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GENERALIZED LOCAL SEPARATION PROPERTIES

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Earlier various generalizations of the usual separation properties of topology and separation properties at a point p are defined for an arbitrary topological category over sets[1]. In this paper, under certain conditions a generalized separation property implies that property at p and that a separation property at p implies another separation property at p.

INTRODUCTION

Various generalizations of the usual separation axioms of topology and separation properties at p are given in Baran[1]. These generalizations are given at a point, p, i.e., locally, then they are generalized to point free definitions by using the generic elelment[6] method of topos theory. These generalizations include two notions of each of T_0 at p and T_2 at p, one notion of T_1 at p, and four notions of each of T_3 at p and T_4 at p. Each of these notions is studied in Baran[1] for the case of topological spaces. One of the other uses of local separation properties is to define the notion of closed subsets of an object of a topological category (see Baran[4]).

The main object of this paper is to investigate the relationships among various forms of generalized separation properties at p.

Let E be a category and Set be the category of sets. Let $U : E \to$ Set be a topological functor[5]. Each such topological functor has a left adjoint D : Set $\to E$ called the discrete functor, where $D(B)$ is the discrete object of $U^{-1}(B)$, the fiber over B, and is characterized as the minimum element[7], of the fibers of U. Recall[7] that an object B in E is discrete if and only if every map $U(B) \to U(C)$ lifts to a morphism $B \to C$ for each object C in E. A topological functor U is said to be normalized if there is only one structure on the empty set and on a point.

Let B be a set and $B^2 \vee_\Delta B^2$ be the wedge product of B^2 (i.e. two distinct copies of B^2 identified along the diagonal, Δ). A point (x, y) in $B^2 \vee_\Delta B^2$ will be denoted by $(x, y)_1$, $((x, y)_2)$ if (x, y) is in the first (resp. second) component of $B^2 \vee_\Delta B^2$.

Clearly, $(x, y)_1 = (x, y)_2$ iff $x = y$. Recall, Baran[1] p.337, that the principal axis map, $A : B^2 \vee_\Delta B^2 \to B^3$ is defined by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map, $S : B^2 \vee_\Delta B^2 \to B^3$ is given by $S(x, y)_1 = (x, y, y)$ and $S(x, y)_2 = (x, x, y)$, and the fold map, $\nabla : B^2 \vee_\Delta B^2 \to B^2$ is given by $\nabla(x, y)_1 = (x, y)$ for $i = 1, 2$.

Let B be a set, $p \in B$, and $B \vee_p B$ be the wedge product of B. A point x in $B \vee_p B$ will be denoted by $x_1, (x_2)$ if x is in the first (resp. second) component of $B \vee_p B$. Recall, Baran[1] p.334, that the principal p axis map $A_p : B \vee_p B \to B^2$ is given by $A_p(x_1) = (x_1, p)$ and $A_p(x_2) = (p, x_2)$. The skewed p axis map $S_p : B \vee_p B \to B^2$ is given by $S_p(x_1) = (x_1, x_1)$ and $S_p(x_2) = (p, x_2)$, and the fold map at p, $\nabla_p : B \vee_p B \to B$ is given by $\nabla_p(x_i) = x$ for $i = 1, 2$. Note that $\pi_1 A_p = p_1 = \pi_1 S_p$, $\pi_2 A_p = p_2$, and $\pi_2 S_p = \nabla_p$.

Let $U : E \to$ Set be a topological functor, X an object in E, p a point in $U(X) = B$ and D : Set $\to E$ be the discrete functor.

Definitions 1.1 — (See Baran[1], pp.334 and 338) — (1) X is $\overline{T}_\theta$ iff the initial lift of the U-source $\{A : B^2 \vee_\Delta B^2 \to U(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to UD(B^2) = B^2\}$ is discrete.

(2) X is T_1 iff the initial lift of the U-source $\{S : B^2 \vee_\Delta B^2 \to U(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to UD(B^2) = B^2\}$ is discrete.

