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WEAK PRIMARY DECOMPOSITION THEOREM FOR RIGHT NOETHERIAN SEMIRINGS

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We extend the primary decomposition theorem for commutative Noetherian rings to non-commutative right Noetherian, right k-semirings. We also prove that if M is a monic prime ideal in the polynomial semiring $R[x]$ then each coefficient ideal M_i is also prime.

1. INTRODUCTION

A set R together with two associative binary operations called addition and multiplication is called a semiring if (i) addition is commutative operation (ii) there exists $0 \in R$ such that $x + 0 = x = 0 + x$, $x \cdot 0 = 0 = 0x$ for each x in R and (iii) multiplication distributes over addition both from left and from right. A semiring R is called semiring with identity element if there exists $1 \in R$ such that $x \cdot 1 = x = 1 \cdot x$ for each x in R.

Throughout R will represent a semiring with an identity element and Z^+ will denote the semiring of non-negative integers under usual addition and multiplication. The concept of ideal (or two sided ideal), right ideal, left ideal, right Noetherian semiring, irreducible right ideal can be defined on the semilar lines as in rings. An ideal (right ideal) I of a semiring R is called k-ideal (k-right ideal) if $a, a + b \in I, b \in R$ implies $b \in I$. An ideal I of a semiring R is called prime if $aRb \subseteq I$ then $a \in I$ or $b \in I$. A right ideal I of a semiring (or ring) R is called primary if $ab \in I$ then $a^n \in I$ or $b \in I$, for some $n \geq 1$. We say that a right ideal I of a semiring (or ring) R is weak primary if $ab \in I$, $aI \subseteq I$ then $a^n \in I$ or $b \in I$, for some $n \geq 1$. Clearly every primary right ideal is weak primary but converse is not true (see Example 2.9, 2.10). In the commutative case both the notions coincide. It is clear

that a two sided ideal in a semiring (or ring) (not necessarily commutative) is primary if and only if it is weak primary. The following theorem can be proved on the similar lines as for rings.

Theorem 1.1 — The following statements for a semiring R are equivalent :

(i) R is right Noetherian.

(ii) R satisfies a.c.c. on right ideals of R.

(iii) Any non-empty collection of right ideals of R has a maximal element.

We denote $R_{n\times n} = \{[a_{ij}] : a_{ij} \in R\}$, the set of $n \times n$ matrices over a semiring R. Under addition and multiplication of matrices $R_{n\times n}$ forms a non-commutative semiring with identity element. A is an ideal in $R_{n\times n}$ if and only if $A = I_{n\times n}$, the set of $n \times n$ matrices over an ideal I in R (Dale[5], 165).

If R is a semiring and x is indeterminate which commutes with every element of R then $R[x]$ will denote the semiring of polynomials over R in the indeterminate x. An ideal M in $R[x]$ is called monic if $\sum_{\text{finite}} a_i x^i \in M$ implies $a_i x^i \in M$, for each i.

2. Weak Primary Decomposition Theorem for Right Noetherian Semirings

Definition 2.1 — A semiring R will be called a k-semiring (right k-semiring) if every ideal (right ideal) of R is a k-ideal (k-right ideal).

Example 2.2 — Let $R = (Z^+ \bigcup \{\infty\}, \oplus, \mathrm{O})$, where $a \oplus b = \max\{a, b\}$, $a\ \mathrm{O}\ b = \min\{a, b\}$, for all a, b in R. Then R forms a commutative semiring with identity ∞. Clearly I is an ideal in R if and only if $I = R$ or $I = Z^+$ or $I = \{0, 1, 2, ..., t\} = M_t$ (say) for some t in Z^+. Also each ideal of R is a k-ideal. Hence R is a k-semiring. Also R is not Noetherian since $M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots$ is an ascending chain of ideals in R which is not stationary.

Example 2.3 — Every ring is a right k-semiring.

Example 2.4 — Z^+ is not a (right) k-semiring. Also Z^+ is a Noetherian semiring (Theorem 22 of Dale and Allen[2]).

The following lemma can be proved easily.

Lemma 2.5 — Let R be a semiring. Then

(i) I is a k-ideal in R if and only if $I_{n\times n}$ is a k-ideal in $R_{n\times n}$.

(ii) If I, J are ideals in R then $(I \bigcap J)_{n\times n} = I_{n\times n} \bigcap J_{n\times n}$.

(iii) I is an irreducible ideal in R if and only if $I_{n\times n}$ is an irreducible ideal in $R_{n\times n}$.

