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CYLINDRICALLY SYMMETRIC SOLUTION OF RELATIVISTIC MAGNETO-FLUID IN SEN-DUNN THEORY OF GRAVITATION

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A class of exact solution is obtained for perfect magneto-fluid or thermo-dynamical perfect fluid with infinite electric-conductivity and constant magnetic permeability in Sen-Dunn theory of gravitation. It has been observed that due to the $E - R$ metric the magnetic field vector vanishes and magnetic fluid reduces to only Zeldovich or stiff-fluid, even in Sen-Dunn scalar-tensor theory of gravitation. Various physical consequences arising out of the solutions are discussed

Key Words : Exact Solution; Einstein-Rosen Metric; Sen-Dunn Theory of Gravitation

INTRODUCTION

In the scalar tensor theory, proposed by Sen-Dunn[1] the scalar field is characterized by the function $x^0 = x^0 (x^i), x^i$ being the co-ordinates in a four-dimensional Lyra manifold and the tensor field is identified with the metric tensor g_{ij} of the manifold. In their study Sen and Dunn[1] have obtained only a series type solution to the static vacuum field equation. Later Halford[2, 3] considered the vacuum field equations corresponding to an isotropic metric and obtained an exact solution. Reddy[4,5,6] has established a result that an analogue of Birkoff's theorem of Einstein theory exists in this new theory of gravitation also provided that the scalar interaction function x^0 is independent of time. We have also studied considering the matter-free region in one of our papers Roy & Chatterjee[7] taking cylindrically symmetric Marder's metric.

The theory of relativistic magneto-hydrodynamics (RMHD) has been developed to study various astronomical systems like neutron-stars etc. A thermally conducting, viscous fluid with infinite electrical conductivity and magnetic permeability befits theoretical consideration pertaining to such astronomical system. Date[8] has discussed various consequences of fluid equations corresponding to relativistic magneto-fluids. Prasad and Sinha[9] have developed a set of necessary and sufficient conditions for the existence of a space-like Killing vector collinear to the magnetic field vector.

In the present paper we obtain a class of exact solutions of the non-vacuum field equation of Sen-Dunn theory of gravitation corresponding to the cylindrically symmetric Einstein Rosen metric by considering the matter as relativistic magneto-fluid proposed by Lichnerowicz[10].

Here it may be noted that like the paper[11] in Einstein theory of this chosen cylindrical-symmetry, even in the Sen-Dunn scalar tensor theory the magnetic-field vector vanishes and the magneto-fluid reduces to Zel'dovich fluid or stiff-fluid with rest mass density becoming equal to the pressure. The gravitational interaction is found to be located along the z-axis and at infinity. The presence of scalar-interaction x^0 in Sen-Dunn theory still allows the solutions to possess a wave-like character, like in Einstein theory.

FIELD EQUATIONS

The field equations of Sen-Dunn theory of gravitation are as follows

$$G_{ij}=R_{ij}-\frac{1}{2}Rg_{ij}=-\frac{8\pi G}{(x^0)^2}T_{ij}+\frac{w}{(x^0)^2}\left(x_i^0\,x_J^0-\frac{1}{2}g_{ij}x^{0K}x_K^0\right) \quad \text{... (1)}$$

where T_{ij} are the energy-momentum tensors corresponding to perfect-magneto-fluid given by Lichnerowiz's[7] tensor

$$T_{ij}=(D+p+\mu|\bar{h}|^2)\,u_i u_j-\left(p+\frac{1}{2}\mu|\bar{h}|^2\right)g_{ij}-\mu\,h_i h_j \quad \text{... (2)}$$

with the Maxwell equations

$$(u^i h_J-u^J h_i)_{;\,J}=0 \quad \text{... (3)}$$

Here

x^0 ≡ the scalar interaction function

p ≡ the isotropic pressure of the fluid

D ≡ the matter energy density of the fluid

μ ≡ the constant magnetic permeability

and h^α ≡ the magnetic field vector

which satisfying

$$|\bar{h}|^2=-h_i h^i \quad \text{... (4)}$$

and

$$u^i h_i=0 \quad \text{... (5)}$$

where u_i is the four velocity vector. The other symbols have their usual meaning as in Einstein theory.

We consider the cylindrically symmetric Einstein-Rosen metric as

$$ds^2=e^{2\alpha-2\beta}(dt^2-dr^2)-r^2e^{-2\beta}d\phi^2-e^{2\beta}dz^2 \quad \text{... (6)}$$

where α and β are function of r and t only, and r,ϕ,z,t correspond respectively to x^1,x^2,x^3,x^4 co-ordinates.

