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SQUEEZING FLOW BETWEEN VARYING-PERMEABILITY POROUS RECTANGULAR PLATES

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The problem of the squeeze film between rectangular plates, when one of the plates has a varying permeability porous facing, is set up as two systems of equations (to be solved simultaneously) in pressure fields of the porous and film regions. The existence of a unique pair of solutions is thus proved for the weak formulation of the problem. In addition, a priori estimate is obtained for the weak pair of solutions. Finally, the solution of a typical physical problem is compared with its case of average permeability.

NOMENCLATURE

u, v, w	velocity components in the porous matrix
u^*, v^*, w	velocity components in the film region
p_1	pressure in the porous matrix
p^*	pressure in the film region
p	$p_1 + p^*$
h	film thickness
h	bearing surfaces' relative normal velocity
n	absolute viscosity of the lubricant
Φ	varying permeability
Φ_h	$\Phi\ (x, y, h)$
H	porous facing thickness
a	length of bearing
b	width of bearing
r	a/b
$\overline{H}$	H/a

U_h	$\Phi_h H/h^3$
W	varying-permeability-problem's load capacity
W_a	average-permeability-problem's load capacity
s, t	natural numbers
C	$s^2 + r^2 t^2$
$\mathcal{E}$	$(C\pi\bar{H})^2$.

1. Introduction

The analysis concerned with porous metal bearings have up to date been confined to constant-permeability porous plates. Such analysis had been carried out by authors like Morgan and Cameron[1], Prakash and Tiwari[2], Wu[3] and Bhat and Patel[4].

As described by Morgan and Cameron[1] the manufacture of metal bearing is done by compacting metal powders in essentially cylindrical dies, with pressure applied along the axis of the components, thereby resulting in lower permeability at the ends and higher permeability at the centre. This strongly suggests the absence of constant permeability in practice.

In practice, average permeability is utilized. The present work is motivated out of the desire to know to what extent the use of average permeability can affect the working of a porous bearing.

The physics of the problem of the squeeze film between two rectangular plates (see Fig. 1), when one plate has an arbitrarily varying-permeability porous facing, is given in section 2. The governing equations are given in section 3. The existence of a solution is established in section 4, and in addition a priori estimate obtained. Finally, in section 5, a typical physical problem is solved and compared with its case of average permeability. Significant deviations were observed for certain ranges of some parameters. The estimates of a permeability variation parameter for which

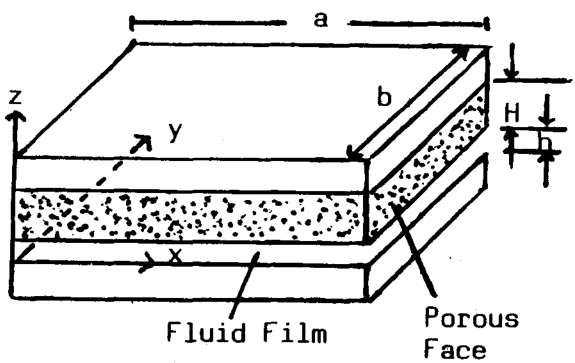

FIG. 1. Geometry and Coordinates of the problem

the use of average permeability is correct to a maximum of 1% deviation are obtained for various values of a permeability parameter and the ratio of the porous matrix thickness to the bearing length are obtained for the typical problem.

2. The Physics of the Problem

The rectilinear squeezing flow of a fluid film separating two flat rectangular plates (one porous faced) approaching each other is considered.

A fluid film separating two surfaces cannot be squeezed out instantaneously. It takes time for the two surfaces to meet. Within such time, the fluid actually supports the load on it. Pressure is thus said to have been developed hydrostatically. A physical example is seen in frictional devices such as clutch plates in automatic transmissions[5].

With the introduction of two plates with one porous faced, the flow pattern changes. Only a part of the fluid is squeezed out, while the remaining part flows into the porous region[3].

