Indian J. pure appl. Math., 34(1) : 3-15, January 2003

FREE VIBRATION OF A RECTANGULAR BEAM WITH TWO LINEAR VARIATIONS IN THICKNESS

A. P. GUPTA* AND NAVEEN SHARMA**

**Department of Mathematics, Indian Institute of Technology, Roorkee* 247 667
***Department of Mathematics, D.A.V. (P.G.) College, Muzaffarnagar* 251 001

(*Received* 4 *July* 2001; *after revision* 8 *March* 2002; *accepted* 21 *June* 2002)

Free transverse vibration of a rectangular beam having two linear variations in thickness along the length is analyzed by classical theory of beams. The solution of the equations of motion is obtained by Frobenius method. Numerical results for natural frequencies and normalized mode shapes are computed for first four normal modes for a clamped-clamped and cantilever beam.

Key Words : Classical Theory of Beams; Two Bilinear Variation in Thickness; Forbenius Method

1. INTRODUCTION

The free vibration analysis of beams with stepped/bilinear/discontinuous variation in thickness has been performed by utilizing many analytical and numerical approaches, in an attempt to simulate the dynamics of a number of structural and mechanical components. Such structural elements are used in rotating machinery for better mechanical balancing, to study the high velocity centrifuge of molecular biology, diagonal braces in civil engineering and the jacket legs in offshore engineering[1].

Greco & Laura[2] have used the Ritz method to examine the free vibration of circular plates of bilinearly varying thickness. Gutierrez *et al.*[3] have presented the fundamental frequency of rectangular plate having a thickness which varies in a bilinear fashion. Greco & Laura[4] have used simple polynomial approximations and the Ritz method to study the forced vibration of a circular plate with thickness varying in a bilinear fashion. Aksu & Al-Kaabi[5] have applied the finite difference energy method to examine the flexural vibration chracteristics of rectangular Mindlin plate with bilinearly varying thickness. Laura *et al.*[6] have investigated the frequency coefficient of isotropic rectangular plate of bilinearly varying thickness. Singh & Saxena[7] have examined the axisymmetric vibration of circular plate with double linear variable thickness. Auciello & Nale[8] have determined the free vibration frequencies of a beam composed of two tapered beam sections with different physical characteristics and a mass at its ends by exact as well as approximate method.

In this paper, free vibration of a rectangular beam having two linear variations in thickness along the length is analyzed using classical theory of beams. The beam is assumed to be made up of two beam elements joined end to end. The beam elements are having in general different linear variation in thickness. Frobenius method is used to solve the equations of motion. The arbitrary constant arising in the solution are solved by end and continuity conditions. Numerical results for natural frequencies and normalized mode shapes for first four normal modes for a beam clamped at both the ends and cantilever beam are plotted in graphs for various value of taper constant. The variation in thickness is taken in such a way that average thickness of the beam remains constant. The frequencies computed as a particular case for a rectangular beam of constant thickness are compared with that of Soni[9].

2. Equations of Motion

An isotropic rectangular beam of length a and breadth b is referred to Cartesian co-ordinates by taking the x, y and z axes along the length, breadth and thickness respectively. The middle plane and the two ends of the beam are taken in the planes $z = 0$, $x = 0$ and $x = a$ respectively. The beam is assumed to be made up of two beam elements joined end to end with their middle plane lying in plane $z = 0$ and having in general different linear variation in thickness. The breadth, Young's modulus, Poisson ratio, density, length and thickness of kth beam element ($k = 1, 2$) are taken as b, E, ν, ρ, a_k and $h_k(x)$ respectively and it lies from $x = x_{k-1}$ to $x = x_k$ where $x_k - x_{k-1} = a_k$, $x_0 = 0$ and $x_2 = a$.

For free vibration analysis, the equations of motion of the beam elements are

$$\left.\begin{aligned} & h_k^3 w_{k,xxxx} + 6h_k^2 h_{k,x} w_{k,xxx} + 3(2h_k h_{k,x}^2 + h_k^2 h_{k,xx}) w_{k,xx} \\ & + \frac{12(1-\nu^2)}{E} \rho h_k w_{k,tt} = 0, \quad x_l \le x \le x_k,\ l = k-1;\ k = 1, 2. \end{aligned}\right\} \quad \ldots (1)$$

where w_k and t are displacement component of kth beam element in the direction of z-axis and time, respectively. The comma followed by the variable suffix denotes differentiation with respect to that variable.

The linearly varying thicknesses are taken as

$$h_k(x) = h_0 [b_k + \beta_k (x - x_l)/a];\ b_1 = 1, b_2 = 1 + \beta_1 x_1/a, \quad \ldots (2)$$

where β_1, β_2 are taper constants and h_0 is the thickness of the beam at $x = 0$.

Making eq. (1) non-dimensional, one gets

$$H_k^3 \overline{W}_{k,XXXX} + 6 H_k^2 H_{k,X} \overline{W}_{k,XXXX} + 3(2H_k H_{k,X}^2 + H_k^2 H_{k,XX}) \overline{W}_{k,XX}$$

$$+ 12(1-\nu^2) H_k \overline{W}_{k,TT} = 0,\ X_l \le X \le X_k,\ l = k-1;\ k = 1, 2, \quad \ldots (3)$$

where $X = x/a,\ X_k = x_k/a,\ H_k = h_k/a,\ \overline{W}_k = w_k/a,$

$T = t\sqrt{E/\rho a^2},\ H_0 = h_0/a,\ X_0 = 0,\ X_2 = 1.$

3. Solution

For harmonic solution, $\overline{W}_k$ is taken as

$$\overline{W}_k(X, T) = W_k(X) e^{i\Omega T}, \quad \ldots (4)$$

where Ω is the circular frequency of vibration.