(iv) I is a prime in R if and only if $I_{n\times n}$ is a prime ideal in $R_{n\times n}$.

Lemma 2.6 — Let I be a right ideal in R. If $I_{n\times n}$ is weak primary then I is also weak primary.

PROOF : Let $I_{n\times n}$ be weak primary. Let $ab \in I$ and $aI \subseteq I$. Let

$$A = \begin{bmatrix} a & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

Then $AB \in I_{n\times n}$ and $AI_{n\times n} \subseteq I_{n\times m}$, since $I_{n\times n}$ is weak primary.

$A^m \in I_{n\times n}$ or $B \in I_{n\times n}$ for some $m \geq 1$. Now $a^m \in I$ or $b \in I$.

The converse of the above lemma is not true.

Example 2.7 — Let $I = \langle 2, 3 \rangle$ in Z^+. Then clearly I is a (weak) primary ideal. But $I_{2\times 2}$ is not (weak) primary. For, let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then $AB \in I_{2\times 2}$ but $A^m \notin I_{2\times 2}$ and $B \notin I_{2\times 2}$ for all $m \geq 1$.

Let I be a right ideal of a semiring R and $a \in R$. We denote $(I : a) = \{r \in R : ar \in I\}$.

Theorem 2.8 — Let R be a right Noetherian semiring. Then every irreducible, k-right ideal of R is weak primary.

PROOF : Let I be irreducible, k-right ideal of R. Let $ab \in I$ and $aI \subseteq I$. Clearly $(I : a) \subseteq (I : a^2) \subseteq (I : a^3) \subseteq \ldots$ is an ascending chain of right ideals in R. Thus there exists $n \geq 1$ such that $(I : a^n) = (I : a^k)$ for all $k \geq n$. Let $A = I + \langle a^n \rangle_r$, $B = I + \langle b \rangle_r$. We claim that $A \cap B = I$. Clearly $I \subseteq A \cap B$. Let $x \in A \cap B$. Then $x = a_1 + a^n r_1$ and $x = a_2 + br_2$ for some $a_1, a_2 \in I$ and $r_1, r_2 \in R$. It can be shown easily that if $y \in I$ then $a^t y \in I$ for all $t \geq 1$.

Now $a^{n+1}x = a^{n+1}a_2 + a^{n+1}br_2 = a^{n+1}a_2 + a^n abr_2 \in I$. Thus $a^{n+1}a_1 + a^{n+1+n}r_1 = a^{n+1}(a_1 + a^n r_1) = a^{n+1}x \in I$. Also $a^{n+1}a_1 \in I$. Hence $a^{n+1+n}r_1 \in I$ since I is a k-right ideal. Therefore $r_1 \in (I : a^{n+n+1}) = (I : a^n)$. Therefore $a^n r_1 \in I$. Thus $x = a_1 + a^n r_1 \in I$. Now $A \cap B = I$. Thus $A = I$ or $B = I$ since I is irreducible right ideal. Hence $a^n \in I$ or $b \in I$.

Example 2.9 — Let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in Z_2, \text{ the ring of integers modulo } 2 \right\}$. Then clearly R is a right Noetherian ring. Let $I = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in Z_2 \right\}$. Then I is a right ideal of R which is weak primary but not primary. For let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in R$. Then $AB \in I$ but $A^n \notin I$ and $B \notin I$ for all $n \geq 1$. It can be shown that if J is a right ideal of R such that $I \subseteq J$ then J contains the identity element and hence $J = R$. Now I is irreducible ideal. Also I is a k-right

ideal. Hence by Theorem 2.8, I is weak primary.

Example 2.10 — Let $R = (\{0, 1\}, \max, \min)$ and $I = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in R \right\}$. Then $R_{2\times 2}$ is a right Noetherian semiring and I is a right ideal of $R_{2\times 2}$ which is weak primary but not primary. For let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $AB \in I$ but $A^n \notin I$ and $B \notin I$ for all $n \geq 1$. It can be shown that if J is a right ideal of $R_{2\times 2}$ such that $I \subsetneq J$ then $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in J$. Hence I is irreducible ideal. Clearly I is a k-right ideal. Hence by Theorem 2.8, I is weak primary.

Corollary 2.11 — Let R be a right Noetherian ring. Then every irreducible right ideal of R is weak primary.

Corollary 2.12 — Let R be a commutative Noetherian Semiring.

Then every irreducible k-ideal of R is primary.