The equations governing the fluid motion are best expressed by using D'Arcy's law[6] and the continuity equation in the porous matrix; and Navier stokes and the continuity equations in the film region.

3. The Governing Equations

The governing equations are derived on the basis of the following simplifying assumptions :

1. The flow in the film region is slow, i.e. the flow with small Reynolds number is considered, such that the flow is laminar and the inertial effects negligible; the pressure is independent of the z-coordinate; and the z-derivative of the velocity components dominates.

2. The fluid has constant properties and is incompressible.

3. The porous facing is of uniform thickness.

4. In the porous medium, Darcy's law

$$V = (0, 0, h) - \frac{\Phi(x, y, z)}{n} \nabla p_1 \qquad \text{... (3.1)}$$

is satisfied, where $V = (u, v, w)$ is the volume rate of the flow per unit area normal to the pressure gradient; h the relative normal velocity of the bearing surfaces; p_1 the pressure in the porous region; n is the fluid viscosity; and Φ the varying permeability.

From Appendix I, the governing equations are :

$$\nabla\ (\Phi\ Vp_1) = 0 \qquad \text{... (3.2)}$$

and

$$\nabla p^* = \frac{12n}{h^3}\left[h - \frac{\Phi_h}{n}\left[\frac{\delta p_1}{\delta z}\right]_{z=h} \right] \qquad \text{... (3.3)}$$

where p^* is the film pressure field and h is the film thickness.

Setting $D^* = (0, a) \times (0, b)$ and $D = D^* \times (h, h+H)$,
the boundary conditions to be satisfied by p_1 and p^* are :

$$p_1 = 0 \text{ on } \Gamma_D := D^* \times \{z = h\}$$

$$p^* = 0 \text{ on } D^*$$

$$\frac{\delta p_1}{\delta z}\bigg|_{z=h+H} = 0$$

and the matching condition $p_1 = p^*$ on $z = h$.

The problem together with the associated boundary conditions is solved analytically in series form[3], for the constant permeability case.

If $\qquad H_D^{1,2} := \{q \in H^{1,2}(D) : q = 0 \text{ on } \Gamma_D\};$

$$H_C^{1,2} := H^{1,2}(D^*) \times H^{1,2}(D)$$

and $\qquad H_L^{1,2} := H_0^{1,2}(D^*) \times H_0^{1,2}(D)$, then

the weak formulation of the problem as obtained from Appendix II is :

Find a pair $(p^*, p) \in H_L^{1,2}$ such that

$$a < (q^*, q), (p^*, p) > :$$

$$= \frac{h^3}{12}\int_{D^*} \nabla q^* \cdot \nabla p^* + \int_D \Phi \nabla (q^* + q) \cdot \nabla (p^* + p)$$

$$= -\int_{D^*} n q^* h$$

for all $(S^*, S) \in H_L^{1,2}$, where $p_1 = p + p^*$.

4. Existence of a Solution

Proposition 4.1 — The vector space V : with the scalar product $\langle \ldots \rangle_v := a\langle \ldots \rangle$ is a Hilbert space, and there exists constants $0 < c \le C < \infty$ such that

$$c \,||\, (p^*, p) \,||^2_{H_c^{1,2}} \le a\langle (p^*, p), (p^*, p)\rangle \le C \,||\, (p^*, p) \,||^2_{H_c^{1,2}}$$

where $\quad || (p^*,p) ||_{H_c^{1,2}} := || p^* ||_{H^{1,2}(D^*)} + || p ||_{H^{1,2}(D)}.$

