\right]$$
$$\le C\left(1 + |u_{sx}|_2^2\right). \quad \text{... (2.11)}$$

Thus, integrating the above inequality with respect to t we get

$$|u_{sx}|_2^2 \le C + C\int_0^t |u_{sx}(\tau)|_2^2 d\tau,\ t \in [0, T]. \quad \text{... (2.12)}$$

Therefore, applying Gronwall's inequality to (2.12) gives

$$|u_{sx}|_2^2 \le C, \quad t \in [0, T]$$

which completes the proof of the proposition.

3. Proof of Theorem A

The proof depends on a series of Lemmas. We first establish the existence of a unique global solution $u^\varepsilon(x, t)$ to the initial value problem

$$u_t + \varepsilon\,(-u_{6x} + u_{4x} - u_2)$$
$$= -i\left[u_{4x} + 4|u|^2 u_{2x} + u^2 u_{2x}^* + 3u^*(u_x)^2 + 2|u_x|^2 u + \frac{3}{2}|u|^4 u\right],$$
$$0 < \varepsilon < 1 \quad \text{... (3.1)}$$

with the initial data $u(x, 0) = \varphi(x)$. We reduce (3.1) to a problem of finding fixed points of completely continuous maps by considering the family of nonlinear problems,

$$u_t + \varepsilon\,(-u_{6x} + u_{4x} - u_{2x})$$
$$= -i\tau\left[u_{4x} + 4|u|^2 u_{2x} + u^2 u_{2x} + 3u^*(u_x)^2 + 2|u_x|^2 u + \frac{3}{2}|u|^4 u\right],$$
$$0 < \varepsilon < 1,\ 0 \le \tau \le 1, \quad \text{... (3.2)}$$

and the related linear problem

$$u_t + \varepsilon\,(-u_{6x} + u_{4x} - u_{2x})$$

(*equation continued on p. 592*)

$$= -i\tau\left[v_{4x} + 4|v|^2 v_{2x} + v^2 v_{2x} + 3v^*(v_x)^2 + 2|v_x|^2 v + \frac{3}{2}|v|^4 v\right],$$

$$0 < \varepsilon < 1,\ 0 \leq \tau \leq 1. \quad \dots (3.3)$$

We introduce two sets $X(n, T)$, $Y(n, T)$ of functions on $Rx[0, T]$ with finite norms

$$\langle\langle v \rangle\rangle^2_{X(n,T)} = \int_0^T |v_t|_2^2\,dt + \int_0^T ||v(t)||^2_{n+3}\,dt + \operatorname*{Sup}_{0\leq t\leq T} ||v(t)||^2_n$$

and

$$\langle\langle v \rangle\rangle^2_{Y(n,T)} = \int_0^T ||v(t)||^2_{n+1}\,dt + \operatorname*{Sup}_{0\leq t\leq T} ||v(t)||^2_{n-2}$$

respectively. Let $n \geq 4$. If $v \in Y(n, T)$ there exists a global solution $u(x, t)$ in $X(n, T)$ to (3.3) which is uniquely determined by its initial values. Thus problem (3.3) defines a nonlinear operator $u^\tau = \Phi(v; \tau)$, whose fixed points are solutions of (3.2). The fixed points of Φ for $\tau = 1$ are solutions of (3.1). In the following we first apply the Leray-Schauder theorem[21] to determine the fixed points of Φ. Then we show that the solution u^ε to problem (3.1) converges to a solution of (1). Now this limiting procedure is well-known. Hence, in this part we are mainly concerned with the solvability of (3.1) and the global estimates of its solutions. The following statements are aimed at verifying the conditions of Leray-Schauder theorem.

Lemma 3.1 — Let $n \geq 4$. If $u^\tau(x, t)$ is a solution in $X(n, T)$ of (3.2), then

$$\int_0^T |\partial_t u^\tau(t)|_2^2\,dt \text{ and } \varepsilon\int_0^T ||u^\tau(t)||^2_{n+3}\,dt + \operatorname*{Sup}_{0\leq t\leq T} ||u^\tau(t)||^2_n \leq C \quad \dots (3.4)$$

where $C > 0$ is a constant depending only on $||\varphi||_n$ and $T < +\infty$, most importantly it is indpendent of τ.

PROOF : It should be pointed out that in the process of the proof we use the property of complete integrability of NLS equation as a Hamiltonian system[1, 28].

(1) Multiply equation (3.2) by u^*. Integrate over R and take the real part of the resulting expression to obtain

$$\frac{d}{dt}|u(t)|_2^2 + 2\varepsilon\sum_{j=1}^{3} |D_x^j u|_2^2 = 0$$

which leads to

$$|u(t)|_2^2 + 2\varepsilon\int_0^t \sum_{j=1}^{3} |D_x^j u(t)|_2^2\,dt = |\varphi|_2^2. \quad \dots (3.5)$$

(2) Multiply equation (3.2) by $-u^*_{2x} - |u|^2 u^*$, proceed as in (1) to get

$$\frac{dE_1}{dt} + \varepsilon \int_0^t \sum_{j=1}^{3} |D_x^{j+1} u(t)|_2^2 \, dt = \varepsilon \operatorname{Re} \int (-u_{6x} + u_{4x} - u_{2x}) \, |u|^2 u^* \, dx. \quad \text{... (3.6)}$$