Substitution of solutions (4) in eqs. (3), after the use of relation (1) yields,

$$\{1+C_k(X-X_l)\}^2 W_{k,XXXX}+6C_k\{1+C_k(X-X_l)\}$$

$$W_{k,XXX}-6C_k^2 W_{k,XX}-\omega_k^2 W_k=0,$$

$$C_k=\beta_k/b_k,\ \omega_k^2=\frac{12(1-\nu^2)\Omega^2}{H_0^2 b_k^2},\ k=1,2. \quad \dots (5)$$

The above equations, governing the free vibration are solved by Frobenius method. The solutions are assumed as

$$W_k=\Sigma\, d_{nk}(X-X_l)^{n+\lambda-1},\ d_{1k}\neq 0, \quad \dots (6)$$

where the summation over n is taken from 1 to ∞.

Substitution for W_k from (6) in (5), the roots of the indicial equation are found to be λ = 0, 1, 2, 3, 4. The recurrence relation is obtained to be

$$d_{n+4;k}=\Sigma\, d_{pk}f^p_{n+4;k}, \quad \dots (7)$$

where

$$f^p_{n+4;k}=\left[-2C_k(n+2)f^p_{n+3;k}-C_k^2(n+1)f^p_{n+2;k}+\omega_k^2 f^p_{nk}/(n(n+1)(n+2))\right]/(n+3) \quad \dots (8)$$

$f^p_{qk}=\delta^p_q$, q = 1, 2, 3, 4; δ^p_q is the Krönecker delta.

The summation over p is taken from 1 to 4 and d_{pk}, $p=1,2,3,4$, are arbitrary constants.

Therefore, the solution of eqs. (5) becomes

$$W_k(X)=\sum_{p=1}^{4} d_{pk}f^p_k(X-X_l), \quad \dots (9)$$

where $$f^p_k(X-X_l)=\sum_{n=1}^{\infty} f^p_{nk}(X-X_l)^{n-1}.$$

3.1. Continuity Conditions

The continuity conditions at $X = X_1$ are taken as

In eqs. (7) and (8) , ; is used to separate the subscripts.

$$\left.\begin{aligned} &W_1(X_1)=W_2(X_1); \qquad W_{1,X}(X_1)=W_{2,X}(X_1) \\ &W_{1,XX}(X_1)=W_{2,XX}(X_1); \quad W_{1,XXX}(X_1)=W_{2,XXX}(X_1) \end{aligned}\right\}. \qquad \text{... (10)}$$

Using above conditions, one gets

$$\left.\begin{aligned} &\Sigma\, d_{p1} f_1^p(X_1)=d_{12}; \qquad \Sigma\, d_{p1} f_{1,X}^p(X_1)=d_{22} \\ &\frac{1}{2}\Sigma\, d_{p1} f_{1,XX}^p(X_1)=d_{32}; \quad \frac{1}{6}\Sigma\, d_{p1} f_{1\,XXX}^p(X)_1=d_{42} \end{aligned}\right\}. \qquad \text{... (11)}$$

In this way the 8 constant arising in solutions (9) reduce to 4.

3.2. *End Conditions*

The following two types of end conditions are considered :

3.2.1. *Clamped-clapmed Beam (C-C Beam)*

For a beam clamped at both the ends $X = 0$ and $X = 1$, we have

$$W_1 = W_{1,X} = 0 \text{ at } X = 0 \text{ and } W_2 = W_{2,X} = 0 \text{ at } X = 1. \qquad \text{... (12)}$$

3.2.2. *Cantilever beam (C-F Beam)*

For a beam clamped at $X = 0$ and free at $X = 1$, one has

$$W_1 = W_{1,X} = 0 \text{ at } X = 0 \text{ and } W_{2,XX} = W_{2,XXX} = 0 \text{ at } X = 1. \qquad \text{... (13)}$$

Use of relation (11) and condition (12) or (13) (as the case may be), gives

$$d_{11}=0,\, d_{21}=0,\, A_2(1-X_1)\, A_1(X_1)\, D_1=0, \qquad \text{... (14)}$$

where $D_1 = [d_{31}\ d_{41}]'$,

$$A_1(X)=\begin{bmatrix} f_1^3(X) & f_1^4(X) \\ f_{1,X}^3(X) & f_{1,X}^4(X) \\ \frac{1}{2} f_{1,XX}^3(X) & \frac{1}{2} f_{1,XX}^4(X) \\ \frac{1}{6} f_{1,XXX}^3(X) & \frac{1}{6} f_{1,XXX}^4(X) \end{bmatrix}, \qquad \text{... (15)}$$

$$A_2(X)=\begin{bmatrix} f_2^1(X) & f_2^2(X) & f_2^3(X) & f_2^4(X) \\ f_{2,X}^1(X) & f_{2,X}^2(X) & f_{2,X}^3(X) & f_{2,X}^4(X) \end{bmatrix} \text{ for C-C beam}$$

and $$A_2(X) = \begin{bmatrix} f^1_{2,XX}(X) & f^2_{2,XX}(X) & f^3_{2,XX}(X) & f^4_{2,XX}(X) \\ f^1_{2,XXX}(X) & f^2_{2,XXX}(X) & f^3_{2,XXX}(X) & f^4_{2,XXX}(X) \end{bmatrix} \text{ for } C\text{-}F \text{ beam} \quad \ldots (16)$$

3.3 Frequency Determinant

For non-trivial solution of homogeneous system (14)

$$\det\,[A_2(1 - X_1)\,A_1(X_1)] = 0. \quad \ldots (17)$$

The countably infinite roots of eq. (17) are the natural frequencies Ω for various modes of vibration.