PROOF : a is bilinear. Hence

$$a \langle (p^*,p),(p^*,p) \rangle \le \frac{h^3}{12}$$

$$|| \nabla p^* ||^2_{L^2(D^*)} + || \Phi ||_{L^\infty(D)} || \nabla (p^*+p) ||^2_{L^2(D)}$$

$$\le \frac{h^3}{12} || \nabla p^* ||^2_{L^2(D^*)} + || \Phi ||_{L^\infty(D)}$$

$$\left[|| \nabla p^* ||^2_{L^2(D)} + || \nabla p ||^2_{L^2(D)} + 2 || \nabla p^* ||_{L^2(D^*)} || \nabla p ||_{L^2(D)} \right]$$

$$\le \frac{h^3}{12} || \nabla p^* ||^2_{L^2(D^*)} + 2 || \Phi ||_{L^\infty(D)} || \nabla p^* ||^2_{L^2(D^*)}$$

$$+ 2 || \Phi ||_{L^\alpha(D)} || \nabla p ||^2_{L^2(D)}$$

$$\le \left(\frac{h^3}{12} + 2 || \Phi ||_{L^\alpha(D)} \right) \left[|| \nabla p^* ||^2_{L^2(D^*)} + || \nabla p ||^2_{L^2(D)} \right]$$

$$\le C || (p^*,p) ||^2_{H_c^{1,2}}.$$

Further, there exists constants $a_0 > 0$ and $b_0 > 0$ such that

$$a \langle (p^*,p),(p^*,p) \rangle \ge a_0 \int_{D^*} | \nabla p^* | 2 + b_0 \int_D | \nabla (p^*+p) |^2$$

$$\ge m_0 \left[\int_{D^*} | \nabla p^* |^2 + \int_D | \nabla (p^*+p) |^2 \right]$$

where $m_0 := \min (a_0, b_0)$

$$\ge m_0 \left[\int_{D^*} | \nabla p^* |^2 - \int_D || \nabla p^* | - k | \nabla p ||^2 + k \int_D | \nabla p^* | - | \nabla p ||^2 \right]$$

for some $k \in (0, 1)$

$$= m_0 \Big[2k \int_D | \nabla p^* | \, | \nabla p | - k^2 \int_D | \nabla p |^2 + k \int_D | \nabla p^* |^2 + k \int_D | \nabla p |^2$$

$$- 2k \int_D | \nabla p^* | \, | \nabla p | \Big]$$

(equation continued on p. 670)

$$\geq c_0 \left[\int_{D^*} | \nabla p^* |^2 + \int_D | \nabla p |^2 \right] \text{ where } c_0 = m_0 (k - k^2).$$

By Poincare's inequality, there exists a_1 and b_1 which depend on D^* and D respectively such that

$$\int_{D^*} | p^* |^2 + \int_D | p |^2 \leq a_1 \int_{D^*} | \nabla p^* |^2 + b_1 \int_D^* | \nabla p |^2$$

$$\leq c_1 \left[\int_{D^*} | \nabla p^* |^2 + \int_D | \nabla p |^2 \right].$$

Hence

$$\frac{1}{2} \| (p^*, p) \|_{H_c^{1,2}} \leq \int_{D^*} | p^* |^2 + \int_D | p |^2 + \int_{D^*} | \nabla p^* |^2 + \int_D | \nabla p |^2$$

$$\leq (c_1 + 1) \left[\int_{D^*} | \nabla p^* |^2 + \int_D | \nabla p |^2 \right]$$

$$\leq \frac{(c_1 + 1)}{c_0} a \langle (p^*, p), (p^*, p) \rangle$$

i.e. $\quad c \| (p^*, p) \|^2_{H_c^{1,2}} \leq a \langle (p^*, p), (p^*, p) \rangle$

where $\quad c = \dfrac{c_0}{2(c_1 + 1)}$.

The proved estimates show that $a \langle \dots \rangle^{1/2}$ is an equivalent norm on $H_l^{1,2}$ to the $H_c^{1,2}$ norm. Thus V is complete.