For the terms in the right-hand side of (3.6) we have

$$\varepsilon \left| \operatorname{Re} \int u_{6x} \, |u|^2 u^* \, dx \right| = \varepsilon \left| \operatorname{Re} \int u_{4x} \, (|u|^2 u^*)_{2x} \, dx \right|$$

$$\leq \frac{\varepsilon}{2} \, |u_{4x}|_2^2 + C\,(|\varphi|_2)\, \varepsilon \, |u_{2x}|_2^2 \, |u_x|_2^2$$

$$\varepsilon \left| \operatorname{Re} \int u_{4x} \, |u|^2 u^* \, dx \right| = \varepsilon \left| \operatorname{Re} \int u_{2x} \, (|u|^2 u^*)_{2x} \, dx \right|$$

$$\leq \frac{\varepsilon}{2} \, |u_{2x}|_2^2 + C\,(|\varphi|_2)\, \varepsilon \, |u_{2x}|_2^2 \, |u_x|_2^2$$

$$\varepsilon \left| \operatorname{Re} \int u_{2x} \, |u|^2 u^* \, dx \right| = \varepsilon \left| \operatorname{Re} \int u_x \, (|u|^2 u^*)_x \, dx \right|$$

$$\leq C\,(\|\varphi\|_1)\, \varepsilon \, |u_x|_2^2 \, .$$

Therefore

$$\frac{dE_1}{dt} + \frac{\varepsilon}{2} \sum_{j=1}^{3} |D_x^{j+1} u|_2^2 \leq C\,(|\varphi|_2)\, \varepsilon \, |u_{2x}|_2^2 \, |u_x|_2^2$$

$$+ C\,(\|\varphi\|_1)\, \varepsilon \, |u_x|_2^2 \, .$$

From the inequality above and (3.5) there follows that

$$E_1 + \frac{\varepsilon}{2} \int_0^t \sum_{j=1}^{3} |D_x^{j+1} u(t)|_2^2 \, dt \leq C\,(\|\varphi\|_1)$$

$$+ C\,(|\varphi|_2) \int_0^t \varepsilon \, |u_{2x}|_2^2 \, |u_x|_2^2 \, dx \quad \text{... (3.7)}$$

Using Young's inequality we have

$$\int |u|^4 \, dx \leq |u_x|^2 + C(|\varphi|_2). \quad \text{... (3.8)}$$

By (3.7) and (3.8) we have

$$|u_x|_2^2 + \varepsilon \int_0^t \sum_{j=1}^{3} \left| D_x^{i+1} u(t) \right|_2^2 \, dt$$

$$\leq C((\|\varphi\|_1) + C(|\varphi|_2) \int_0^t \varepsilon \, |u_{2x}|_2^2 \, |u_x|_2^2 \, dt. \quad \text{... (3.9)}$$

Applying Gronwall's inequality to (3.9) to yield

$$|u_x|_2^2+\varepsilon\int_0^t\sum_{j=1}^3 \left|D_x^{i+1}u(t)\right|_2^2 dt \le C(||\varphi||_1). \qquad ...\ (3.10)$$

(3) Using the complete integrability or direct calculation we can obtain

$$\frac{dE_2}{dt}+2\varepsilon\,\mathrm{Re}\int(-u_{6x}+u_{4x}-u_{2x})\frac{\delta E_2}{\delta u}dx=0 \qquad ...\ (3.11)$$

$$\frac{dE_3}{dt}+2\varepsilon\,\mathrm{Re}\int(-u_{6x}+u_{4x}-u_{2x})\frac{\delta E_3}{\delta u}dx=0. \qquad ...\ (3.12)$$

Proceed as in (1) and (2), use (3.5), (3.10), Sobolev embedding theorem and Gronwall's inequality to obtain

$$|u_{2x}|_2^2+\varepsilon\int_0^t\sum_{j=1}^3 \left|D_x^{j+2}u(t)\right|_2^2 dt \le C(||\varphi||_2) \qquad ...\ (3.13)$$

$$|u_{3x}|_2^2+\varepsilon\int_0^t\sum_{j=1}^3 \left|D_x^{j+3}u(t)\right|_2^2 dt \le C(||\varphi||_3). \qquad ...\ (3.14)$$

(4) Assume that the second result in (3.4) is proved for all values less than or equal to $s-1$. We now prove it for s. To this end, use (2.11) in Proposition 2.2 to obtain

$$\frac{dE_s}{dt}+2\varepsilon\,\mathrm{Re}\int(-u_{6x}+u_{4x}-u_{2x})\frac{\delta E_s}{\delta u}dx \le C(1+|u_{sx}|_2^2) \qquad ...\ (3.15)$$

where $$E_s(t)=\int\left\{\frac{1}{2}|u_{sx}|^2-s|u|^2\left|D_x^{s-1}u^*\right|^2 -\frac{1}{2}\mathrm{Re}\,(u^2(D_x^{s-1}u^*)^2)\right\}dx.$$

By definition we have

$$\frac{\delta E}{\delta u}=\frac{1}{2}\Big\{(-1)^s u^*_{2sx}-2s\left|D_x^{s-1}u\right|^2 u^* + (-1)^s 2sD_x^{s-1}\left(|u|^2 D_x^{s-1}u^*\right) - 2u(D_x^{s-1}u^*)^2+(-1)^s 2D_x^{s-1}((u^*)^2 D_x^{s-1}u)\Big\}.$$

Now we have

$$2\varepsilon \ \mathrm{Re} \int (-u_{6x} + u_{4x} - u_{2x}) \ \frac{\delta E_s}{\delta u} \, dx$$

$$= \varepsilon \sum_{j=1}^{3} \left| D_x^{s+j} \ u(t) \right|_2^2 + \varepsilon \, \mathrm{Re} \int (-u_{6x} + u_{4x} - u_{2x})$$

$$\{(-1)^s \, 2s D_x^{s-1} \left(|u|^2 D_x^{s-1} u^* \right)$$

$$+ (-1)^s \, 2 D_x^{s-1} \left((u^*)^2 D_x^{s-1} u \right) - 2s \left| D_x^{s-1} u \right|^2 u^* - 2u \left(D_x^{s-1} u^* \right)^2 \} dx. \qquad \text{... (3.16)}$$

