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MATRIX TRANSFORMATIONS BETWEEN CESARO SEQUENCE SPACES

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The main purpose of this paper is to characterize the matrices in the classes (Ces (p, s), cs) and (Ces(p, s), bs), where cs is the space of convergent series and bs is the space of bounded series.

1. Introduction

Let X and Y be any two non-empty subsets of the space of all sequences of complex numbers and let $A = (a_{nk})$, $(n, k = 1, 2, ...)$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ converges for each n. If $x = (x_k) \in X \Rightarrow Ax = (A_n(x)) \in Y$, we say that A defines a matrix transformation from X into Y and we denote it by $A : X \rightarrow Y$. By (X, Y) we mean the class of matrices A such that $A : X \rightarrow Y$.

The main purpose of this paper is to characterize the matrices in the classes (Ces(p, s), cs), (Ces(p, s), bs) and for $p = (p_r)$ with $\inf p_r > 0$ we define (see Khan and Khan[1]) for $s \geq 0$

$$\text{Ces}(p, s) = \left\{ x = (x_k) : \sum_{r=0}^{\infty} (2^r)^{-s} \left(\frac{1}{2^r} \sum_r |x_k| \right)^{P_r} < \infty \right\}$$

where Σ denotes a sum over the ranges $2^r \leq k < 2^{r+1}$.

These spaces i.e. Ces(p, s) can be viewed as Ces(p) spaces with weights, generalizing Ces(p) spaces. Obviously, the space Ces(p), which has been investigated by Lim[2] is a special case of Ces(p, s) with $s = 0$.

Space Ces(p, s) is paranormed by

$$g(x) = \left(\sum_{r=0}^{\infty} (2^r)^{-s} \left(\frac{1}{2^r} \sum_r |x_k| \right)^{P_r} \right)^{1/M}$$

if $H = \sup_r p_r < \infty$ and $M = \max(1, H)$ (see Khan and Khan[1]).

Now we define (see Stieglitz and Tietz[4])

$$cs = \left\{ x : \left(\sum_{i=1}^{n} x_i \right) \in c \right\}$$

$$c_0(s) = \left\{ x : \left(\sum_{i=1}^{n} x_i \right) \in c_0 \right\}$$

$$bs = \left\{ x : \left(\sum_{i=1}^{n} x_i \right) \in l_\infty \right\}.$$

We state the following inequality (see Maddox[3]) which will be used later. For any $C > 0$ and any two complex numbers a, b,

$$|ab| \leq C\left(|a|^q C^{-q} + |b|^p \right) \quad \text{... (1)}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

2. Matrix Transformations on Ces(p, s)

The following notations are used throughout for all integers $n \geq 1$, we write

$$t_n(Ax) = \sum_{i=1}^{n} A_i(x) = \sum_{k=1}^{\infty} b_{nk} x_k$$

where

$$b_{nk} = \sum_{i=1}^{n} a_{ik} .$$

We now prove :

Theorem 1 — Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (\text{Ces}(p, s), cs)$ iff (i) there exists an integer $E > 1$ such that

$$T = \sup_n (U_n) < \infty$$

where $U_n = \sum_{r=0}^{\infty} \left(\max_r |b_{nk}| \, 2^r \right)^{q_r} (2^r)^{s(q_r - 1)} \, E^{-q_r}$

and $\frac{1}{p^r} + \frac{1}{q^r} = 1$, r = 0, 1, 2, ...

(ii) $\lim_{n \to \infty} b_{nk} = \alpha_k$ for all k.

PROOF : *Necessity* — Suppose $A \in$ (Ces(p, s), cs). Now $t_n(Ax)$ exists for each n and $x \in$ Ces(p, s). If we put $\sigma_n(x) = t_n(Ax)$, then $\{\sigma_n\}_n$ is a sequence of continuous real functions on Ces(p, s) and further $\sup_n |t_n(Ax)| < \infty$ on Ces(p, s). Now arguing with uniform boundedness principle (see Khan and Khan[1], Th.3) we have condition (i).

The condition (ii) is obtained by taking $x = e_k \in$ (Ces(p, s)) where e_k = (0, 0, ..., 1, 0, 0, ...), where 1 appears at k'th place.