Proposition 4.2 — $a \langle (q^*, q), (p^*, p) \rangle$ is a sesquilinear form from $H_c^{1,2} \times H_c^{1,2} \to \mathbf{R}$ such that

$$| a \langle (q^*, q), (p^*, p) \rangle | \leq C \| (q^*, q) \|_{H_c^{1,2}} \| (p^*, p) \|_{H_c^{1,2}}.$$

PROOF : $| a \langle (q^*, q), (p^*, p) \rangle | \leq \dfrac{h^3}{12} \| Vq^* \|_{L^2(D^*)} \| Vp^* \|_{L^2(D^*)}$

$$+ \| \Phi \|_{L^\infty(D)} \Big[\| \nabla q^* \|_{L^2(D^*)} \| \nabla p^* \|_{L^2(D^*)}$$

$$+ \| \nabla q \|_{L^2(D)} \| \nabla p \|_{L^2(D)} + \| \nabla q^* \|_{L^2(D^*)} \| \nabla p \|_{L^2(D)}$$

$$+ \| \nabla q \|_{L^2(D)} \| \nabla p^* \|_{L^2(D^*)} \Big]$$

(*equation continued on p.* 671)

$$\leq \left(\frac{h^3}{12} + || \Phi ||_{L^\infty(D)}\right)\left[|| \nabla q^* ||_{L^2(D^*)} + || \nabla q ||_{L^2(D)} \right] \times$$

$$\leq c \, || (q^*, q) ||_{H_c^{1,2}} \, || (p^*, p) ||_{H_c^{1,2}} .$$

Proposition 4.3 — $H(q^*, q) := -\int_{D^*} \dot{n} h q^*$ belongs to V', the dual space of V, for all $(q^*, q) \in V$.

PROOF : $| H(q^*, q) | \leq | \dot{n}h | \; || q^* ||_{L^2(D^*)}$

$$\leq | \dot{n}h | \; || q^* ||_{H^{1,2}(D^*)}$$

$$\leq | \dot{n}h | \; || q^*, q ||_{H_c^{1,2}}$$

$$\leq \frac{1}{c} | \dot{n}h | \; || q^*, q ||_v \text{ (by proposition 3.1).}$$

Theorem 4.4 — There exists a unique solution $(p^*, p) \in V$ such that $\langle (q^*, q), (p^*, p) \rangle_v = H(q^*, q)$ for all $(q^*, q) \in V$.

PROOF : The existence of a unique solution follows from the use of Proposition 4.3 and the Riesz Representation Theorem. Further, using Propositions 4.1 and 4.2, we have by Lax Milgram Theorem that exactly one bijective map $T \in L(H_c^{1,2})$ (continuous operators on $H_c^{1,2}$) exists such that

$$a \langle (q^*, q), (p^*, p) \rangle = \langle (q^*, q), (p^*, p) \rangle_{H_c^{1,2}}$$

for all $(q^*, q), (p^*, p) \in H_c^{1,2}$, Hence for the weak solution (p^*, p) we have for all $(q^*, q) \in V$:

$$H(q^*, q) = a \langle (q^*, q), (p^*, p) \rangle = \langle (q^*, q), T(p^*, p) \rangle_{H_c^{1,2}}.$$

Thus by Proposition 3.1, we have a priori estimate

$$c \, || (p^*, p) ||^2_{H_c^{1,2}} \leq || (p^*, p) ||_{H_c^{1,2}} \, || T(p^*, p) ||_{H_c^{1,2}}$$

$$= || H ||_{H_c^{1,2}} \, || (p^*, p) ||_{H_c^{1,2}}$$

i.e. $\quad || (p^*, p) ||_{H_c^{1,2}} \leq m' \, | \dot{n}h |$

where $\quad m' = \frac{1}{c}$.

5. Comparison of a Typical Physical Problem with its Correspondence Case of Average Permeability

5.1. *General Analysis*

The compacting of metal powders as described in (1) suggests a significant permeability variation along one direction. Hence, we assume without loss of generality, for our example, a linearly varying permeability

$$\Phi(z) = [m(z-h)/H + 1]\,\Phi_h \qquad \dots (5.1)$$

where $\Phi_h = \Phi(h)$ and m shall be referred to as permeability variation parameter.