The field equations (1) for the metric (6) are as follows

$$G_{11}=\left(\beta_1^2+\beta_4^2-\frac{\alpha_1}{r}\right)=\frac{w}{2}(K_1^2+K_4^2)-\frac{8\pi G}{e^{2K}}T_{11} \quad \text{.... (7)}$$

$$G_{22}=r^2e^{-2\alpha}(-\alpha_{11}+\alpha_{44}-\beta_1^2+\beta_4^2)$$

$$= -\frac{w}{2}(K_1^2 - K_4^2)\, r^2 e^{-2\alpha} - \frac{8\pi G}{2^{2K}} T_{22} \qquad \dots (8)$$

$$G_{33} = e^{4\beta - 2\alpha}\left(2\beta_{11} - 2\beta_{44} - \alpha_{11} + \alpha_{44} + \frac{2\beta_1}{r} - \beta_1^2 + \beta_4^2 \right)$$

$$= -\frac{w}{2} e^{4\beta - 2\alpha}(K_1^2 - K_4^2) - \frac{8\pi G}{e^{2K}} T_{33} \qquad \dots (9)$$

$$G_{44} = \left(\beta_1^2 + \beta_4^2 - \frac{\alpha_1}{r} \right) = \frac{w}{2}(K_1^2 + K_4^2) - \frac{8\pi G}{e^{2K}} T_{44} \qquad \dots (10)$$

$$G_{14} = \left(2\beta_1 \beta_4 - \frac{\alpha_4}{r} \right) = wK_1 K_4 - \frac{8\pi G}{e^{2K}} T_{14} \qquad \dots (11)$$

here for convenience we consider $x^0 = e^{2K}$.

From eq. (7) and (10) we have

$$T_{11} - T_{44} = 0 \qquad \dots (12)$$

Case 1 — Assuming the co-ordinates to be comoving i.e. the four-velocity vector u^i is assumed to satisfy the relations,

$$u^4 u_4 = 1 \text{ and } u_1 = u_2 = u_3 = 0 \qquad \dots (13)$$

We have for the metric (6) the energy-momentum-tensor or Lichnerowicz's tensor given by eq. (2) is as follows :

$$T_{11} = e^{2\alpha - 2\beta}\left(p + \frac{1}{2} u \,|\bar{h}|^2 \right) - \mu h_1^2, \qquad \dots (14)$$

$$T_{22} = r^2 e^{-2\beta}\left(p + \frac{1}{2}\mu \,|\bar{h}|^2 \right), \qquad \dots (15)$$

$$T_{33} = e^{2\beta}\left(p + \frac{1}{2}\mu \,|\bar{h}|^2 \right), \qquad \dots (16)$$

$$T_{44} = e^{2\alpha - 2\beta}\left(D + \frac{1}{2}\mu \,|\bar{h}|^2 \right) \qquad \dots (17)$$

and

$$T_{14} = -\mu h_1 h_4 \qquad \dots (18)$$

From (5) we have $u^4 h_4 = 0$

but since $u^4 \neq 0$ hence $h_4 = 0$ ... (19)

Using (14), (17) and (19) we have from eq. (12)

$$e^{2\alpha - 2\beta}(D - p) + \mu (h_1^2) = 0$$

since $\mu \neq 0$ and $e^{2\alpha - 2\beta} \neq 0$ hence we have

$$D = p \qquad \text{... (20)}$$

and $$h_1 = 0. \qquad \text{... (21)}$$

Thus due to the $E - R$ metric tee magnetic-fluid becomes stiff fluid.

Case 2 — In general when the system is not comoving i.e.,

$$u_2 = u_3 = 0, u_1 \neq 0 \text{ and } u_4 \neq 0 \qquad \text{... (22)}$$

the energy momentum, tensor (2) is as follows :

$$T_{11} = (D + p + \mu |\bar{h}|^2)\, u_1^2 - \left(p + \frac{1}{2} \mu |\bar{h}|^2 \right) g_{11} - \mu h_1^2 \qquad \text{... (23)}$$

$$T_{22} = -\left(p + \frac{1}{2} \mu |\bar{h}|^2 \right) g_{22} \qquad \text{... (24)}$$

$$T_{33} = -\left(p + \frac{1}{2} \mu |\bar{h}|^2 \right) g_{33} \qquad \text{... (25)}$$

$$T_{44} = (D + p + \mu |\bar{h}|^2)\, u_4^2 - \left(p + \frac{1}{2} \mu |\bar{h}|^2 \right) g_{44} - \mu h_4^2 \qquad \text{... (26)}$$

and $$T_{14} = (D + p + \mu |\bar{h}|^2)\, u_1 u_4 - \mu h_1 h_4 \qquad \text{... (27)}$$

Using eq. (23) and (26) we have from (12)

$$(D = p + \mu |\bar{h}|^2)(u_1^2 - u_4^2) + 2e^{2\alpha - 2\beta} \left(p + \frac{1}{2} \mu |\bar{h}|^2 \right) - \mu (h_1^2 - h_4^2) = 0. \qquad \text{... (28)}$$