4. Results and Discussion

The variation in thickness is taken in such a way that the average thickness of the beam, h_a, remain constant, therefore

$$\int_0^{a_1} h_0(b_1 + \beta_1 x/a)\,dx + \int_{a_1}^{a} h_0(b_2 + \beta_2 x/a)\,dx = ah_a, \quad \ldots (18)$$

which leads to $H_0 = 2H_a/\left\{2 + \beta_2 + X_1(\beta_1 - \beta_2)(2 - X_1)\right\}$, where $H_a = h_a/a$.

All the series involved in the analysis are summed upto 15 terms which gives an accuracy of four decimal places. Numerical results for natural frequencies and mode shapes are computed for ν = 0.3 and H_a = 0.06.

Ω vs X_1 for *C-C* beams, for first four modes of vibration is plotted in Fig. 1 for various negative values of β_2 when β_1 = 0.8. This is the case when the beam is thin at the ends and thick in the middle. It is seen that the value of Ω fluctuates when X_1 increases from zerto to one. The number of fluctuations increases as one goes to higher modes. It is also seen that Ω decreases with the decrease in β_2. This can be attributed to the fact that as β_2 decreases the thickness of the right clamped end decreases and consequently rigidity of the beam decreases. The decrease in Ω is more prominent in the first mode when X_1 is 0.5 and in second and higher modes when X_1 = 0. It can be due to the fact that in first mode there is no nodal point in the beam except at the clamped edges whereas in higher modes nodal points lie in the middle of the beam also. All the curves converge to same value of Ω when X_1 is 1.0. It is because of the fact that when X_1 is 1.0 the thickness of the beam varies with only one taper constant β_1 throughout the whole length.

Ω vs X_1 for C-C beams, for various positive values of β_2 when $\beta_1 = -0.8$ is plotted in Fig. 2. This is the case when the beam is thick at the ends and thin in the middle. Here also the value of Ω fluctuates with the increase in modes of vibration. When $X_1 = 0$ the value of Ω decrease with the increase in β_2 but this variation get reserved for non-zero values of X_1. In all the cases, the value of Ω increases when X_1 lies in the middle and decreases when X_1 approaches towards the ends.