(3) X is PreT$_2$ iff the initial lift of the U-sources $A : B^2 \vee_\Delta B^2 \to U(X^3) = B^3$ and $S : B^2 \vee_\Delta B^2 \to U(X^3) = B^3$ agree.

(4) X is $\overline{T}_2$ iff X is $\overline{T}_\theta$ and Pre$\overline{T}_2$.

(5) X is $\overline{T}_\theta$ at p iff the initial lift of the U-source $\{A_p : B \vee_p B \to U(X^2) = B^2$ and $\nabla_p : B \vee_p B \to UD(B) = B\}$ is discrete.

(6) X is T_1 at p iff the initial lift of the U-source $\{S_p : B \vee_p B \to U(X^2) = B^2$ and $\nabla_p : B \vee_p B \to UD(B) = B\}$ is discrete.

(7) X is T_θ' at p iff the initial lift of the U-source $\{id : B \vee_p B \to U(B \vee_p B)' = B \vee_p B$ and $\nabla_p : B \vee_p B \to UD(B) = B\}$ is discrete, where $(B \vee_p B)'$ is the wedge in E, i.e., the final lift of the U-sink $\{i_i, i_2 : U(X) = B \to B \vee_p B\}$ where i_1, i_2 denote the canonical injections.

(8) X is Pre$\overline{T}_2$ p at iff the initial list of the U-sources $A_p : B \vee_p B \to U(X^2) = B^2$ and $S_p : B \vee_p B \to U(X^2) = B^2$ agree.

(9) X is PreT_2' p at iff the initial lift of the U-sources $S_p : B \vee_p B \to U(X^2) = B^2$ and the final lift of the U-sink $\{i_1, i_2 : U(X) \to B \vee_p B\}$ agree.

(10) X is $\overline{T}_2$ at p iff X is $\overline{T}_\theta$ at p and Pre$\overline{T}_2$ at p.

(11) X is T_2' at p iff X is T_θ' at p and PreT_2' at p.

Lemma 1.2 — If $gh : X \to Y \to Z$ is a final (initial) lift, then so is g(resp. h).

PROOF : It follows easily.

2. MAIN RESULTS

In this section, we will investigate relationships among various generalized separation properties defined in Definition 1.1. Let 1 denote the one point set.

Lemma 2.1 — Each retract of a discrete object in a topological category $U : E \to$ Set is discrete.

PROOF : Suppose $h : Y \to DA$ is retract, i.e., there exists a map $r : DA \to Y$ such that $rh = id$, where DA is A equipped with a discrete structure. Let Z be any object in E and $g : UY \to UZ$ be any function. Since DA is discrete and $gUr : UDA \to UZ$, there exists a morphism $m : DA \to Z$ such that $Um = gUr$. If $k = mh$, then clearly $Uk = g$ and consequently Y is discrete.

Definition 2.2 — $p \in UX$ is a retract of X if the initial lift $h : \bar{1} \to X$ of the U-source $p : 1 \to UX$ is a retract.

Lemma 2.3 — Let $\bar{f}_i : X \to X_i$ and $\bar{g}_i : Y \to Y_i$ be the initial lifts of the U-sources $f_i : B \to UX_i$ and $g_i : C \to UY_i$, respectively.

(1) If $h_i : UX_i \to UY_i$ and $k : B \to C$ satisfy $h_i f_i = g_i k$, then k lifts to $X \to Y$ if each h_i lifts to $X_i \to Y_i$.

(2) If, further, $h : C \to B$ and $k_i : UY_i \to UX_i$ satisfy $k_i g_i = f_i h$, k_i lifts to $Y_i \to X_i$, and $hk = id$, then k lifts to a retract $X \to Y$.

PROOF : (1) Since $g_i k = h_i f_i$ lifts to $X \to Y_i$, and $\{\bar{g}_i\}$ is an initial lift, k lift to a map $f : X \to Y$.