Using Sobolev embedding theorem and Young's inequality we can obtain the following estimates

$$2s\varepsilon \left| \mathrm{Re} \int (-u_{6x} + u_{4x} - u_{2x}) D_x^{s-1} \left(|u|^2 D_x^{s-1} u^* \right) dx \right|$$

$$\leq 2s\varepsilon \left| \int D_x^{s+3} u \left(|u|^2 D_x^{s-1} u^* \right)_{2x} dx \right|$$

$$+ 2s\varepsilon \left| \int D_x^{s+3} u \left(|u|^2 D_x^{s-1} u^* \right) dx \right|$$

$$+ 2s\varepsilon \left| \int D_x^{s+1} u \left(|u|^2 D_x^{s-1} u^* \right) dx \right|$$

$$\leq \frac{\varepsilon}{4} \left(\left| D_x^{s+3} u \right|_2^2 + \left| D_x^{s+1} u \right|_2^2 \right) + C(\| \varphi \|_3)$$

$$\varepsilon \left(\left| D_x^{s+1} u \right|_2^2 + \left| D_x^{s} u \right|_2^2 + \left| D_x^{s-1} u \right|_2^2 \right), \qquad \text{... (3.17)}$$

$$2\varepsilon \left| \mathrm{Re} \int (-u_{6x} + u_{4x} - u_{2x}) D_x^{s-1} \left((u^*)^2 D_x^{s-1} u \right) dx \right|$$

$$\leq \frac{\varepsilon}{4} \left(\left| D_x^{s+3} u \right|_2^2 + \left| D_x^{s+1} u \right|_2^2 \right) + C(\| \varphi \|_3)$$

$$\varepsilon \left(\left| D_x^{s+1} u \right|_2^2 + \left| D_x^{s} u \right|_2^2 + \left| D_x^{s-1} u \right|_2^2 \right), \qquad \text{... (3.18)}$$

$$2s\varepsilon \left| \mathrm{Re} \int (-u_{6x} + u_{4x} - u_{2x}) \, u^* \left| D_x^{s-1} u \right|^2 dx \right|$$

$$\leq \varepsilon \left(\left| u_{6x} \right|_2^2 + \left| u_{4x} \right|_2^2 + \left| u_{2x} \right|_2^2 \right)$$

$$+ C(|\varphi|_2) \, \varepsilon \left| D_x^{s-1} u \right|_2^2 \left| D_x^{s} u \right|_2^2 \qquad \text{... (3.19)}$$

$$2\varepsilon \left| \operatorname{Re} \int (-u_{6x} + u_{4x} - u_{2x})\, u \left(D_x^{s-1} u^*\right)^2 dx \right|$$

$$\leq \varepsilon \left(\left| u_{6x} \right|_2^2 + \left| u_{4x} \right|_2^2 + \left| u_{2x} \right|_2^2 \right)$$

$$+ C(\left| \varphi \right|_2)\, \varepsilon \left| D_x^{s-1} u \right|_2^2 \left| D_x^s u \right|_2^2 , \qquad \dots (3.20)$$

From (3.16) to (3.20) there appears that

$$E_s(t) + \frac{\varepsilon}{2} \int_0^t \sum_{j=1}^{3} \left| D_x^{s+j} u(t) \right|_2^2 dt$$

$$\leq E_s(0) + C(\| \varphi \|_3)$$

$$\int_0^t \varepsilon \left(\left| u_{2x} \right|_2^2 + \left| u_{4x} \right|_2^2 + \left| u_{6x} \right|_2^2 + \left| D_x^{s-1} u \right|_2^2 + \left| D_x^{s} u \right|_2^2 + \left| D_x^{s+1} u \right|_2^2 \right) dt$$

$$+ C \int_0^t \left(\varepsilon \left| D_x^{s-1} u \right|_2^2 + 1 \right) \left| D_x^s u \right|_2^2 dt$$

$$\leq C(\| \varphi \|_3, T) + C \int_0^T \left(\varepsilon \left| D_x^{s-1} u \right|_2^2 + 1 \right) \left| D_x^s u \right|_2^2 dt . \qquad \dots (3.21)$$

By the expression of $E_s(t)$ and Young's inequality, from (3.21) it follows that

$$\left| u_{sx}(t) \right|_2^2 + \varepsilon \int_0^t \sum_{j=1}^{3} \left| D_x^{s+j} u(t) \right|_2^2 dt$$

$$\leq C(\| \varphi \|_3, T) + C \int_0^T \left(\varepsilon \left| D_x^{s-1} u \right|_2^2 + 1 \right) \left| D_x^2 u \right|_2^2 dt . \qquad \dots (3.22)$$

Now applying Gronwall's inequality to (3.22) to complete the proof of the second inequality. The first inequality in (3.4) can be easily obtained by multiplying (3.2) by u_t^*, integrating in x and t and taking the real part. This finishes the proof.

Lemma 3.2 — If $n \geq 4$ and $u \in X(n, T)$ solves (3.3), then

$$\left| u_t \right|_2^2 + \varepsilon \int_0^1 \sum_{j=1}^{3} \left| D_x^j u(t) \right|_2^2 dt \leq C(\langle\langle v \rangle\rangle_{Y(n,T)}^2 + \left| \varphi \right|_2^2)$$

$$\int_0^T \left| u_t \right|_2^2 dt + \varepsilon \sum_{j+1}^{3} \left| D_x^j u(t) \right|_2^2 \leq C(\langle\langle v \rangle\rangle_{Y(n,T)}^2 + \| \varphi \|_3^2),$$

$$|D_x^h u(t)|_2^2 + \varepsilon \int_0^T \sum_{j=1}^3 |D_x^{j+h} u(t)|_2^2 dt$$

$$\leq C(\langle\langle v \rangle\rangle_{Y(n,T)}^2 + ||\varphi||_h^2), \quad h \leq n$$

where C depends on ε, T and the size of v.