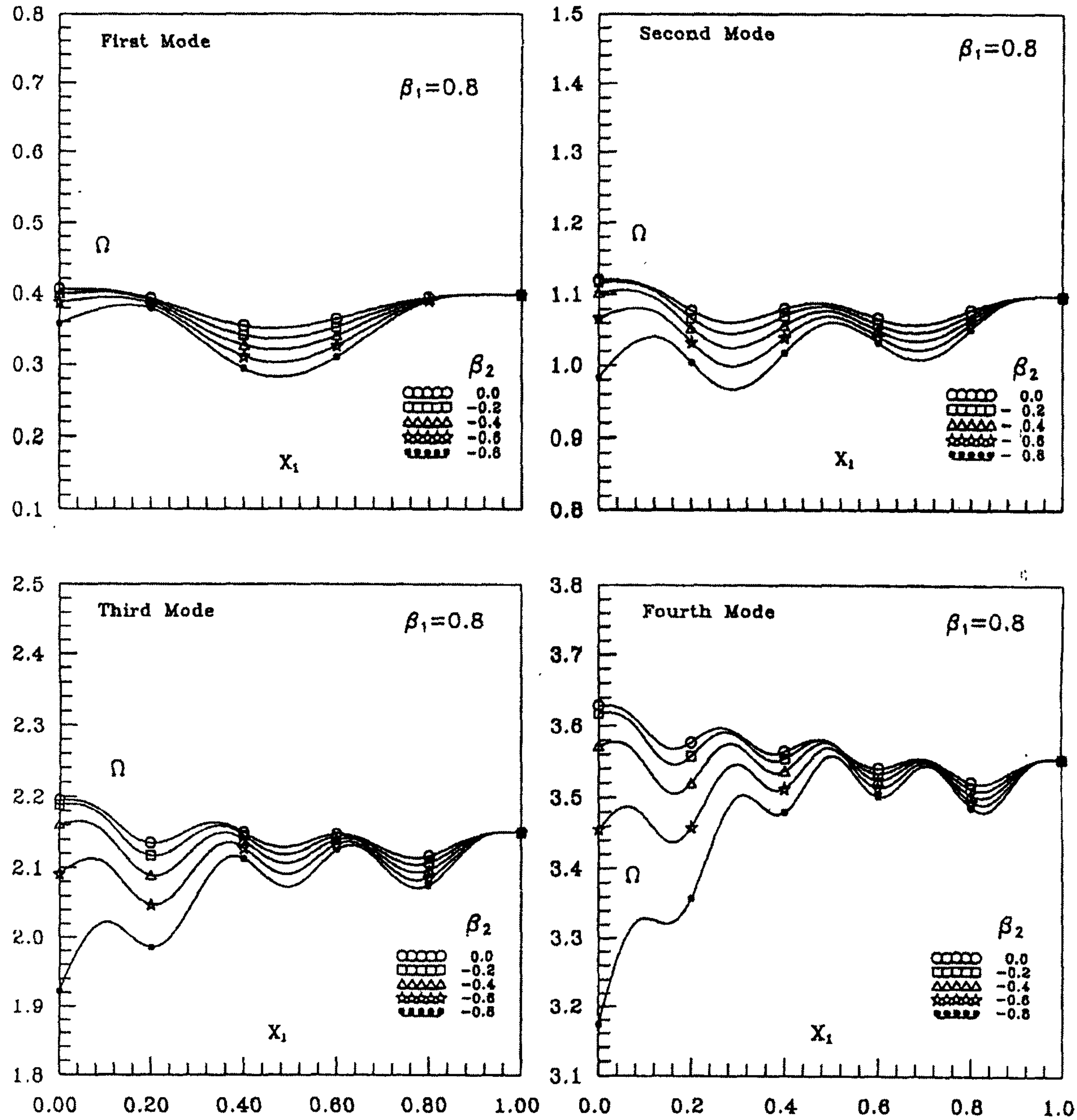

FIG. 1. Ω vs X_1 for C-C beam for various values of β_2

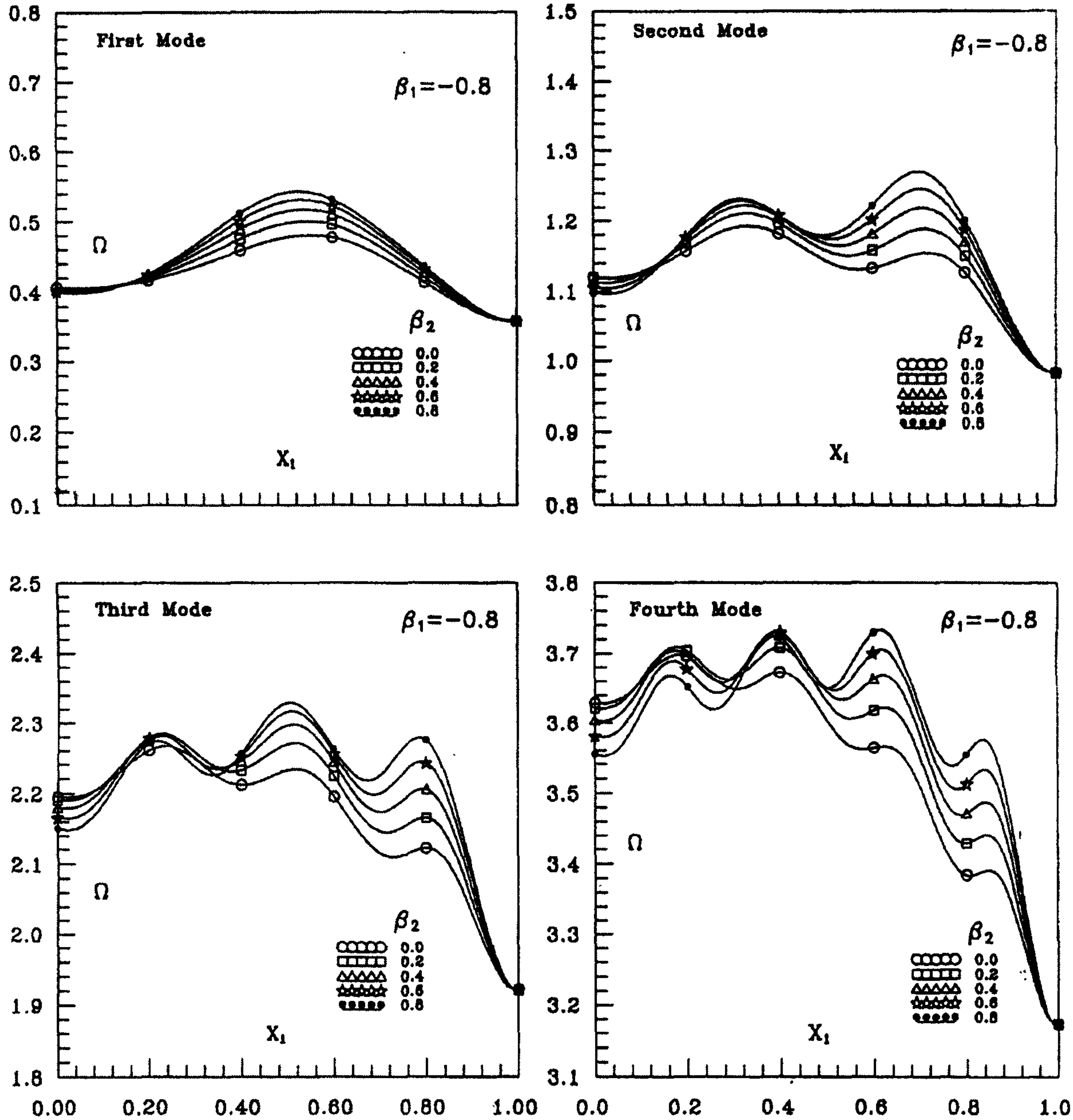

FIG. 2. Ω vs X_1 for C-C beam for various values of β_2

It may be noted from Figs. 1 and 2, when $X_1 = 0$, i.e., when the *C-C* beam is having one linear thickness variation, the frequency is maximum when it is of constant thickness and it decreases with the increase or decrease in the taper constant. It means, the *C-C* beam shows more rigidity whn it is of constant thickness in comparison to when it is taper even though the average thickness remains the same.

Ω vs β_2 for *C-C* beams, is plotted in Fig. 3 for various values of β_1 when $X_1 = 0.5$. It is seen that Ω and the rate of increase of Ω with the increase in β_2 as well as in the number of