The following lemma can be proved easily.

Lemma 2.13 — Let R be a right Noetherian semiring. Then every right ideal of R can be represented as a finite intersection of irreducible right ideals.

Definition 2.14 — If a right ideal I of a semiring (or ring) R is written as a finite intersection of weak primary right ideals of R then this representation of I is called a weak primary decomposition of I.

Theorem 2.15 (Weak primary decomposition) — Let R be a right Noetherian and right k-semiring. Then every right ideal of R has a weak primary decomposition.

PROOF : It follows from Lemma 2.13 and Theorem 2.8.

Corollary 2.16 — Let R be a right Noetherian ring. Then every right ideal of R has a weak primary decomposition.

Corollary 2.17 — Let R be a commutative Noetherian k-semiring. Then every ideal of R has a primary decomposition.

Proposition 2.18 — A semiring R satisfies a.c.c. on two sided ideals of R if and only if $R_{n\times n}$ satisfies a.c.c. on two sided ideals of $R_{n\times n}$.

PROOF : Suppose that R satisfies a.c.c. on ideals of R.

Let $J_1 \subseteq J_2 \subseteq J_3 \subseteq \ldots$ be any ascending chain of ideals of $R_{n\times n}$. Then $J_t = (I_t)_{n\times n}$, for some ideal I_t in R, for all $t \geq 1$. Clearly $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$. Since R satisfies a.c.c. on ideals of R, there exists $m \geq 1$ such that $I_m = I_t$, for all $t \geq m$. Now $J_m = J_t$ for all $t \geq m$. On the other hand let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ be an ascending chain of ideals in R.

Then $(I_1)_{n\times n} \subseteq (I_2)_{n\times n} \subseteq \ldots$ is an ascending chain of ideals of $R_{n\times n}$. Hence $(I_t)_{n\times n} = (I_m)_{n\times n}$, for some $m \geq 1$, for all $t \geq m$. Hence $I_t = I_m$ for all $t \geq m$.

It is not known that if R is right Noetherian semiring then $R_{n\times n}$ is right Noetherian.

Proposition 2.19 — Let I be an ideal of Z^+. Then I is a k-ideal if and only if I is a principal ideal.

PROOF : Suppose I is a k-ideal. If $I = <0>$ then we are through. Suppose $I \neq \langle 0\rangle$. Choose m, the smallest positive integer in I. We claim that $I = \langle m\rangle$. Let $x \in I$. By division algorithm there exist unique $q, r \in Z^+$ such that $x = qm + r$ where $0 \leq r < m$. Now $qm + r$ and $qm \in I$ and I is a k-ideal hence $r \in I$. Which implies that $r = 0$. Thus $x = q^m \in \langle m \rangle$. Hence $I = \langle m \rangle$. Conversely let $I = \langle m \rangle$ be a principal ideal. Let $x, x + y \in I$. Then $x = qm$ and $x + y = rm$ for some $q, r \in Z^+$. Hence $qm + y = rm$. Now y is a multiple of m. Thus $y \in I$.

Example 2.20 — Let $R = Z^+$. Then R satisfies a.c.c. on two sided ideals of R. Thus $R_{n\times n}$ satisfies a.c.c. on two sided ideals of $R_{n\times n}$.

(i) Let $I = <20, 30>$. Then $I = I_1 \bigcap I_2$, where $I_1 = <10>$, $I_2 = \{3n : 0 \leq n \leq 6\} \bigcup \{m : m \geq 20\}$ are ideals in R. Thus I is reducible. By Lemma 2.5, $I_{n\times n}$ is a reducible ideal in $R_{n\times n}$. By Proposition 2.19, I is not a k-ideal. Hence by Lemma 2.5, $I_{n\times n}$ is not a k-ideal. Also $I_{n\times n}$ is not a (weak) primary ideal. For, let

$$A = \begin{bmatrix} 4 & 0 & \ldots & 0 \\ 0 & 0 & .... & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 & \ldots & 0 \\ 0 & 0 & .... & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

Then $AB \in I_{n\times n}$ but $A^m \notin I_{n\times n}$ and $B \notin I_{n\times n}$ for all $m \geq 1$.