Using (5.1) and the transformation $z' = z - h/H$ in (II.1) one obtains

$$\overset{\circ}{\nabla}{}^2 p + \nabla^2 p^* + \frac{m}{(mz'+1)H^2}\,\frac{\delta p}{\delta z} = 0$$

where

$$\overset{\circ}{\nabla}{}^2 := \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{1}{H^2}\,\frac{\delta^2}{\delta z^2}\,. \qquad \dots (5.2)$$

Applying the sine transform

$$= f(s,t) = \int_0^a \int_0^b f(x,y)\,\sin\frac{s\pi x}{a}\,\sin\frac{t\pi y}{b}\,dxdy$$

on (5.2) and (II.2) one obtains respectively the ordinary linear differential equation

$$\frac{d^2\bar{\bar{p}}}{dz'^2} + \frac{m}{(mz'+1)}\,\frac{d\bar{\bar{p}}}{dz'} + (C\bar{H}\pi)^2\bar{\bar{p}} = (C\bar{H}\pi)^2\bar{\bar{p}}^{\,*} \qquad \dots (5.3)$$

with the boundary conditions : $p(0) = 0$ and $p'(1) = 0$ and the algebraic equation

$$\frac{(C\pi)^2\bar{\bar{p}}^{\,*}}{a^2} = \frac{48nh}{sth^3} - \frac{12\Phi h}{Hh^3}\left[\frac{d\bar{\bar{p}}}{dz'}\right]_{z'=0} \quad , \; s,\ t \text{ odd}$$

$$\frac{(C\pi)^2\bar{\bar{p}}^{\,*}}{a^2} = \frac{12\Phi h}{Hh^3}\left[\frac{d\bar{\bar{p}}}{dz'}\right]_{z'=0} \quad , \; s,\ t \text{ even}$$

where $\quad C = s^2 + r^2 t^2,\ r = \dfrac{a}{b}$ and $\bar{H} \doteq \dfrac{H}{a}$.

Using Galerkin Method and the base functions

$$g_0(z') = 0$$

$$g_i(z') = z'^i (z'-1)^2 (mz'+1), i = 1, 2, ...$$

and approximate solution of (5.3) has the form

$$g(z') = \frac{-E(q_1 B_2 - q_2 B_1) p^* z'}{A_1 B_2 \ \ A_2 B_1} (z'-1)^2 \ (mz'+1) \quad ... (5.4)$$

$$\frac{-E(q_2 A_1 - q_1 A_2) p^* z'^2}{A_1 B_2 \ \ A_2 B_1} (z'-1)^2 \ (mz'+1) \quad ... (5.5)$$

where $E = (C\pi\overline{H})^2$

$$A_1 = \frac{1}{210}(3m^2 + 14m + 28) + \frac{E}{1260}(2m^2 + 9m + 12)$$

$$B_1 = \frac{1}{840}(5m^2 + 20m + 28) + \frac{E}{2520}(2m^2 + 8m + 9)$$

$$A_2 = \frac{1}{210}(2m^2 + 7m + 7) + \frac{E}{2520}(2m^2 + 8m + 9)$$

$$B_2 = \frac{1}{1260}(7m^2 + 24m + 24) + \frac{E}{6930}(3m^2 + 11m + 11)$$

$$q_1 = \frac{1}{60}(2m + 5)$$

$$q_2 = \frac{1}{60}(m + 2).$$

Substituting (5.5) into (5.4) we have after rearrangement :

$$p^* = \left[\begin{array}{l} \dfrac{-48nh}{sth^3 \pi^2 C^2 a^{-2} [1 + 12U_h F(m, C, H)]}, s, t \text{ odd} \\ 0 \end{array} \right.$$

where $U_h = \frac{\Phi_h H}{h^3}$; $F(m, C, H) = \frac{q_1 B_2 - q_2 B_1}{A_1 B_2 - A_2 B_1}$.