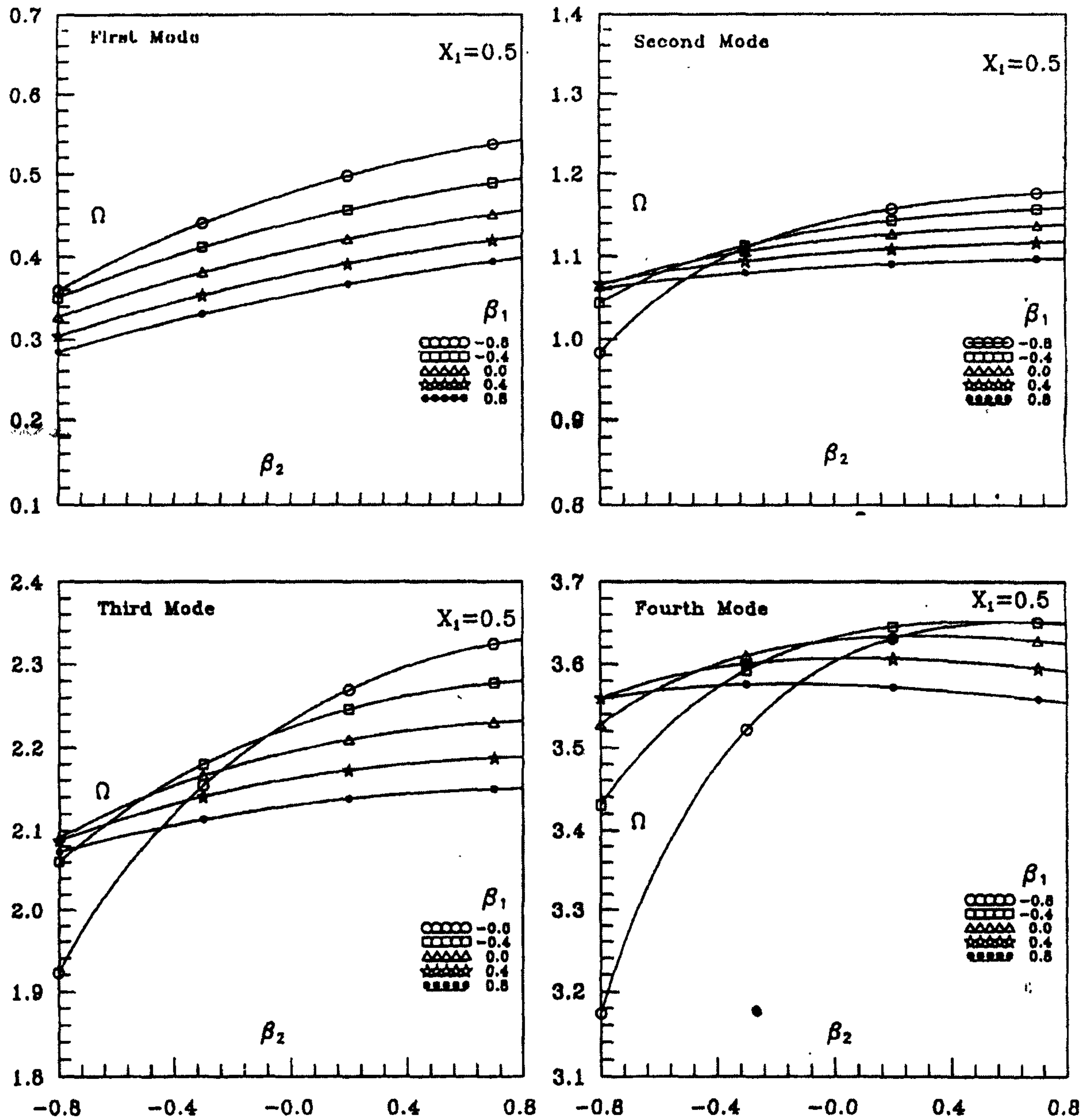

FIG. 3. Ω vs β_2 for C-C beam for various values of β_1

modes. It means keeping the average thickness constant, if the ends of a *C-C* beam are made thicker and the middle portion is made thinner, the frequency increases.

Ω vs X_1 for *C-F* beams, for various negative values of β_2 when β_1 = 0.8is plotted in Fig. 4. It shows that if the thickness of the beam from clamped end is increased upto a certain