(2) By (1) $\bar{h}$ lifts to $Y \to X$. Note that $U(\bar{h}\,\bar{k}) = id$ and since U is topological, $\bar{h}\,\bar{k} = id$, i.e., $\bar{k}$ is a retract.

Lemma 2.4 — Let $\bar{f}_i : X_i \to X$ be the final lift of the U-sink $f_i : UX_i \to B$ and $\bar{g}_j : Y \to Y_j$ be the initial lift of the U-source $g_j : B \to UY_j$. All the maps $g_j f_i$ lift to the map $X_i \to Y_i$ iff there exists a map $f : X \to Y$ with $U(f) = id$.

PROOF : Let h_{ij} be the lifts of $g_j f_i$, i.e, $U(h_{ij}) = g_j f_i$. Since $\bar{g}_j$ is initial, there exists a map $k_i : X_i \to Y$ such that $U(k_i) = f_i$ and consequently $f : X \to Y$ exists (since $\bar{f}_i : X_i \to X$ is the final lift). Conversely, if there exists $f : X \to Y$ with $U(f) = id$, the identity map $B \to B$, then let $h_{ij} = \bar{g}_j f \bar{f}_i : X_i \to Y_j$. Clearly, $U(h_{ij}) = g_j f_i$ i.e. $g_j f_i$ lifts.

Lemma 2.5 — The constant map $f : UX \to UX$ at p lifts to $\bar{f} : X \to X$ iff p is a retract of X.

PROOF : Suppose $\bar{f} : X \to X$ is a lift of $f : UX \to UX$, the constant map at p. Since $\bar{p} : \bar{1} \to X$ is an initial lift, there exists a map $r : X \to \bar{1}$ such that $Ur : UX \to 1$. Clearly, $r\bar{p} = id$, i.e., p is a retract of X. Conversely, if p is a retract of X, then there exists $r : X \to \bar{1}$. Let $(\bar{f} : X \to X) = (\bar{p}r : X \to \bar{1} \to X)$. Clearly, $U(\bar{f}) = f$, the constant map at p.

Theorem 2.6 — Suppose $U : E \to$ Sets is topological. Let X be an object in E and let p be a retract of X. Then

(1) X is $\overline{T}_0$ at p if X is $\overline{T}_0$.

(2) X is T_1 at p if X is T_1.

(3) X is pre$\overline{T}_2$ at p if X is Pre$\overline{T}_2$.

(4) X is $\overline{T}_2$ at p if X is $\overline{T}_2$.

PROOF : (1) In Lemma 2.3, take $\overline{f}_i : X \to X_i$ to be the initial lift of the U-source $\{A_p : B \vee_p B \to U(X^2),\ \nabla_p : B \vee_p B \to U(DB)\}$, $\overline{g}_i : Y \to Y_i$ as the initial lift of the U-source $\{A : B^2 \vee_p B^2 \to U(X^3),\quad \nabla : B^2 \vee_p B^2 \to U(D\,B)\}$, $h_i : U(X_i) \to U(Y_i)$ as $\{h_i : B^2 \to B^3,\ h_2 : B \to B^2$, where $h_1(a, b) = (p, a, b)$ and $h_2(a) = (p, a)\}$, $k_i : U(Y_i) \to U(X_i)$ as $\{\pi_{23}\, B^3 \to B^2, \pi_2 : B^2 \to B$ where $\pi_{23}(a, b, c) = (b, c)$ for any a, b, c in $B\}$, $h = \pi_2 + \pi_2 : B^2 \vee_\Delta B^2 \to B \vee_p B$, and $k : B \vee_p B \to B^2 \vee_\Delta B^2$ given by $k(x) = (p, x)$ for any x in $B \vee_p B$. Clearly, $h_i f_i = g_i k$. Note that h_2 lifts to $\overline{h}_2 : DB \to DB^2$ since DB is discrete, h_i lifts to $\overline{h}_i : X^2 \to X^3$ since p is a retract (2.5), and consequently, by Lemma (2.3) (1), there exists a lift $\overline{k}$ of k. Since X is $\overline{T}_0$, Y is discrete, i.e., $Y = D(B^2 \vee_\Delta B^2)$ and consequently h lifts to h' (see introduction). Since $hk = id$ and U is topological (faitful), $h'\overline{k} = id$, i.e., $\overline{k} : \overline{B\ V_p\ B} \to D(B^2 \vee_\Delta B^2)$ is a retract. By Lemma 2.1, $B\ V_p\ B$ is discrete. Hence, this shows (1). The proof of (2) similar to the proof of the first part by replacing S_p and S in place of A_p and A, respectively.