PROOF : Multiply eqn. (3.3) by u^*, integrate and take the real part of the resultant expression to obtain

$$|u(t)|_2^2 + \varepsilon \int_0^T \sum_{j=1}^3 |D_x^j u(t)|_2^2 dt \leq |\varphi|_2^2 + \frac{1}{2}\int_0^T |u(t)|_2^2 dt$$

$$+ \frac{1}{2}\int_0^T |K(v)|_2^2 dt$$

which implies (by Gronwall's inequality)

$$|u(t)|_2^2 + \varepsilon \int_0^T \sum_{j=1}^3 |D_x^j u(t)|_2^2 dt \leq C\left(|\varphi|_2^2 + \int_0^T |K(v)|_2^2 dt\right)$$

where $K(v) = v_{4x} + 4|v|^2 v_{2x} + v^2 v_{2x}^* + 3v^*(v_x)^2 + 2|v_x|^2 v + \frac{3}{2}|v|^4 v.$

By the standard estimates and the definition of $Y(n, T)$ we have

$$|u(t)|_2^2 + \varepsilon \int_0^T \sum_{j=1}^3 |D_x^j u(t)|_2^2 dt \leq C\left(|\varphi|_2^2 + \langle\langle v \rangle\rangle_{Y(n,T)}^2\right). \quad \ldots (3.23)$$

To obtain the second inequality of the lemma, multiply equation (3.3) by u_t^*, integrate in x and t, take the real part and proceed as before to finish the proof.

The final estimate is established by first taking D_x^h of (3.3), multiplying by u_{hx}^*. Then integrate the resulting expression in x and t and take the real part to obtain

$$|D_x^h u(t)|_2^2 + 2\varepsilon \int_0^T \sum_{j=1}^3 |D_x^{h+1} u(t)|_2^2 dt \leq |D_x^h \varphi|_2^2$$

$$+ \left|\int_0^T \int D_x^h u^* D_x^h K(v)\, dx\, dt\right| \quad \ldots (3.24)$$

where $K(v)$ is the same as above. Obviously,

$$\left|\int_0^T \int D_x^h u^* D_x^{h+4} v dxdt\right| = \left|\int_0^T \int D_x^{h+3} u^* D_x^{h+1} v dxdt\right|$$

$$\leq \frac{\varepsilon}{2}\int_0^T |D_x^{h+1}u(t)|_2^2\,dt + C_\varepsilon \int_0^T |D_x^{h+1}v(t)|_2^2\,dt. \quad \ldots (3.25)$$

For $1 \leq h \leq 2$ we have

$$\left|\int_0^T\int D_x^h u^* D_x^h K_i(v)dxdt\right| = \left|\int_0^T\int D_x^{h+1}u^* D_x^{h-1}K_i(v)dxdt\right|$$

$$\leq \frac{\varepsilon}{2}\int_0^T |D_x^{h+1}u(t)|_2^2\,dt + C\langle\langle v\rangle\rangle^2_{Y(n,T)} \quad \ldots (3.26)$$

where $K_1(v) = 4\,|\,v\,|\,^2 v_{2x} + v^2 v_{2x}^* + 3v^*(v_x)^2 v + 2\,|\,v_x\,|^2 v + \frac{3}{2}\,|\,v\,|\,^4 v$.

For $h \geq 3$ we have

$$\left|\int_0^T\int D_x^h u^* D_x^h K_i(v)dxdt\right| = \left|\int_0^T\int D_x^{h+3}u^* D_x^{h-3}K_i(v)dx\right|$$

$$\leq \frac{\varepsilon}{2}\int_0^T |D_x^{h+3}u(t)|_2^2\,dt + C\langle\langle v\rangle\rangle^2_{Y(n,T)}. \quad \ldots (3.27)$$

Now the final result follows from (3.24)-(3.27).

Now we turn to the proof of the solvability of equation (3.1). If $n \geq 4$ and $\varphi \in H^n$, then it is known that there exists a unique solution $u \in X(n,T)$ to the linear parabolic equation (3.3) for each $\tau \in [0, 1]$ and $v \in Y(n,T)$. This defines a nonlinear operator $u = \Phi(v;\tau)$, which for each $\tau \in [0, 1]$ determines the solution to (3.3). To obtain the existence of fixed points of Φ we apply Leray-Schauder theorem. In what follows we proceed as in Leray and Schauder[21] and Schwartz[24] to verify the conditions of Leray-Schauder theorem.

Consider $\Phi(v;\tau)$ on the subset B of $Y(n,T)$ consisting of functions $v \in Y(n,T)$ satisfying the inequalities (3.4) with the bound increased by adding the positive number $\gamma \geq 0$. We shall verify that $\Phi(v;\tau)$ on $B \times [0, 1]$ is

(a) continuous in v, uniformly in τ,

(b) continuous in τ, uniformly in v,

(c) completely continuous on $B \times [0, 1]$.

We also show that all fixed points of Φ lie strictly in the interior in the set B and that $v - \Phi(v;0)$ has nonzero degree. We then apply Leray-Schauder theorem for existence of fixed points of Φ.

(I) $\Phi(v;\tau)$ is continuous with respect to v in $Y(n, T)$ uniformly in $\tau \in$ [0 1]. Begin with two elements $v, \tilde{v}$ from B so that $u = \Phi(v;\tau)$ and $\tilde{u} = \Phi(\tilde{v};\tau)$. Then $U = u - \tilde{u}$ and $V = v - \tilde{v}$ satisfy

$$U_t + \varepsilon(-U_{6x} + U_{4x} - U_{2x}) = -i\tau\,[K(v) - K(\tilde{v})] \quad \ldots (3.28)$$

$U(x, 0) = 0.$

Proceeding as in the proof of Lemma 3.2 we can obtain

$$| U(t) |_2^2 + \varepsilon \int_0^T \sum_{j=1}^3 \left| D_x^j U(t) \right|_2^2 dt \leq C \langle\langle V \rangle\rangle_{Y(n,T)}^2$$

$$\int_0^T | U_t |_2^2 dt + \varepsilon \sum_{j=1}^3 \left| D_x^j U(t) \right|_3^2 \leq C \langle\langle V \rangle\rangle_{Y(n,T)}^2$$

$$\left| D_x^h U(t) \right|_2^2 + \varepsilon \int_0^T \sum_{j=1}^3 \left| D_x^{h+j} U(t) \right|_2^2 dt \leq C \langle\langle V \rangle\rangle_{Y(n,T)}^2$$

where the C does not depend on $\tau \in [0, 1]$. The above three equalities give the proof of (a).