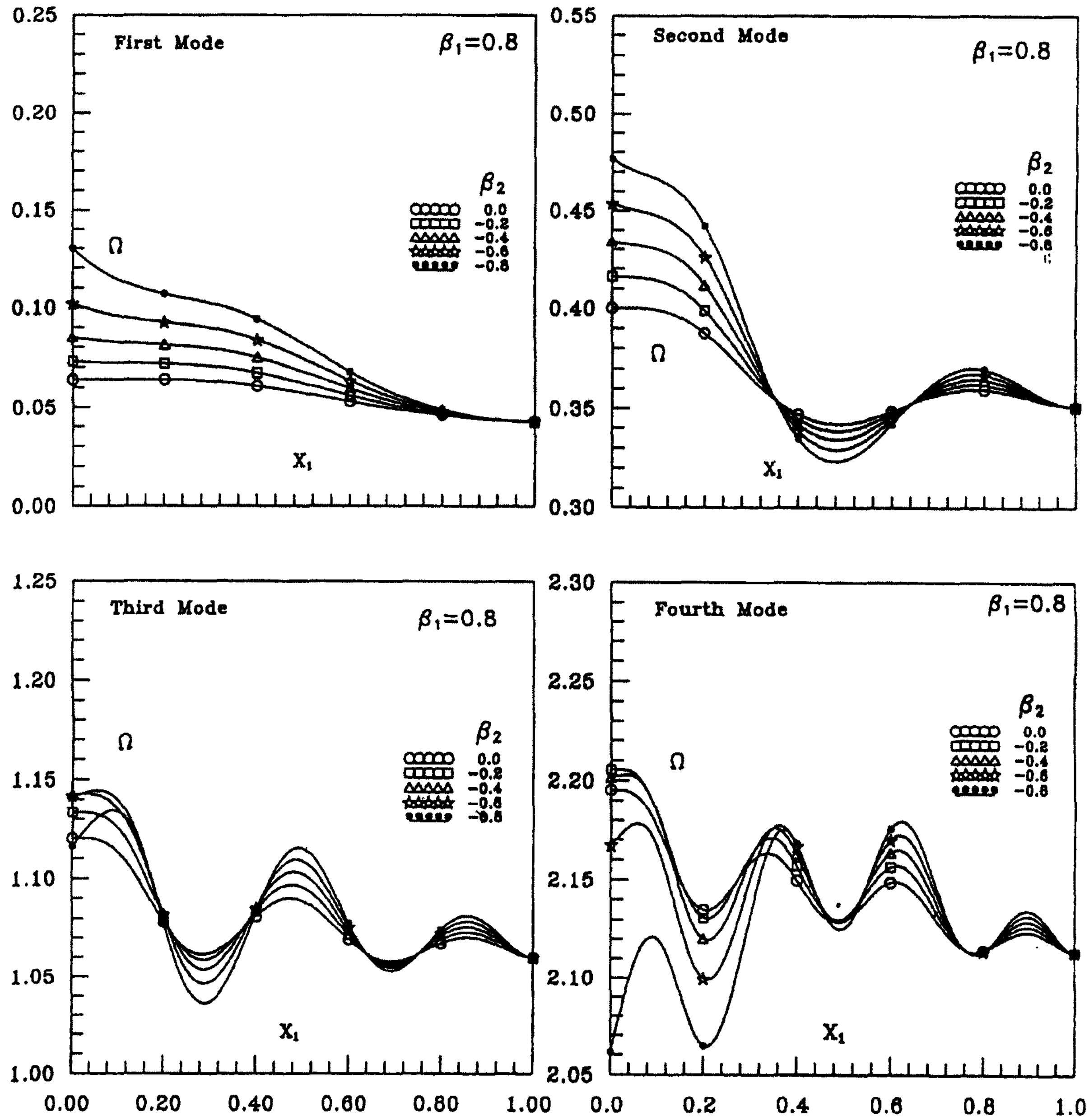

FIG. 4. Ω vs X_1 for C-F beam for various values of β_2

point X_1 of the beam and then decreased and point X_1 is moved towards free end, Ω decreases throughout, in the first mode. It decreases till $X_1 = 0.5$ and then increases, in the second mode. The value of X_1 for minimum Ω decreases and the number of fluctuations increases as one goes to higher modes.

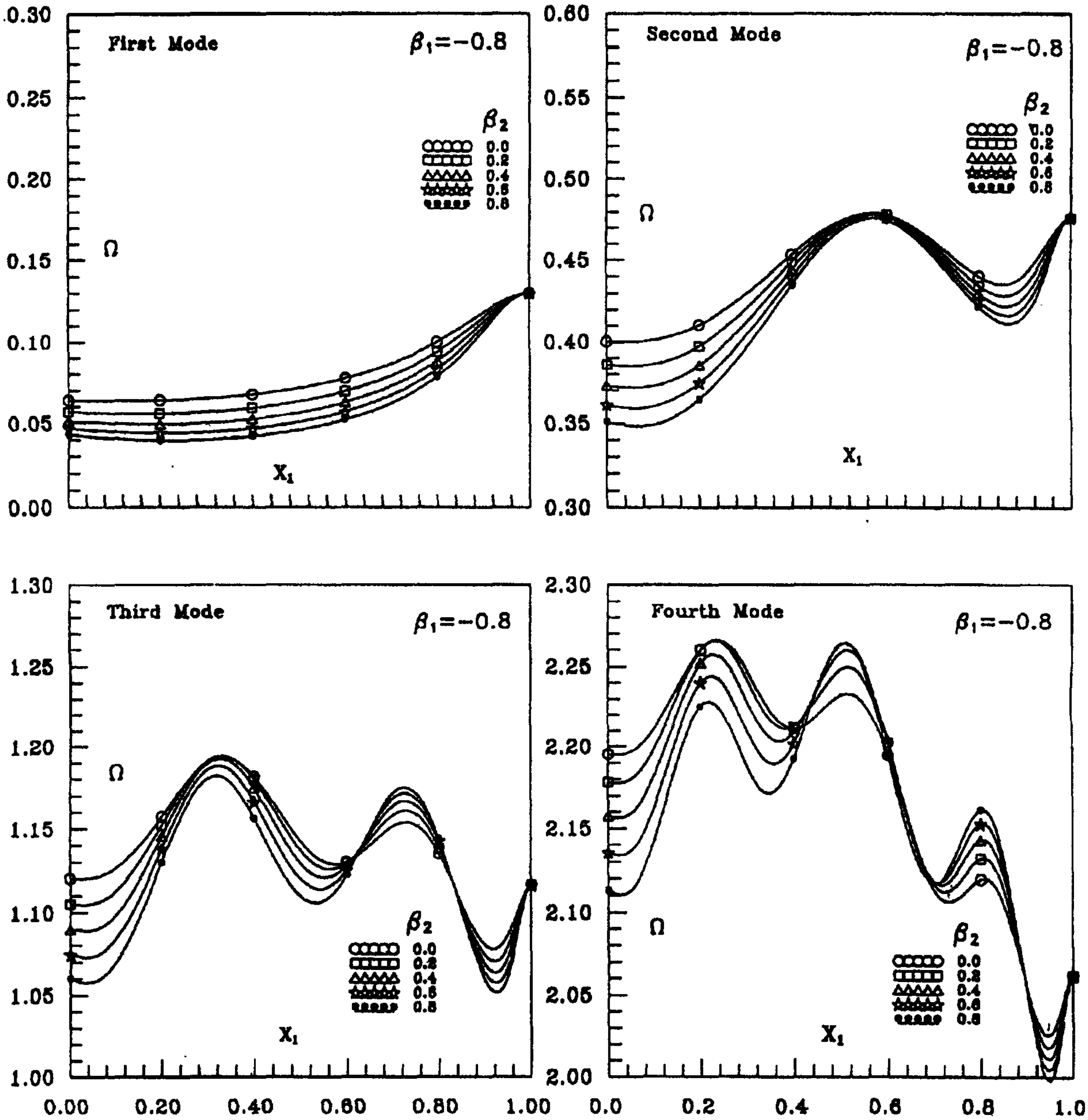

FIG. 5. Ω vs X_1 for C-F beam for various values of β_2

Ω vs β_2 for *C-F* beams, for various values of β_1 when $X_1 = 0.5$ is plotted in Fig. 6. In first mode, Ω decrease with the increase in β_2. In second mode, Ω increase with the increase in β_2 but the rate of increase in Ω is small and it further decreases as β_1 decreases. In third mode,