Using the inversion theorem, the dimensionless pressure in the film region is

$$\frac{-h^2 p^*}{na^2 h} = \frac{192}{\pi^4} \sum_{s,t}^{\infty} \int_{odd}^{\infty} \frac{\sin \frac{s\pi x}{a} \sin \frac{t\pi y}{b}}{stc^2 [1 + 12U_h F(m, C, H)]}. \quad ... (5.6)$$

Hence we obtain the dimensionless load capacity as

$$\overline{W} = \frac{-h^3 w}{bna^3 h} = \frac{768}{\pi^6} \sum_{s,t}^{\infty} \int_{odd}^{\infty} \frac{1}{s^2 t^2 C [1 + 12U_h F(m, C, \overline{H})]} . \quad \ldots (5.7)$$

Using (4.1), the average permeability is

$$\frac{1}{H} \int_h^{h+H} \Phi(z)\, dz = 1/2\,(m+2)\,\Phi_h.$$

Thus, the dimensionless load capacity for the average permeability case

$$\overline{W}_a = \frac{-h^3 w}{bna^3 h} = \frac{768}{\pi^6} \sum_{s,t}^{\infty} \sum_{odd}^{\infty} \frac{1}{s^2 t^2 C [1 + 6U_h F(O, C, \overline{H})]} .$$

5.2. Results and Discussions

A perusal of Table I reveals that the deviation between $\overline{W}$ and $\overline{W}_a$ increases with increase in the value of the variation permeability m for a given value of U_h. In addition, the deviation tends to zero as m tends to zero for any given value of U_h, because $\overline{W}$ and $\overline{W}_a$ tend to the case with constant permeability as m tends to zero.

Again, from Table I, the deviation is observed to increase with increase in the value of U_h for a given value of the parameter m, while it tends to zero as U_h tends to zero, because W and W_a both tend to the solution for the non-porous case.

In using average permeability in the analysis, the term $\nabla \Phi \cdot \nabla p_1$ is neglected in solving the equation satisfied by the pressure (p_1) in the porous region as $\Phi(z)$ is replaced by a constant average permeability Φ_a. But for the matching condition $p_1(x, y, h) = p^*(x, y)$, the pressure in the porous medium could have been completely independent of the average permeability. In an experimental situation, where $\nabla \Phi$ is not negligible, substantial deviations from the prediction of values of lubricating characteristics by mathematical analysis when average permeability is used is expected.

Within the range of values of parameters considered, a deviation of about 20% is observed to occur between $\overline{W}$ and $\overline{W}_a$ for the values of the parameters $\overline{H}$ = 0.04, r = 1, U_h = 1.0 and m = 1.4, while a negligible deviation of about 4.46 × 10^{-4} occurs for parameters $\overline{H}$ = 0.04, r = 1, U_h = 10^{-4} and m = 0.2. Thus, for substantially slow-varying-permeability or low-permeability porous medium, the use of average permeability presents negligible error, while a significant error results with substantially high-varying-permeability or high-permeability porous medium.

TABLE I

Comparison of $\overline{W}$ and $\overline{W}_a$ for various values of m and U_h, taking $H = 0.04$, $r = 1$

U_h	$m = 0.2$			$m = 0.8$			$m = 1.4$		
	w	w_a	$d\%$	w	w_a	$d\%$	w	w_a	$d\%$
10^{-4}	0.421518	0.421516	4.47×10^{-4}	0.42149	0.42147	5.98×10^{-3}	0.42148	0.42142	1.56×10^{-2}
10^{-3}	0.41988	0.41986	4.45×10^{-3}	0.41961	0.41936	5.95×10^{-2}	0.41951	0.41886	0.155
10^{-2}	0.40416	0.40399	4.28×10^{-2}	0.40169	0.39941	0.567	0.40078	0.39494	1.46
10^{-1}	0.29407	0.29316	0.311	0.28147	0.27066	3.84	0.27709	0.25137	9.28
1.0	0.078973	0.078317	0.831	0.070498	0.064085	9.10	0.067814	0.054230	20.0