(II) $\Phi(v; \tau)$ is continuous in τ, uniformly in v on $B \times [0, 1]$. Let $u^{\tau+\Delta\tau} = \Phi(v; \tau + \Delta\tau)$,$ = \Phi(v; \tau)$ and $u = u^{\tau+\Delta\tau} - u^\tau$. By eqn. (3.3) we obtain

$$u_t + \varepsilon(-u_{6x} + u_{4x} - u_{2x}) = -i\Delta\tau k(v).$$

By standard estimates and proceeding as in the proof of Lemma 3.2 it is easy to know that

$$| u(t) |_2^2 + \varepsilon \int_0^T \sum_{j=1}^3 \left| D_x^j u \right|_2^2 dt \leq C \Delta\tau$$

$$\int_0^T | u(t) |_2^2 dt + \varepsilon \sum_{j=1}^3 \left| D_x^j u \right|_2^2 \leq C \Delta\tau$$

$$\left| D_x^h u(t) \right|_2^2 + \varepsilon \int_0^T \sum_{j=1}^3 \left| D_x^{h+j} u(t) \right|_2^2 dt \leq C \Delta\tau$$

which give

$$\langle\langle u(t) \rangle\rangle_{X(n,T)}^2 \leq C \Delta\tau.$$

This completes the proof of b).

(III) Φ on B is completely continuous. The preceding results show that Φ is continuous at every point in $B \times [0, 1]$. By Lemma 3.2 $\Phi(v; \tau)$ for each $\tau \in [0, 1]$ maps a set B in $Y(n, T)$ into a set $\{u\}$ with

$$\langle\langle u \rangle\rangle_{X(n,T)} \leq C \langle\langle v \rangle\rangle_{Y(n,T)}$$

where C is independent of $\tau \in [0, 1]$. From Lions[22], Schwartz[24] or an easy direct proof we know that a bounded set in $X(n, T)$ is compact in $Y(n, T)$. Thus, Φ maps an arbitrary set $(v; \tau)$ in $Y(n, T) \times [0, 1]$ into a set that is compact in $Y(n, T)$. This shows that the set of values of $\Phi(v;\tau)$ on $B \times [0, 1]$ is compact.

(IV) All possible fixed points u lie strictly in the interior of the set B. The a priori estimates for the solution ot (3.2) in Lemma 3.1, and the definition of B show that all possible fixed points u of Φ lie strictly in the interior of the set B.

(V) $v-\Phi(v;\tau)$ has nonzero degree. For $\tau = 0$, Φ maps B into a single point u, the unique solution to $u_t + \varepsilon(-u_{6x} + u_{4x} - u_{2x}) = 0$. Thus $v - \Phi(v; 0)$ is invertible and has nonvanishing degree.

Now the application of Leray-Schauder theorem tells us that for each $\tau \in [0, 1]$ there exists at least one fixed point $u(x, t)$ for Φ, which for $\tau = 1$ is a solution to (3.1). The uniqueness of solutions to (3.1) can be proved by the standard energy estimates. Obviously, for $\varepsilon \in (0, 1)$ fixed and $\varphi \in H^\infty$ there exists a unique solution $u \in C^\infty([0, T]; H^\infty)$ to (3.1), which satisfies the estimate (3.4) for every integer n.

Proof of Theorem A — Here, only an outline is given. From the arguments above we know that for $\varphi \in H^s$, $s \geq 4$, there exists a unique solution $u^s(x, t)$ to (3.1) for each $\varepsilon \in (0, 1)$, which satisfies the estimate (3.4) for $n = s$ and the bound is independent of $\varepsilon \in (0, 1)$. By the standard limiting process $\varepsilon \to 0$ we know that there exists a subsequence $u^{\varepsilon'}$ converging weakly star in $L^\infty(0, T; H^s)$ to some u, which is a desired solution to (1). In view of equation (1) one sees that $u_t \in L^\infty(0, T; H^{s-4})$. In this way we can obtain $u \in \bigcap_{k+4h \leq s} W^h_\infty(0, T; H^k(R))$.

4. Proof of Theorem B

It is of interest to know that eqn. (1) has solutions decreasing faster than $H^s(R)$ convergence as x tends to infinity, in particular, solutions in the Schwartz space $S(R)$, for each t provided that its initial data is in $S(R)$. This can be realized by considering the intial value problem (1), (2) in the weighted Sobolev space $J^s_r(R)$. For the $J^s_r(R)$ convergence of solutions to problem (1), (2) we have Theorem B and Corollary C. In the proof of our result we employ the same method as that in Tsutsumi[25].

Since $S(R)$ is dense in $J^s_r(R)$, there exists a sequence $\{\varphi^k\}$ $cS(R)$ such that $\{\varphi^k\} \to \varphi$ strongly in J^s_r as $k \to +\infty$. ... (4.1)

We first consider the parabolic regularization of equation (1) : for $k \in Z\backslash 0$.

$$iu_t = \frac{i}{k}[u_{6x} - u_{4x} + u_{2x}] + u_{4x} + 4|u|^2 u_{2x} + u^2 u^*_{2x} + 3u^*(u_x)^2 + 2|u_x|^2 u + \frac{3}{2}|u|^4 u \quad \text{... (4.2)}$$

with

$$u(x, 0) = \varphi^k(x) \quad \text{... (4.3)}$$

For problem (4.2), (4.3) we have

Lemma 4.1 — For every fixed $k \in Z\backslash 0$, problem (4.2), (4.3) has a unique global solution $u^k \in C^\infty([0, T]; S(R))$, $T > 0$.