(ii) Let $I = <10>$. Then I is a k-ideal in R and hence $I_{n\times n}$ is a k-ideal in $R_{n\times n}$. Let $I_1 = <2>$, $I_2 = <5>$. Then $I = I_1 \bigcap I_2$ and $I \not\subseteq I_1$, $I \not\subseteq I_2$. Thus I is a reducible ideal in R and hence $I_{n\times n}$ is a reducible ideal in $R_{n\times n}$. Also $I_{n\times n}$ is not (weak) primary. For let

$$A = \begin{bmatrix} 2 & 0 & \ldots & 0 \\ 0 & 0 & .... & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 & \ldots & 0 \\ 0 & 0 & .... & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

Then $AB \in I_{n\times n}$ but $A^m \notin I_{n\times n}$ and $B \notin I_{n\times n}$ for all $m \geq 1$.

(iii) Let $I = <2, 3>$. Then I is an irreducible ideal and not a k-ideal in R. Hence $I_{n\times n}$ is irreducible ideal and not a k-ideal in $R_{n\times n}$. Also $I_{n\times n}$ is not weak primary.

For let

$$A = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

Then $AB \in I_{n\times n}$ but $A^m \notin I_{n\times n}$ and $B \notin I_{n\times n}$ for all $m \geq 1$.

3. Polynomial Semirings

Dale has proved that if A is an ideal in $R[x]$ and $A_i = \{a \in R$: there is an $f \in A$ such that ax^i is a term of $f\}$ then $\{A_n\}$ is an ascending chain of ideals in R (Theorem 2.2 of Dale). The ideals $\{A_n\}$ are called coefficient ideals. Dale has also proved that if R is a commutative semiring with identity 1 then a monic ideal M in $R[x]$ is a k-ideal if and only if M_i is a k-ideal in R for each i. (Theorem 3.4 of Dale[3]). In this Section we prove the following theorem.

Theorem 3.1 — Let M be a monic, prime ideal in a polynomial semiring $R[x]$. Then each coefficient ideal M_i is a prime ideal of R.

Proof : Let $aRb \subseteq M_i$. We claim that $aR[x]bx^i \subseteq M$. Let $h(x) = \sum_j a_j x^j \in R[x]$. Now $aa_j b \in M_i \subseteq M_{i+j}$ for all j, since $\{M_n\}$ is an ascending chain of ideals in R (Theorem 2.2 of Dale[5]). Then there exists $f_j(x) \in M$ such that $aa_j bx^{i+j}$ is a term of $f_j(x)$, for all j. Therefore $aa_j bx^{i+j} \in M$ since M is a monic ideal. Hence $ah(x)bx^i = a \sum_j a_j x^j bx^i = \sum_j aa_j bx^{i+j} \in M$. Now $aR[x]bx^i \subseteq M$. Therefore $a \in M$ or $bx^i \in M$ since M is a prime ideal. Therefore $a \in M_0 \subseteq M_i$ or $b \in M_i$.

The converse of the above theorem is not true.

Example 3.2 — Let $R = (Z^+ \bigcup \{\infty\}, \oplus, \odot)$, where $a \oplus b = \max\{a, b\}$ and $a \odot b = \min\{a, b\}$, for all a, b in R. Then R is a commutative semiring with identity ∞. Let $S = R_{2\times 2}$. Define $M = \{f(x)x^2 \in S[x] : f(x) = \Sigma A_i x^i \in S[x]$, where $A_i = \begin{bmatrix} a_{i1} & a_{i2} \\ a_{i3} & a_{i4} \end{bmatrix}$, $a_{ij} \leq i+1$, $j = 1, 2, 3, 4$ for all $i\}$. Then we will show that M is a monic ideal and the coefficient ideals M_i's are prime ideals but M is not a prime ideal.

(i) *M is an ideal* : Let $f(x) = \Sigma A_i x^i x^2$, $g(x) = \Sigma B_i x^i x^2 \in M$, where

$A_i = \begin{bmatrix} a_{i1} & a_{i2} \\ a_{i3} & a_{i4} \end{bmatrix}$, $B_i = \begin{bmatrix} b_{i1} & b_{i2} \\ b_{i3} & b_{i4} \end{bmatrix}$. Therefore $A_i + B_i = \begin{bmatrix} a_{i1} \oplus b_{i1} & a_{i2} \oplus b_{i2} \\ a_{i3} \oplus b_{i3} & a_{i4} \oplus b_{i4} \end{bmatrix}$ and each entry of $A_i + B_i \le i + 1$ since each entry of A_i and $B_i \le i + 1$. Hence $f(x) + g(x) = \Sigma (A_i + B_i) x^i x^2 \in M$. Let $h(x) \Sigma D_j x^j \in S[x]$ and $f(x) = \Sigma A_i x^i x^2 \in M$, where

$$A_i = \begin{bmatrix} a_{i1} & a_{i2} \\ a_{i3} & a_{i4} \end{bmatrix}, \quad D_j = \begin{bmatrix} d_{j1} & d_{j2} \\ d_{j3} & d_{j4} \end{bmatrix}.$$