TABLE II

Estimates (α), $0 \le m \le \alpha$, for which $\overline{w}_a$ is correct to at most 1% deviation, for some values of U_h and $\overline{H}$, taking $r = 1$

H U_h	0.0001	0.001	0.01	0.1	1.0
0.04	49.099	5.2508	1.1126	0.37307	0.22118
0.08	49.571	5.2871	1.1142	0.37043	0.21745
0.12	50.340	5.3464	1.1169	0.36628	0.21147
0.16	51.387	5.4276	1.1206	0.36093	0.20358
0.20	52.699	5.5298	1.1255	0.35470	0.19425
0.24	54.264	5.6522	1.1318	0.34788	0.18397
0.28	56.072	5.7946	1.1393	0.34074	0.17326
0.32	58.120	5.9569	1.1481	0.33351	0.16255
0.36	60.403	6.1391	1.1583	0.32634	0.15218
0.40	62.918	6.3412	1.1699	0.31942	0.14239

5.3. CONCLUSION

The use of average permeability in bearing analyses is suitable for substantially low-varying-permeability or low-permeability porous media, while it is unsuitable for substantially high-varying-permeability or highly permeable porous media.

Presented in Table II are estimate (α), $0 \le m \le \alpha$, for which the load capacity obtained by the use of average permeability is correct to at most 1% deviation for a linearly-varying-permeability porous bearing, and for various values of U_h and $\overline{H}$, taking $r = 1$. It is noteworthy that these estimates are correct to less than 2.5% for all values of r.

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APPENDIX I

Classical Formulation of Governing Equations

Using (2.1) one obtains,

$$n\nabla \cdot V = -\nabla \cdot (\Phi V p_1) = 0$$

for the continuity of an incompressible flow. Hence, one has

$$= \nabla \cdot (\Phi \nabla p_1) = 0 \qquad \text{... (I.1)}$$

as the equation satisfied by the pressure in the porous region.

Applying appropriate assumptions, the flow in the film region is governed by the equations

$$\frac{\delta^2 u^*}{\delta z^2} = \frac{1}{2n} \frac{\delta p^*}{\delta x} \qquad \text{.... (I.2)}$$

$$\frac{\delta^2 v^*}{\delta z^2} = \frac{1}{2n} \frac{\delta p^*}{\delta y} \qquad \text{.... (I.3)}$$

where u^*, v^* are the velocity components in the x, y directions and p^* is the film pressure field. Solving (I.2) and (I.3) with the no-slip conditions on both surfaces to obtain u^* and v^*, and substituting these into the continuity equation, one obtains, after integrating the ensuring equation across the film thickness, the equation

$$W^* = \frac{h^3}{12n} \nabla^2 p^* \qquad \text{... (I.4)}$$

where w^* is the normal component of the velocity field of the film, $w^*(x, y, h) := w^*$ and $w^*(x, y, 0) = 0$, as the lower plate is non-porous.

For the continuity of velocity components w and w^* at the porous plate-film interface, we have using (2.1) the condition

$$w_h^* = h - \frac{\Phi_h}{n} \left[\frac{\delta p_1}{\delta z}\right]_{z=h} \qquad \text{... (I.5)}$$

Using (I.4) and (I.5), the modified Reynolds equations satisfied by the pressure field in the film region is

$$\nabla^2 p^* = \frac{12n}{h^3} \left[h - \frac{\Phi_h}{n} \left[\frac{\delta p_1}{\delta z}\right]_{z=h} \right]. \qquad \text{... (I.6)}$$

Let $D^* := (0, a) \times (0, b)$ and $D := D^* x(h, h + H)$. Then the classical formulation of the problem is to solve (I.1) and (I.6) in the D and D^* respectively with the conditions $p_1 = 0$ on $D; p^* = 0$ on D^* (i.e. the no side leakage assumptions);

$$\left| \frac{\delta p_1}{\delta z} \right|_{z=h+H} = 0$$

as the porous matrix is bounded at $z = h + H$ by an impermeable wall; and the matching condition $p_1 = p^*$ on $z = h$ for the continuity of pressure at the place film interface.