PROOF : By the results of section 3 we know that problem (4.2), (4.3) has a unique global solution $u^k \in C^\infty([0, T]; H^\infty)$ and

$$\sup_{0 \le t \le T} \| u^k(t) \|_s^2 + \frac{1}{k} \int_0^T \left(\sum_{j=1}^3 | D^{j+s} u^k(t) |_2^2 \right) dt \le C \quad \ldots (4.4)$$

where C is independent of k but depends on the size of φ and T. In order to prove the assertion of this lemma, it suffices to show that

$$u^k \in L^\infty(0, T; J_r^0(R)) \quad \ldots (4.5)$$

for every $r \in Z$. The proof of (4.5) together with

$$D^j u^k \in L^2(0, T; J_r^0(R)), \qquad j = 1, 2, 3 \quad \ldots (4.6)$$

is done by induction on r, when $r = 0$, it is obvious. Assume that the result is known for all values less than or equal to $r - 1$ $(r \ge 1)$. We prove it for r. Let $\alpha(x)$, $\alpha_\varepsilon(x)$ be as given in Theorem 1.2. For simplicity we sometimes suppress k in u^k. Differentiating
$\| \alpha_\varepsilon u^k(t) \|_{r,0}^2$ with respect to t, using eqn. (4.2) and integrating by parts we have

$$\frac{1}{2}\frac{d}{dt} \| \alpha_\varepsilon u(t) \|_{r,0}^2 + \frac{1}{k} \sum_{m=1}^3 \| \alpha_\varepsilon D^m u(t) \|_{r,0}^2$$

$$+ \frac{1}{k} \sum_{m=1}^3 \mathrm{Re} \int D^m u \left(\sum_{\substack{d+j+n+h=m \\ h<m}} \frac{m!}{d!\,j!\,n!\,h!} D^d \omega^{2r} D^j \alpha_\varepsilon D^n \alpha_\varepsilon D^h u^* \right) dx$$

$$+ \mathrm{Im} \int (\alpha_\varepsilon)^2 \omega^{2r} u^* (u_{4x} + 4 | u |^2 u_{2x} + u^2 u_{2x}^* + 3u^* (u_x)^2$$

$$+ 2 | u_x |^2 u + \frac{3}{2} | u |^4 u)\, dx = 0. \quad \ldots (4.7)$$

Since $| D^d \omega^{2r} | \le C(d)(\omega(x))^{2r-d}$ and $| D^j \alpha_\varepsilon(x) | \le C(j)(\omega(x))^{-j}$, we obtain

$$\frac{1}{k} \left| \sum_{m=1}^3 \mathrm{Re} \int D^m u \left(\sum_{\substack{d+j+n+h<m \\ h<m}} \frac{m!}{d!\,j!\,n!\,h!} D^d \omega^{2r} D^j \alpha_\varepsilon D^n \alpha_\varepsilon D^h u^* \right) dx \right|$$

$$\le \frac{1}{6k} \sum_{m=1}^3 \| \alpha_\varepsilon D^m u \|_{r,0}^2 + C \sum_{j=0}^3 \| D^j u \|_{r-1,0}^2 . \quad \ldots (4.8)$$

Now consider

$$\mathrm{Im} \int (\alpha_\varepsilon)^2 \omega^{2r} u^* (u_{4x} + 4 | u |^2 u_{2x} + u^2 u_{2x}^* + 3u^* (u_x)^2$$

$$+ 2u \mid u_x \mid^2 + \frac{3}{2} \mid u \mid^4 u)dx$$

$$= \operatorname{Im} \int (\alpha_\varepsilon)^2 \omega^{2r} u^* u_{4x}\, dx + \operatorname{Im} \int (\alpha_\varepsilon)^2 \omega^{2r} u^* \Big[4 \mid u \mid^2 u_{2x} + u^2 u_{2x}^*$$

$$+ 3u^* (u_x)^2 + 2u \mid u_x \mid^2 + \frac{3}{2} \mid u \mid^4 u \Big]\, dx . \quad \text{... (4.9)}$$

For the second term in the right-hand side of (4.9) we have

$$\mid \text{the second term} \mid \le \frac{1}{6k} \sum_{m=1}^{2} \| \alpha_\varepsilon D^m u \|_{r,0}^2 + C(k) \| \alpha_\varepsilon u \|_{r,0}^2 . \quad \text{... (4.10)}$$

For the first term in the right-hand side of (4.9) we have

$$\operatorname{Im} \int (\alpha_\varepsilon)^2 \omega^{2r} u^* u_{4x}\, dx$$

$$= \operatorname{Im} \int u_{2x} \left(\sum_{\substack{d+j+n+h=2 \\ h<2}} \frac{2}{d!\,j!\,n!\,h!} D^d \alpha_\varepsilon D^j \alpha_\varepsilon D^n \omega^{2r} D^h u^* \right) dx$$

$$\le \frac{1}{6k} \sum_{m=1}^{2} \| \alpha_\varepsilon D^m u \|_{r,0}^2 + C(k) \sum_{m=0}^{2} \| D^m u \|_{r-1,0}^2 \quad \text{... (4.11)}$$

Considering (4.7)-(4.11) we get

$$\frac{d}{dt} \| \alpha_\varepsilon u(t) \|_{r,0}^2 + \frac{1}{k} \sum_{j=1}^{3} \| \alpha_\varepsilon D^j u(t) \|_{r,0}^2$$

$$\le C(k) \| \alpha_\varepsilon u \|_{r,0}^2 + C(k) \sum_{j=0}^{3} \| D^j u \|_{r-1,0}^2 \quad \text{... (4.12)}$$

where $C(k)$ is a positive constant independent of ε. We integrate (4.12) with respect to t and use the assumptions of the induction. Then Gronwall's inequality yields that

$$\operatorname{Sup}_{0 \le t \le T} \| \alpha_\varepsilon u^k(t) \|_{r,0}^2 \le C \quad \text{... (4.13)}$$

$$\int_0^T \| \alpha_\varepsilon D^j u^k(t) \|_{r,0}^2\, dt \le C, \qquad j = 1, 2, 3 \quad \text{... (4.14)}$$

where C is a positive constant independent of ε. Therefore, $\{\alpha_\varepsilon u^k\}$ remains in a bounded set of $L^\infty(0, T; J_r^0(R))$. So, taking the limit as $\varepsilon \to 0$, we see that $\alpha_\varepsilon u^k \to u^k$ weakly star in $L^\infty(0, T; J_r^0(R))$ and the assertions (4.5), (4.6) hold for r

since $L^\infty(0,T;J_r^0) = (L^1(0,T;H_{-r}^0 + L^2))'$. This ends the proof of the lemma.

