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STRONGLY SUMMABLE SEQUENCE SPACES DEFINED BY A MODULUS

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The purpose of this paper is to introduce some new sequence spaces and to examine their basic properties, which generalizes the results of Maddox.

By $\mathbb{C}$ and l_∞ we denote the set of complex numbers and bounded sequences, respectively.

A sequence $x = (x_k)$ is said to be summable $(c, 1)$ if and only if $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k$ exists.

Spaces of strongly Cesaro summable sequences were discussed by Kuttner[1] and some others. The class of sequences which are strongly Cesaro summable with respect to a modulus was introduced by Maddox[5] as an extension of the definition of strongly Cesaro summable.

Recall[5,6] that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that (i) $f(x) = 0$ if and only if $x = 0$, (ii) $f(x + y) \leq f(x) + f(y)$ for $x \geq 0$, $y \geq 0$, (iii) f is increasing, (iv) f is continuous from the right at 0. Hence f must be continuous everywhere on $[0, \infty)$. A modulus may be unbounded or bounded.

In the present paper we introduce and examine some properties of three sequence spaces defined by using a sequence of strictly positive real numbers $p = (p_k)$, which generalize the spaces $w(f)$, $w_0(f)$, and $w_\infty(f)$ of Maddox[5].

We now introduce the generalizations of the class of sequences which are strongly Cesaro summable with respect to a modulus that was introduced by Maddox[5].

Definition 1 — Let $p = (p_k)$ be a sequence of strictly positive real number and f be a modulus. Denote by s the space of all complex sequences $x = (x_k)$. Define

$$w(f, p) = \left\{ x : \frac{1}{n} \sum_{k=1}^{n} f(|x_k - l|)^{p_k} \to 0 \text{ as } n \to \infty \text{ for some } l \right\},$$

$$w_0(f,p) = \left\{ x : \frac{1}{n} \sum_{k=1}^{n} f(|x_k|)^{p_k} \to 0 \text{ as } n \to \infty \right\}$$

$$w_\infty(f,p) = \left\{ x : \sup_n \frac{1}{n} \sum_{k=1}^{n} f(|x_k|)^{p_k} < \infty \right\}$$

where, for convenience, we put $f(|x_k|)^{p_k}$ instead of $\{f(|x_k|)\}^{p_k}$.

If $x \in w(f,p)$ with $\frac{1}{n} \sum_{k=1}^{n} f(|x_k - l|)^{p_k} \to 0$ as $n \to \infty$, then we will write $x_k \to l\,(w(f,p))$.

The case $f(x) = x$ was discussed by Maddox[2,3]. If $f(x) = x$ we write $w(p)$, $w_0(p)$, and $w_\infty(p)$ for $w(f, p)$, $w_0(f,p)$, and $w_\infty(f,p)$, respectively. Hence $w(p)$, $w_0(p)$ and $w_\infty(p)$ are the same as the spaces $[C, I, p]$, $[C, I, p]_0$ and $[C, I, p]_\infty$ of Maddox[2,3], respectively.

If $p_k = 1$ for all k, we write $w(f)$, $w_0(f)$ and $w_\infty(f)$ for $w(f, p)$ $w_0(f,p)$, and $w_\infty(f,p)$, respectively. Hence $w(f)$, $w_0(f)$ and $w_\infty(f)$ are the same as the spaces $w(f)$, $w_0(f)$ and $w_\infty(f)$ of Maddox[5], respectively.

The following inequality will be used frequently throughout the paper :

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k}) \qquad \dots (1)$$

where $a_k, b_k \in \mathbb{C}$, $0 < p_k \leq \sup p_k = H$, $C = \max(1, 2^{H-1})$ (Maddox[4]).

Let $p = (p_k)$ be boubded. It is easy to see that $W(f, p)$, $w_0(f,p)$ and $w_\infty(f,p)$ are linear spaces.

Also we can now show that the space $w_0(f,p)$ is paranormed by

$$g(x) = \sup_n \left\{ \frac{1}{n} \sum_{k=1}^{n} f(|x_k|)^{p_k} \right\}^{1/M}$$

where $H = \sup p_k < \infty$, and $M = \max(1, H)$.