In case of side leakage, the non-side-leakage assumptions are replaced by $p = p_{atm}$ on D and $p^* = p_{atm}$ on D^*, where p_{atm} is the atmospheric pressure. Transformations $p_t = p - p_{atm}$ and $p_t^* = p^* - p_{atm}$ render this problem analogous to solving the problem with the no-side leakage assumption.

APPENDIX II

Weak Formulation of Governing Equations

Setting $p_1 = p + p^*$ in (I.1) and (I.2) yields

$$\nabla \cdot (\Phi V p) + \nabla \cdot (\Phi V p^*) = 0 \text{ in } D \qquad \text{... (II.1)}$$

$$\nabla^2 p^* = \frac{12n}{h^3} \left[h - \frac{\Phi_h}{n} \left[\frac{\delta p_1}{\delta z}\right]_{z=h} \right] \text{ in } D^* \qquad \text{... (II.2)}$$

with the conditions

$$p = 0 \text{ on } \Gamma_D := \dot{D}^* x \{z = h\}$$

$$\left| \frac{\delta p}{\delta z} \right|_{z=h} = 0$$

$$p^* = 0 \text{ on } \dot{D}^*.$$

Let $S \in C_D^\infty := \{t \in C_0^\infty(D) : t = 0 \text{ on } \Gamma_D\}$.

Then we have for all $S \in C_D^\infty$, using (II.1) and the condition

$$\left| \frac{\delta p}{\delta z} \right|_{z=h+H} = 0$$

the equation $\int_D (\Phi \nabla S \cdot \nabla_p + \Phi \nabla S \cdot \nabla p^*) = 0.$ ... (II.3)

Multiplying (II.2) by $S^* \in C_0^\infty(D)$ we have, using the boundary condition

$$\left| \frac{\delta p}{\delta z} \right|_{z=h+H} = 0,$$

the equation

$$\frac{h^3}{12} \int_{D^*} \nabla S^* \cdot \nabla p^* + \int_D \Phi \nabla S^* \cdot \nabla p + \int_D \Phi \nabla S^* \cdot \nabla p^* + \int_{D^*} n S^* h^{\cdot} = 0. \quad \text{... (II.4)}$$

Addint (II.3) and (II.4) gives

$$\frac{h^3}{12} \int_{D^*} \nabla S^* \cdot \nabla p^* + \int_D \Phi \nabla (S^* + S) \cdot \nabla (p^* + p) + \int_{D^*} n S^* h^{\cdot} = 0 \quad \text{... (II.5)}$$

for all $(S^*, S) C\, C_0^\infty(D^*) \times C_D^\infty$.

Let $H_D^{1,2} = \{q C H^{1,2}(D) : q = 0 \text{ on } \Gamma_D\}$,

$$H_C^{1,2} = H^{1,2}(D^*) \times H^{1,2}(D)$$

and $H_L^{1,2} = H_0^{1,2}(D^*) \times H_D^{1,2}(D)$, then the weak formulation of the problem is to find a pair $(p^*, p) C\, H_L^{1,2}$ such that

$$a \langle (q^*, q), (p^*, p) \rangle := h^3 \int_{D^*} \nabla q^* \cdot \nabla p^* + \int_D \Phi \nabla (q^* + q) \cdot \nabla (p^* + p)$$

$$= - \int_{D^*} n q^* h^{\cdot}$$

for all $(S^*, S) C\, H_L^{1,2}$.