Proof of Theorem B — With Lemma 4.1 we now consider the convergence of u^k as $k \to \infty$. From Lemma 4.1 we know that for any given $T > 0$,

$$\sup_{0 \le t \le T} \| u^k(t) \|_{r,0}^2 \le C \quad \text{... (4.15)}$$

holds for all integers s. From this it follows that $\{u^k\}$ forms a bounded set of $L^\infty(0,T;J_0^s(R))$. We next show that for $r > 0$, $\{u^k\}$ remains bounded in $L^\infty(0,T;J_0^s(R))$. In fact, we have

$$\frac{1}{2}\frac{d}{dt}\| u^k(t) \|_{r,0}^2 = \mathrm{Re}\int \omega^{2r} u^* u_t \, dx$$

$$= \frac{1}{k}\mathrm{Re}\int \omega^{2r} u^* (u_{6x} - u_{4x} + u_{2x})\, dx$$

$$- \mathrm{Im}\int \omega^{2r} u^* \Big[u_{4x} + 4|u|^2 u_{2x} + u^2 u_{2x}^* + 3u^*(u_x)^2$$

$$+ 2u|u_x|^2 + \frac{3}{2}|u|^4 u \Big] dx$$

$$= \frac{1}{k}\mathrm{Re}\int \omega^{2r} u^* u_{6x}\, dx - \frac{1}{k}\mathrm{Re}\int \omega^{2r} u^*$$

$$u_{4x}\, dx + \frac{1}{k}\mathrm{Re}\int \omega^{2r} u^* u_{2x}\, dx$$

$$- \mathrm{Im}\int \omega^{2r} u^* u_{4x}\, dx - \mathrm{Im}\int \omega^{2r} u^*$$

$$\Big[4|u_x|^2 u_{2x} + u^2 u_{2x}^* + 3u^*(u_x)^2$$

$$+ 2|u_x|^2 u + \frac{3}{2}|u_x|^4 u \Big] dx$$

$$= B_1 + B_2 + B_3 + B_4 + B_5 \quad \text{... (4.16)}$$

In what follows we bound each term B_j. Using Theorem 1.3 and (4.15) we obtain

$$|B_5| \le C \| u \|_{r,0}^2 \quad \text{... (4.17)}$$

Integrating by parts and using Theorem 1.3 and (4.15) we have

$$|B_4| = \Big| \mathrm{Im}\int u_{2x}(2r\omega^{2(r-1)} u^* + 4r(r-1)\,\omega^{2(r-2)}$$

$$x^2 u^* + 4rx\omega^{2(r-1)} u_x^*)\, dx \Big|$$

$$= |\mathrm{Im}\int u_{2x}(2r\omega^{2(r-1)} u^* + 4r(r-1)\,\omega^{2(r-2)} x^2 u^*)\, dx$$

$$- \operatorname{Im} \int (4r\,\omega^{2(r-1)} u^* u_{2x} + 4r\omega^{2(r-1)} u^* u_{3x}$$

$$+ 8r(r-1)\, x^2 \omega^{2(r-2)} u^* u_{2x})\, dx \mid$$

$$= \mid \operatorname{Im} \int (2r\omega^{2(r-1)} u^* u_{2x} + 4rx\omega^{2(r-1)} u^* u_{3x}$$

$$+ 4r(r-1)\, x^2 \omega^{2(r-2)} u^* u_{2x}\, dx \mid$$

$$\le C \left(\| u \|^2_{r,0} + \| u_{2x} \|^2_{r-1,0} + \| u_{3x} \|^2_{r-1,0} \right) \quad \dots (4.18)$$

$$B_3 = \frac{1}{k} \operatorname{Re} \int \omega^{2r} u^* u_{2x}\, dx$$

$$= -\frac{1}{k} \operatorname{Re} \int (\omega^{2r} u^*_x + 2r\omega^{2(r-1)} x u^*)\, u_x\, dx$$

$$\le -\frac{1}{2k} \| u_x \|^2_{r,0} + C \| u \|^2_{r-1,0} \quad \dots (4.19)$$

$$B_2 = -\frac{1}{k} \operatorname{Re} \int \omega^{2r} u^* u_{4x}\, dx$$

$$= -\frac{1}{k} \| u_{2x} \|^2_{r,0} - \frac{1}{k} \operatorname{Re} \int (2r\omega^{2(r-1)} u^* u_{2x} + 4rx\omega^{2(r-1)} u^* u_{3x}$$

$$+ 4r(r-1)\, \omega^{2(r-2)} x^2 u^* u_{2x})\, dx \le -\frac{1}{2k} \| u_{2x} \|^2_{r,0}$$

$$+ C\left(\| u_x \|^2_{r,0} + \| u_{2x} \|^2_{r-1,0} + \| u_{3x} \|^2_{r-1,0} \right) \quad \dots (4.20)$$

$$B_1 = \frac{1}{k} \operatorname{Re} \int \omega^{2r} u^* u_{6x}\, dx = -\frac{1}{k} \| u_{3x} \|^2_{r,0}$$

$$- \frac{1}{k} \operatorname{Re} \int [(\omega^{2r})_{2x} u^* + 3(\omega^{2r})_x u^*_{2x} + 3(\omega^{2r})_{2x} u^*_x)\, u_{3x}\, dx$$

$$\le -\frac{1}{2k} \| u_{3x} \|^2_{r,0} + C \sum_{j=0}^{3} \| D^j u \|^2_{r-1,0}. \quad \dots (4.21)$$