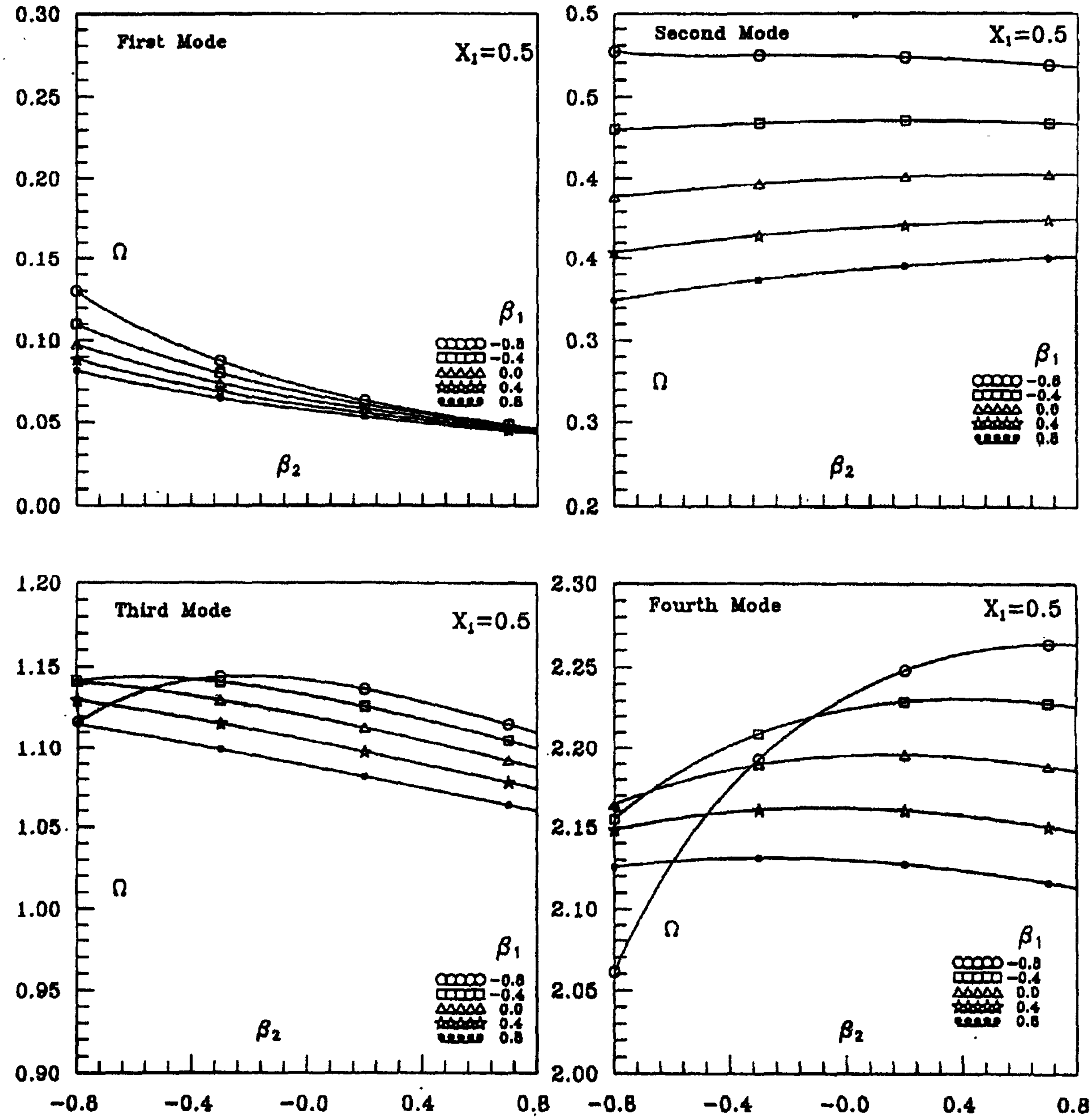

FIG. 6. Ω vs β_2 for C-F beam for various values of β_1

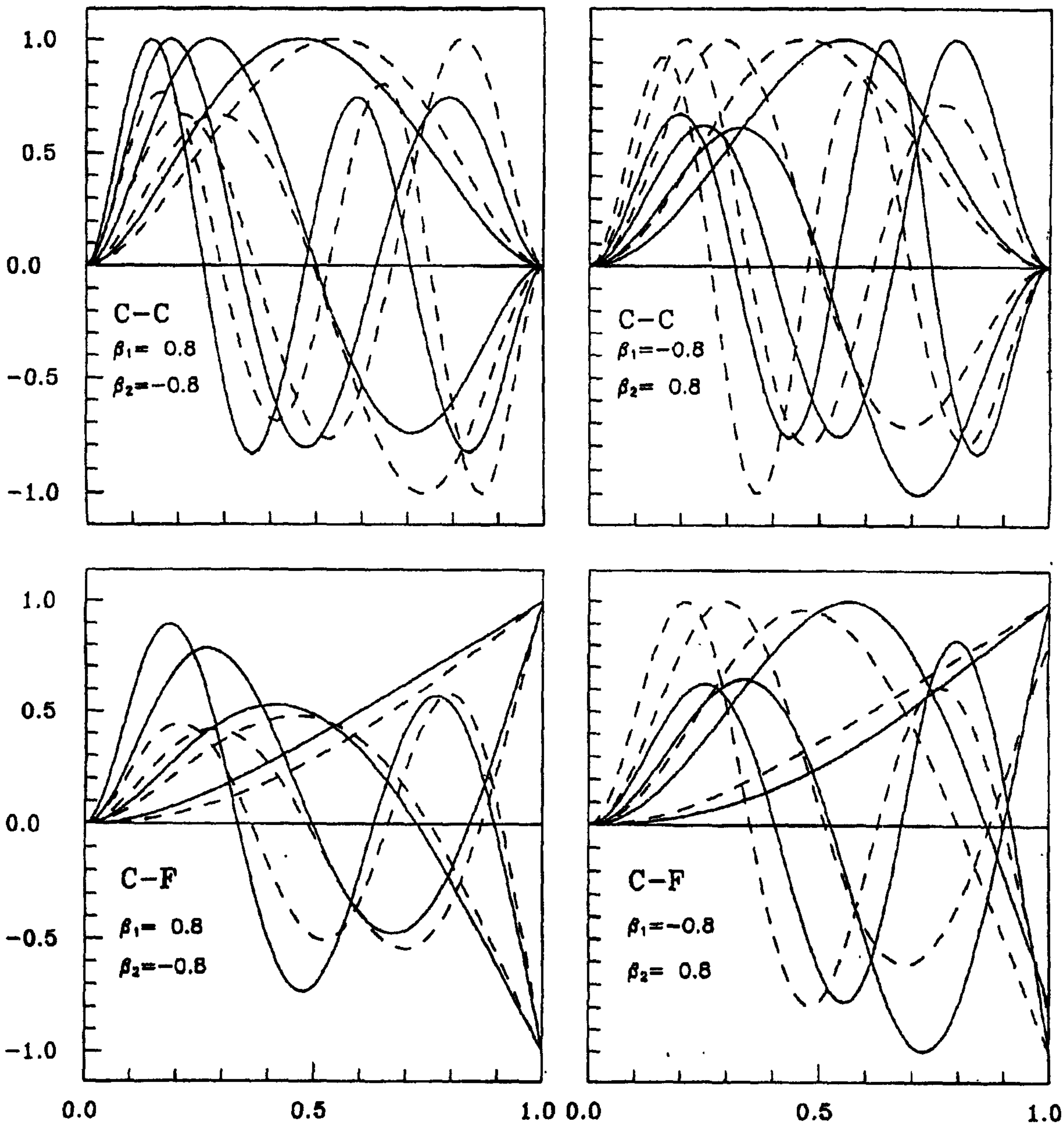

FIG. 7. Normalized transverse deflection in the first four normal modes of vibration of X_1 = 0.7 (---------) and X_1 = 0.3 (- - - - -)

Ω first increase and then decreases for $\beta_1 = -0.8$ whereas it decrease throughout for all other values of β_1. In fourth mode, Ω first increases and then decreases slightly. The rate of increase decreases as β_1 increases.

Normalized mode shapes for first four normal modes are plotted in Fig. 7 for *C-C* as well as *C-F* beams. For a beam thick in the middle and thin towards the ends, as X_1 shifts towards right, the peak of maximum deflection lies in the second half of the beam. Similarly when X_1 lies in the second half of the beam, the peaks of maximum deflection lies in the first half of the beam.

For a beam thin in the middle and thick towards the ends, as X_1 shift towards the right, the peaks also shift towards the right. When X_1 lies in the first half of the beam, the peaks of maximum deflection also lie in the first half of the beam. A similar is the case when X_1 lies in the second half of the beam.