Sufficiency — Suppose condition (i) and (ii) hold. Then the conditions imply that

$$\sum_{r=0}^{\infty} \left\{ 2^r \max_r |\alpha_k| \right\}^{q_r} (2^r)^{s(q_r - 1)} E^{-q_r} = \lim_{n \to \infty} (U_n) \le \sup_n (U_n) < \infty. \quad \ldots (2)$$

Thus the series $\sum_{k=1}^{\infty} b_{nk} x_k$ and $\sum_{k=1}^{\infty} \alpha_k x_k$ converges for each n and $x \in$ Ces(p, s) from (2).

Put $t_{nk} = b_{nk} - \alpha_k$.

Then $$\sum_{k=1}^{\infty} b_{nk} x_k = \sum_{k=1}^{\infty} \alpha_k x_k + \sum_{k=1}^{\infty} t_{nk} x_k .$$

By (ii) for $k_0 \in Z^+$, we have

$$\lim_{n \to \infty} \sum_{k \le 2}^{k_0} t_{nk} x_k = 0.$$

Moreover, for each $x \in$ Ces(p, s), there exists $k_0 \in Z^+$ such that

$$g_{k_0}(x) = \sum_{r=k_0}^{\infty} (2^r)^{-s} \left(\frac{1}{2^r} \sum_r |x_k| \right)^{p_r} < 1.$$

If $g_{k_0}(x) \neq 0$. Then we have

$$\sum_{k=2}^{k_0} |t_{nk}| \, |x_k| / (g_{k_0}(x))^{1/M} = \sum_{r=k_0}^{\infty} \left(\sum_r |t_{nk}| \, |x_k| \right) \Big/ (g_{k_0}(x))^{1/M}$$

$$\le E(V_n + 1) \qquad \text{(from (1))}$$

where $V_n = \sum_{r=k_0}^{\infty} (2^r B_r(n))^{q_r} E^{-q_r} (2^r)^{s(q_r - 1)}$

and $B_r(n) = \max_r |t_{nk}|$.

Thus

$$\sum_{k=2}^{k_0} |t_{nk}|\ |x_k| \leq E(V_n + 1)\,(g_{k_0}(x))^{1/M}.$$

Since

$$V_n \leq \sup_n\ (V_n) \leq 2T \qquad \text{(from (2))}.$$

Therefore $$\lim_{n \to \infty} \sum_{k=2}^{k_0} |t_{nk}|\ |x_k| = 0.$$

Hence $$\lim_{n \to \infty} \sum_{k=1}^{\infty} b_{nk} x_k = \sum_{k=1}^{\infty} \alpha_k x_k$$

and this complets the proof.

Corollary — Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (\text{Ces}(p, s), (c_0)s)$ iff

(i) the condition (i) of above theorem holds.

(ii) $\lim_{n \to \infty} b_{nk} = 0$.

Theorem 2 — Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (\text{Ces}(p, s), bs)$ iff there exists an integer $E > 1$ such that $T < \infty$, where T is defined as in Theorem 1.

PROOF : Necessity follows by using similar argument as in Theorem 1. For sufficiency, suppose that the condition holds and that $x \in \text{Ces}(p, s)$, then by inequality (1) we have

$$|t_n(Ax)| = \left| \sum_{k=1}^{\infty} b_{nk} x_k \right|$$

$$\leq \sum_{k=1}^{\infty} |b_{nk} x_k| = \sum_{r=0}^{\infty} \sum_r |b_{nk} x_k|$$

$$\leq \sum_{r=0}^{\infty} \left[2^r \max_r |b_{nk}|\ (2^r)^{s/p_r} \cdot \frac{1}{2^r} (2^r)^{-s/p_r} \sum_r |x_k| \right]$$

$$\leq \sum_{r=0}^{\infty} E \left[\left\{ 2^r \max_r |b_{nk}| \right\}^{q_r} (2^r)^{s(q_r - 1)} E^{-q_r} \right.$$

$$+ \left\{ (2^r)^{-s} \left(\frac{1}{2^r} \sum_r |x_k| \right)^{p_r} \right\} \Big]$$

$$= E \left[\sum_{r=0}^{\infty} \left(2^r \max_r |b_{nk}| \right)^{q_r} (2^r)^{s(q_r - 1)} E^{-q_r} \right.$$

$$\left. + \sum_{r=0}^{\infty} (2^r)^{-s} \left(\frac{1}{2^r} \sum_r |x_k| \right)^{p_r} \Big) \right] < \infty.$$

Therefore $A \in (\mathrm{Ces}(p, s), bs)$ and this completes the proof. Finally we are thankful to the referee for his valuable remarks.

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