From (4.16)-(4.21) there appears that since $s \ge \max(3r, 4)$ and $\| u \|^2_{0,s}$ is bounded on $[0, T]$, by using (g) of Theorem 1.1 we have

$$\frac{d}{dt} \| u^k(t) \|^2_{r,0} + \frac{1}{k} \sum_{j=1}^{3} \| D^j u^k(t) \|^2_{r,0}$$

$$\le C \,||\, u^k \,||\,^2_{r,0} + C \sum_{j=0}^{3} ||\, D^j u^k \,||\,^2_{r-1,0}$$

$$\le C \,||\, u^k \,||\,^2_{r,0} + C \sum_{j=0}^{3} |||\, u^k \,|||\,^2_{r,rj}$$

$$\le C \,||\, u^k \,||\,^2_{r,0} + C \,||\, u^k \,||\,^2_{0,s} \le C\left(||\, u^k \,||\,^2_{r,0} + 1 \right) \qquad \text{... (4.22)}$$

where C is independent of the natural number k. Integrating (4.22) with respect to t and using Gronwall's inequality give

$$\operatorname*{Sup}_{0 \le t \le T} \left\| u^k(t) \right\|_{r,0} \le C \qquad \text{... (4.23)}$$

with the constant C independent of k. From (4.15) and (4.23) there follows that $\{u^k\}$ forms a bounded sequence in $L^\infty(0, T; J^s_r(R))$. Hence, there exists a subsequence of $\{u^k\}$ (also denoted by $\{u^k\}$) and $u \in L^\infty(0, T; J^s_r(R))$ such that

$$u^k \to u \text{ weakly star in } L^\infty(0, T; J^s_r(R)) .$$

Then, it can be easily seen by the standard argument that u is a desired solution of (1), (2) (see Lions[22]). From (h) of Theorem 1.1 it is shown that

$$u_{4x} \in L^\infty(0, T; J^{s'}_{r'}(R))$$

where $s' = s - 4$ and $r' = \frac{r(s-4)}{s}$. Here, in view of equation (1) we can conclude that

$$u_t \in L^\infty(0, T; J^{s'}_{r'}(R))$$

with s' and r' as above. Continuing in this way and using eqn. (1) we obtain the conclusion of Theorem B.

Corollary C is just a result of Theorem B and (d) of Theorem 1.1 combined.

ACKNOWLEDGEMENT

The author would here like to express his heartfelt thanks to Professor Gao Wei bing, Professor Cheng Mian and Professor Huo Wei for their helpful instruction. Special thanks goes to the anonymous referees for their invaluable suggestion about the improvement of the manuscript.

REFERENCES

1. M. Ablowitz and H. Segur, *SIAM* Philadelphia, 1981.
2. L. Brull and H. Lange, *Math. Mech. Appl. Sci.* **8** (1986), 559-75.
3. X. S. Feng, On global solutions of the fifth order KdV equation, To appear in *"Acta Math. Appl.*

Sinica".

4. X. S. Feng, The asymptotic behaviors of solutions for the KdV hierarchy, Preprint.
5. X. S. Feng, Asymptotic behaviors of solutions for the MKdV hierarchy, Preprint.
6. X. S. Feng, The decay properties of solutions for the higher order BO equation, To appear in "*J. PDEs*".
7. A. Friedman, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.
8. M. V. Goldman, *Rev. Mod. Phys.* **56** (1984), 709-35.
9. B. L. Guo and S. B. Tan, *Proc. Roy. Soc. Edingburgh* **A118** (1991), 1-15.
10. B. L. Guo, *Applied Math.* (A Journal of Chinese Universities), **6** (1991), 173-81 (In Chinese).
11. J. Ginibre and G. Velo, *J. Funct. Anal.* **32** (1979), 1-32, 33-71.
12. J. Ginibre and G. Velo, *Ann. Inst. Henri Poincare* **A28** (1978), 287-316.
13. N. Hayashi, *Math. Z.* **192** (1986), 637-50.
14. N. Hayashi, *Physica D* **55** (1992), 14-36.
15. N. Hayashi, K. Nakamitsu and M. Tsutsumi, *J. Funct Anal.* **71** (1987), 218-45.
16. N. Hayashi, K. Nakamitsu and M. Tsutsumi, *Funfcialai Ekvacioj* **31** (1988), 363-81.
17. K. P. Hedaker and H. Lange, *J. Math. Phys.* **28** (1987), 1091-93.
18. R. Iorio, *J. Math. Anal. Appl.* **157** (1991), 577-90.
19. T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation, *Advances in Mathematies, Supplementary Studies in Applied Mathematics* **8** (1983), 93-128.
20. M. Lakshmanan, K. Porsezian and M. Daniel, *Phys. Lett.* **A133** (1988), 483-87.
21. J. Leray and J. Schauder, *Ann. Sci. Ecole Norm. Sup.* **51** (1934), 45-78.
22. J. L. Lions, *Quelques methodes de resolutions des problemes aux Limites nonlineaires*, Gauthier-Villars, Paris, 1969.
23. Y. Matsuno, *J. Phys. Soc. Japan* **49** (1980), 787-94.
24. M. Schwartz, *Advances in Math.* **54** (1984), 22-56.
25. M. Tsutsumi, *J. Diff. Equations* **42** (1981), 260-81.
26. M. Tsutsumi, *Nonlinear Analysis*, **13** (1989), 1051-56.
27. M. Tsutsumi and N. Hayashi, *Math. Z.* **177** (1981), 217-34.
28. V. E. Zakharov and S. V. Manakov, *Theor. Math. Phys.* **19** (1974), 551-59.
29. V. E. Zakharov and A. B. Shabat, *Sov. Phys. JETP.* **34** (1972), 62-69.