Clearly $g(\theta) = 0$, and $g(x) = g(-x)$. Take any $x, y \in w_0(f,p)$. Since $p_k/M \leq 1$ and $M \geq 1$, using the Minkowski's inequality and definition of f, for each n, we have

$$\left\{ \frac{1}{n} \sum_{k=1}^{n} f(|x_k + y_k|)^{p_k} \right\}^{1/M} \leq \left\{ \frac{1}{n} \sum_{k=1}^{n} \{f(|x_k|) + f(|y_k|)\}^{p_k} \right\}^{1/M}$$

$$\leq \left\{ \frac{1}{n} \sum_{k=1}^{n} f(|x_k|)^{p_k} \right\}^{1/M} + \left\{ \frac{1}{n} \sum_{k=1}^{n} f(|y_k|)^{p_k} \right\}^{1/M}$$

Now it follows that g is subadditive. Finally, to check the continuity of multiplication, let us take any complex λ. By definition of f we have

$$g(\lambda x) = \sup_n \left\{ \frac{1}{n} \sum_{k=1}^{n} f(\mid \lambda x_k \mid)^{p_k} \right\}^{1/M} \leq K^{H/M} g(x)$$

where $[t]$ denotes the integer part of t, and $K = 1 + [|\lambda|]$. Now, let $\lambda \to 0$ for any fixed x with $g(x) \neq 0$. By definition of f, for $|\lambda| < 1$, we have

$$\frac{1}{n} \sum_{k=1}^{n} f(\mid \lambda x_k \mid)^{p_k} < \varepsilon \text{ for } n > N(\varepsilon). \qquad \text{... (2)}$$

Also, for $1 \leq n \leq N$, taking λ small enough, since f is continuous we have

$$\frac{1}{n} \sum_{k=1}^{n} f(\mid \lambda x_k \mid)^{p_k} < \varepsilon \qquad \text{... (3)}$$

(2) and (3) together imply that $g(\lambda x) \to 0$ as $\lambda \to 0$.

We now establish inclusion relation between $w(f, p)$ spaces. Let $p_k = r \;\forall\; k$, $q_K = s \;\forall\; k$ and $0 < r \leq s$. Then it follows from Holder's inequality

$$\frac{1}{n} \sum_{k=1}^{n} f(\mid x_k \mid)^{r} \leq \left\{ \frac{1}{n} \sum_{k=1}^{n} f(\mid x_K \mid)^{s} \right\}^{r/s}$$

and therefore $w(f, q) \subseteq w(f, p)$. We now consider that (p_k) and (q_k) are not constant sequences. We are able to prove $w(f, q) \subseteq w(f, p)$ only under additional conditions. We have

Theorem 2 — Let $0 < p_k \leq q_k$ and let $\dfrac{q_k}{p_k}$ be bounded. Then $w(f, q) \subseteq w(f, p)$.

PROOF : Let $x \in w(f, q)$. Write $t_k = f(|x_k - l|)^{q_k}$ and $\lambda_k = \dfrac{p_k}{q_k}$, so that $0 < \lambda \leq \lambda_k \leq 1$. Define $u_k = t_k \,(t_k \geq 1)$, $= 0 \;(t_k < 1)$ and $v_k = 0 \;(t_k \geq 1)$, $= t_k \,(t_k < 1)$. So $t_k = u_k + v_k$, $t_k^{\lambda_k} \leq u_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that $u_K^{\lambda_k} \leq u_k \leq t_k$, $v_K^{\lambda_k} \leq v_K^{\lambda}$. We have

$$\frac{1}{n} \sum_{k=1}^{n} t_K^{\lambda_k} \leq \frac{1}{n} \sum_{k=1}^{n} t_k + \left\{ \frac{1}{n} \sum_{k=1}^{n} v_k \right\}^{\lambda}$$

Now, it follows that $x \in w(f, p)$ and this completes the proof.

Result 3 — (i) If $0 < \inf p_k \leq p_k \leq 1$, then $w(f) \subseteq w(f, p)$.

(ii) If $1 \leq p_k \leq \sup p_k = H < \infty$, then $w(f, p) \subseteq w(f)$.

In the following theorem we consider the case when $x_k \to l$ implies $x_k \to l$ $[w(f, p)]$. To prove the theorem we require the following :

Lemma — Let $p_k > 0$, $q_k > 0$. Then $c_0(q) \subseteq c_p(p)$ if and only if $\liminf \frac{p_k}{q_k} > 0$, where $c_0(p) = \{x : |x_k|^{p_k} \to 0 \text{ as } k \to \infty\}$ (Maddox[2]).

Theorem 4 — (i) Let $\liminf p_k > 0$. Then $x_k \to l$ implies $x_k \to l$ $[w(f, p)]$.

(ii) Let $\liminf p_k = r > 0$. If $x_k \to l$ $[w(f, p)]$, then l is unique.