(3) Let $\overline{f}_i : X \to X_i$ be the initial lift of either $\{A_p : B \vee_p B \to U(X^2)\}$ or $\{S_p : B \vee_p B \to U(X^2)\}$, $\overline{g}_i : Y \to Y_i$ be the initial lift of either $\{A : B^2 \vee_\Delta B^2 \to U(X^3)\}$ or $\{S : B^2 \vee_\Delta B^2 \to U(X^3)\}$, and $h_1 : B^2 \to B^3$, $k_1 = \pi_{23} : B^3 \to B^2$, $h = \pi_2 + \pi_2\ : B^2 \vee_\Delta B^2 \to B \vee_p B$, and $k : B \vee_p B \to B^2 \vee_\Delta B^2$ be defined as above. We apply 2.3 (1) twice. Clearly, $h_1 A_p = Ak$ and h_1 lifts to $\overline{h}_1 : X^2 \to X^3$ since p is a retract Lemma 2.5. Hence, k lifts to $k' : \overline{\overline{B \vee_p B}} \to \overline{\overline{B^2 \vee_p B^2}}$. Similarly, $h_1 S_p = S_k$ and h_1 lifts to $X^2 \to X^3$. Hence, by Lemma 2.3(1), k lifts to $k'' : \overline{\overline{B \vee_p B}} \to \overline{\overline{B^2 \vee_p B^2}}$. Note that $\overline{h}_i$ is actually the initial lift of h_i. Since $\overline{h}_i \overline{A}_p = \overline{A}\, k'$ is the initial lift (since the composition of the initial lifts is initial) and $\overline{h}_i \overline{A}_p = \overline{A}\ k'$, it follows from Lemma 1.2 that k' is initial. Similarly, k'' is the initial lift of k. Since the initial lifts are unique, it follows that $k' : k''$, i.e. $\overline{B \vee_p B} \to \overline{B \vee_p B}$. Hence, X is Pre$\overline{T}_2$ at p.

Part (4) follows from Definition 1.1 and parts (1) and (3).

Corollary 2.7 — If $U : E \to$ Set is normalized, then $\overline{T}_0$, T_1, Pre$\overline{T}_2$, and $\overline{T}_2$ imply $\overline{T}_0$ at p, T_1 at p, Pre$\overline{T}_2$ at p, and $\overline{T}_2$ at p, respectively.

PROOF : Since U is normalized, p is always a retract of X (since $\overline{1}$ is the terminal). Hence, the results follows from 2.6.

Theorem 2.8 — Suppose that $i_1, i_2 : X \to (B \vee_p B)'$ is the final lift of the U-sink $\{i_1, i_2 : U(X) \to B \vee_p B\}$ and $p_1, p_2 : \overline{B \vee_p B} \to X$, $p_1, \nabla_p : \overline{B \vee_p B} \to X$ are the initial lifts of the U-sources $\{p_1, p_2 : B \vee_p B \to U(X)\}$ and $\{p_1, \nabla_p : B \vee_p B \to U(X)\}$, respectively, where p_1, p_2 and ∇_p are defined in 1.1. Then the following are equivalent.