Therefore $f(x)\ h(x) = \Sigma A_i x^i x^2 \Sigma D_j x^j = \Sigma C_k x^k x^2$, where

$$C_k = \sum_{k=i+j} A_i D_j = \begin{bmatrix} c_{k1} & c_{k2} \\ c_{k3} & c_{k4} \end{bmatrix} \text{ say.}$$

We claim that each entry of $C_k \le k + 1$.

$$\text{Consider } A_i D_j = \begin{bmatrix} a_{i1} & a_{i2} \\ a_{i3} & a_{i4} \end{bmatrix} \begin{bmatrix} d_{j1} & d_{j2} \\ d_{j3} & d_{j4} \end{bmatrix}$$

$$= \begin{bmatrix} a_{i1} \odot d_{j1} \oplus a_{i2} \odot d_{j3} & a_{i1} \odot d_{j2} \oplus a_{i2} \odot d_{j4} \\ a_{i3} \odot d_{j1} \oplus a_{i4} \odot d_{j3} & a_{i3} \odot d_{j2} \oplus a_{i4} \odot d_{j4} \end{bmatrix}.$$

Each entry of $A_i D_j \le i + 1$ since each entry of $A_i \le i + 1$. Hence each entry of $A_i D_j \le i + j + 1$. Thus each entry of $C_k = \sum_{k=i+j} A_i D_j \le i + j + 1 = k + 1$. Now $f(x)\ h(x) \in M$. Similarly $h(x) f(x) \in M$.

(ii) *M is a monic ideal* : Let $f(x) = \Sigma A_i x^i x^2 \in M$. Then each term $A_i x^{i+2} = A_i x^i x^2 \in M$, since each entry of $A_i \le i + 1$.

(iii) *M is not a prime ideal* : Let $f(x) = \Sigma A_i x^i \in S[x]$, where $A_i = \begin{bmatrix} a_{i1} & a_{i2} \\ a_{i3} & a_{i4} \end{bmatrix}$.

$$\text{Consider } xf(x) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = f(x) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x^2 = \Sigma A_i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x^i X^2 = \Sigma \begin{bmatrix} a_{i1} & a_{i2} \\ a_{i3} & a_{i4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x^i x^2$$

$$= \Sigma \begin{bmatrix} a_{i1} \odot 1 & a_{i2} \odot 1 \\ a_{i3} \odot 1 & a_{i4} \odot 1 \end{bmatrix} x^i x^2 \in M,$$

since $a_{ij} \odot 1 = \min \{a_{ij}, 1\} \le 1 \le i + 1$, $j = 1, 2, 3, 4$, for all i. Thus $xS[x] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \in M$. But $x \notin M$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \notin M$.

(iv) *Coefficient ideals M_i are prime ideals* : Denote $I_0 = \{0\} = I_1$, $I_q = \{0, 1, 2, 3, \ldots, q - 1\}$, for all $q \ge 2$. Then clearly $I_0, I_1, I_2, \ldots$ are prime ideals in R. By Lemma 2.5 $(I_t)_{2\times 2}$ are prime ideals in S. Let us find the coefficient ideals M_i. Clearly

$M_0 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} = M_1$

i.e. $M_0 = (I_0)_{2\times 2}, M_1 = (I_1)_{2\times 2}$. For $t \geq 2$,

$M_t = \{ A \in S$: there is a $g(x) = f(x)\, x^2 \in M\, (f(x) \in S[x])$ such that Ax^t is a term of $g(x)\}$

$= \{A \in S$: Ax^t is a term of $g(x) = \Sigma B_i x^i x^2 \in M$, where $f(x) = \Sigma B_i x^i\}$

$= \{A \in S$: Ax^t is a term of $\Sigma\ B_i x^{i+2}$, where each entry of $B_i \leq i + 1\}$

$= \{A \in S$: $A = B_{t-2}$, each entry of $B_{t-2} \leq t-2+1 = t - 1\}$

$= \{A \in S$: each entry of $A \leq t-1\}$

$= (I_t)_{2\times 2}$. Now M_t is a prime ideal in S. Thus each coefficient ideals M_i is a prime ideal in S.

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