PROOF : (i) Let $x_k \to l$. By definition of f we have $f(|x_k - l|) \to 0$. Since $\liminf p_k > 0$, it follows from the above Lemma that $f(|x_k - l|)^{p_k} \to 0$, and consequently $x_k \to l$ $[w(f, p)]$.

(ii) Let $\lim p_k = r > 0$. Suppose that $x_k \to l_1$ $[w(f, p)]$, $x_k \to l_2$ $[w(f, p)]$ and $|l_1 - l_2| = a > 0$. Then from (1) and definition of f we have

$$\frac{1}{n}\sum_{k=1}^{n} f(|l_1 - l_2|)^{p_k} \leq \frac{C}{n}\sum_{k=1}^{n} f(|x_k - l_1|)^{p_k} + \frac{C}{n}\sum_{k=1}^{n} f(|x_k - l_2|)^{p_k}.$$

Thus,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(|l_1 - l_2|)^{p_k} = 0. \qquad \dots (4)$$

Also $f(|l_1 - l_2|)^{p_k} \to f(a)^r$ as $K \to \infty$ and therefore

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(|l_1 - l_2|)^{p_k} = f(a)^r. \qquad \dots (5)$$

From (4) and (5) it follows that $f(a) = 0$ and by definition of f, we have $a = 0$. Hence $l_1 = l_2$ and this completes the proof.

Theorem 5 — If $0 < h = \inf p_k \leq \sup p_k = H < \infty$, then for any modulus f we have

$$w(p) \subseteq w(f, p),\ w_0(p) \subseteq w_0(f, p),\ \text{and}\ w_\infty(p) \subseteq w_\infty(f, p).$$

PROOF : Just consider first inclusion; the others are similar except $\inf p_k > 0$ is not necessary for the third inclusion.

Let $0 < h < \inf p_k \leq \sup p_k = H < \infty$ and $x \in w(p)$, so that

$$s_n(p) = \frac{1}{n}\sum_{k=1}^{n} f(|x_k - l|)^{p_k} \to 0 \ (n \to \infty).$$

Let $1 > \varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \le t \le \delta$. Write $t_k = |x_k - l|$ and consider

$$\sum_{k=1}^{n} f(t_k)^{p_k} = \Sigma_1 + \Sigma_2$$

where the first summation is over $t_k \le \delta$ and the second over $t_k > \delta$. Then $\Sigma_1 \le \varepsilon^h n$, and for $t_k > \delta$, we use the fact that

$$t_k < t_k/\delta < 1 + [t_k/\delta].$$

By definition of f, we have for $t_k > \delta$,

$$f(t_k) < (1 + [t_k/\delta])\, f(1) < 2f(1)\, t_k/\delta.$$

Hence $\Sigma_2 \le \max(\ \{2\delta^{-1} f(1)\}^h\ \{2\delta^{-1} f(1)\}^H)\, n\, s_n(p)$, which together with $\Sigma_1 \le \varepsilon^h n$ yields first inclusion.

Some information on multipliers for $w_\infty(f, p)$ is given below. For any set E of sequences we denote by $M(E)$ the space

$$\{a \in S : ax \in E \text{ for all } x \in E\}.$$

Theorem 6 — Let $H = \sup p_k < \infty$. Then

(i) For any modulus f we have $l_\infty \subseteq M(w_\infty(f, p)) \subseteq w_\infty(f, p)$.

(ii) If f is bounded, then $M(w_\infty(f, p)) = w_\infty(f, p) = s$.

PROOF : (i) $a \in l_\infty$ implies $|a_k| < 1 + [K]$ for some $k > 0$ and all K. Let $H = \sup p_k < \infty$. Hence $x \in w_\infty(f, p)$ implies

$$\frac{1}{n} \sum_{k=1}^{n} f(\,|\,a_k x_k\,|\,)^{p_k} \le \{1 + [K]\}^H \frac{1}{n} \sum_{k=1}^{n} f(\,|\,x_k\,|\,)^{p_k}$$

which gives the first inclusion. Let $a \in M(w_\infty(f, p))$. Since $x = (1, 1, 1, \ldots) \in w_\infty(f, p)$, then we have $a \in w_\infty(f, p)$ which gives the second inclusion.

(ii) Let $H = \sup p_k < \infty$. If f is bounded, then for any $x \in s$,

$$\frac{1}{n} \sum_{k=1}^{n} f(\,|\,x_k\,|\,)^{p_k} \le \max\ \{1,\ (\sup\ \{f(t)\ :\ t \ge 0\})^H\}$$

so that $M(w_\infty(f, p)) = s = w_\infty(f, p)$.

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