The natural frequencies for first four modes of vibration, for a rectangular beam of constant thickness is compared with the value reported by Soni[9] by taking β_1 = 0.0, β_2 = 0.0 and H_0 = 0.01.

Table I shows a very close agreement in all the results.

TABLE I : Comparison of First Four Frequencies with Ref. [9] when $\beta_1 = \beta_2 = 0$ and $H_0 = 0.01$

Mode of Vibration	C-C Beam		C-F Beam	
	Re. [9]	Ours	Ref. [9]	Ours
I	0.06458	0.06458	0.01015	0.01015
II	0.17802	0.17805	0.06369	0.06367
III	0.39902	0.34903	0.17810	0.17813
IV	0.57694	0.57694	0.34910	0.34911

REFERENCES

1. K. K. hu and D. Lal, *J. struct. Engg.*, **112** (1986) 241.
2. B. V. de Greco, P. A. A. Laura, *J. appl. acoust.*, **16** (1981) 393.
3. R. H. Gutierrez, P. P. A. Laura and R. O. Grossi, *J. Sound Vib.*, **75** (1981) 322.
4. B. V. de greco and P. P. A. Laura, *J. Sound Vib.*, **94** (1984) 525.
5. S. A. Al-Kaabi and G. Aksu, *J. Sound Vib.*, **123** (1988) 373.
6. P. A. A. Laura, R. H. Gutierrez and R. E. Rossi, *J. Acoust. Soc. Amer.*, **98** (1995) 2377.
7. B. Singh and V. Saxena, *J. Sound Vib.*, **179** (1995) 879.
8. N. M. Auciello and G. Nale, *J. Sound Vib.*, **214** (1998) 105.
9. S. R. Soni, Ph.D. Theses, *Dept. Math. U. O. R.*, Roorkee (1972)