(1) There exists a map $f : (B \vee_p B)' \to \overline{B \vee_p B}$ such that $U(f) = id$.

(2) p is a retract of X.

(3) There exists a map $g : (B \vee_p B)' \to B \vee_p \overline{B}$ such that $U(g) = id$.

PROOF : To show that (1) iff (2), in 2.4, let $\overline{f_i} : X_i \to X$ be the final lift of the U-sink $\{i_1, i_2 : U(X) \to B \vee_p B\}$ and $\overline{g}_j : Y \to Y_j$ be the initial lift of the U-source $\{p_1, p_2 : B \vee_p B \to U(X)\}$. Note that $p_1 i_1 = id = p_2 i_2$ and $p_1 i_2 = p = p_2 i_1$. Since id always lifts, $\overline{p}$ lifts, by 2.5, i.e, p is retract, iff there exists a map f : $(B \vee_p B)' \to \overline{B \vee_p B}$ such that $Uf = id$ by Lemma 2.4. i.e. (1) iff (2). For (2) iff (3), in Lemma 2.4, let $\overline{g}_j : Y \to Y_j$ be the initial lift of the U-source $\{p_1, \nabla_p : B \vee_p B \to U(X)\}$. Note $p_1 i_1 = id = \nabla_p i_1$, $\nabla_p i_2 = id$, and $p_1 i_2 = p$, i.e. the constant map at p. Since $id : X \to X$ is always lifts, p lifts by Lemma 2.4, i.e, p is a retract Lemma 2.5 iff there exists a map $g : (B \vee_p B)' \to \overline{B \vee_p B}$ such that $U(g) = id$.

Corollary 2.9 — If $U : E \to$Set is normalized, then there are always morphisms $g : (B \vee_p B)' \to \overline{B \vee_p B}$ and $f : (B \vee_p B)' \to \overline{B \vee_p B}$ such that $U(f) = id = U(g)$ since p is always a retract of X.

Theorem 2.10 — (1) If p is a retract , then each of $\overline{T}_\theta$ at p, T_1 at p, and $\overline{T}_2$ at p implies T_θ' at p.

(2) Each of $\overline{T}_2$ at p and T_2' at p implies T_1 at p

PROOF : It follows easily from 1.1, 2.1, and 2.8.

Corollary 2.11 — If $U : E \to$ Set is normalized, then each of $\overline{T}_\theta$ at p, T_1 at p, and $\overline{T}_2$ implies T_θ' at p.

PROOF : Since U is normalized, p is always a retract of X and the results follow from Theorem 2.10.

Remark 2.12 : (1) The retract condition of Theorem 2.6 is sufficient for parts (1) and (2) but is not necessary. As the category CP of pair spaces (see Baran[4] shows; every object of CP is, by Baran[4], both T_θ and $\overline{T}_\theta$ at p, and T_1 and T_1 at p. However, if p is not in B, then the initial lift $1 = (1, \Phi) \to X = (A, B)$ is clearly not a retract.

(2) Since the category RRel of reflexive relation spaces (see Baran[2]) is normalized, it follows that p is retract. In RRel, T_θ' at p does not imply each of $\overline{T}_\theta$ at p, T_1 at p, and $\overline{T}_2$ at p (see Baran[2]) and consequently the converse of Theorem 2.10(1) does not hold.

(3) Neither T_1 at p nor T_θ' at p implies $\overline{T}_2$ at p and T_2' at p, in general. This happens, for example, in the category *PBorn* of prebornological spaces (see Baran[3]). Hence, the converse of Theorem 2.10(2) does not hold, in general. Moreover, in *PBorn* $\overline{T}_2$ at p does not imply T_2' at p (Baran[3]), and both could be equal as can be seen in the category Prord of preordered spaces (see Baran[2]).

(4) It is possible to have all of the separation properties defined in Definition 1.1 to be equivalent. This happens, for example, in the category Born of bornological spaces (see Baran[3]).

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