But since

$$u^i u_i = 1, \text{ we have}$$

$$(u_4^2 - u_1^2) = g_{44} = e^{2\alpha - 2\beta} \qquad \text{... (29)}$$

Again from (4)

$$|\bar{h}|^2 = -h_1 h' - h_4 h^4 = e^{2\beta - 2\alpha} (h_1^2 - h_4^2)$$

or $$(h_1^2 - h_4^2) = e^{2\alpha - 2\beta} |\bar{h}|^2 \qquad \text{... (30)}$$

hence we have from (28) using (29) and (30)

$$e^{2\alpha - 2\beta} [(D - p) + \mu |\bar{h}|^2] = 0$$

Arguing as before we have

$$D = p$$

and $$|\bar{h}| = h_1^2 - h_4^2 = 0$$

or $$h_1 = \pm h_4 \quad \text{... (31)}$$

Again from (5) we have

$$u^1 h_1 + u^4 h_4 = 0$$

which in view of (29) and (31) immediately reduces to

$$h_1 (= \pm h_4) = 0. \quad \text{... (32)}$$

Thus in general we conclude that due to the $E - R$ metric the magnetic field vectors h^i vanish and the magneto-fluid reduces to the Zedovich fluid even in Sen-Dunn scalar tensor theory of gravitation like in Einstein theory of our paper[7].

SOLUTION

The field equations of Sen-Dunn theory of gravitation given by (7) to (11) reduce to the following form

$$\beta_1^2 + \beta_4^2 - \frac{\alpha_1}{r} = \frac{w}{2}(K_1^2 + K_4^2) - \frac{8\pi G}{e^{2K}} e^{2\alpha - 2\beta} p, \quad \text{.... (33)}$$

$$\alpha_{11} - \alpha_{44} + \beta_1^2 - \beta_4^2 = \frac{w}{2}(K_1^2 - K_4^2) + \frac{8\pi G}{e^{2k}} e^{2\alpha - 2\beta} p \quad \text{... (34)}$$

$$\beta_{11} - \beta_{44} + \frac{\beta_1}{r} = 0 \quad \text{... (35)}$$

and $$2\beta_1 \beta_4 - \frac{\alpha_4}{r} = wK_1 K_4 \quad \text{... (36)}$$

Now assuming a functional relationship between the scalar interaction function x^0 and the metric parameter β as given by

$$x^0 = e^K = e^{n\beta} \quad \text{... (37)}$$

eqs. (33) to (36) become

$$\beta_{11} - \beta_{44} + \frac{\beta^1}{r} = 0 \quad \text{... (38)}$$

$$\frac{\alpha_4}{r} = (2 - n^2 w)\beta_1 \beta_4 \quad \text{.... (39)}$$

$$\alpha_{44} - \alpha_{11} + \frac{\alpha_1}{r} = (2 - n^2 w)\beta_1^2 \quad \text{... (40)}$$

$$18\pi G e^{2\alpha - 2\beta(1-n)} p = \alpha_{44} - \alpha_{11} - \frac{\alpha_1}{r} + (2 - n^2 w)\beta_4^2 \quad \text{... (41)}$$

Solution I — Let us consider one of the solutions of (38) as

$$\beta = (c_1\, t + c_2) \log r \qquad \text{... (42)}$$

substituting the value of β from (40) we have

$$\alpha = C_3\, r^2 + (2 - n^2 w)\left[C_1^2 r^2 \left\{ \frac{(\log r)^2}{4} - \frac{\log r}{4} + \frac{1}{8} \right\} + \frac{(C_1\, t + C_2)^2}{2} \log r \right] + C_4 \qquad \text{... (43)}$$

Using the above value of β and α from (41) we have

$$p = \frac{C_3}{8\,\pi\, G}\, e^{2\,\beta\,(1-n) - 2\,\alpha} \qquad \text{... (44)}$$

Thus we have

$$\alpha = (2 - n^2 w)\left[C_1^2 r^2 \left\{ \frac{(\log r)^2}{4} - \frac{\log r}{4} + \frac{1}{8} \right\} + \frac{(C_1\, t + C_2)^2}{2} \log r \right] + C_3\, r^2 + C_4$$

$$\beta = (C_1\, t + C_2) \log r$$

$$x_0 = \exp\,[n\,(C_1\, t + C_2) \log r]$$

$$p = \frac{C_3}{8\,\pi\, G}\, e^{2\,\beta\,(1-n) - 2\,\alpha}$$

Solution II — Following Rosen[12] we now consider the solution of (38) as

$$\beta = C_1\, J_0\,(mr) \cos\,(mt) \qquad \text{... (45)}$$

where $J_0\,(mr)$ is the Bessel's function of order zero, hence from (37) the scalar-interaction function x^0 is given by

$$x^0 = e^K = \exp\,[nc_1\, J_0\,(mr) \cos\,(mt)] \qquad \text{... (46)}$$

Using the value of β from (39) and (40) we have

$$\alpha = (2 - n^2 w)\, \frac{C_1^2\, m}{4}\, rJ_0\,(mr)\, J_0'\,(mr) \cos 2mt + f\,(r) \qquad \text{... (47)}$$

where dash represents differentiation w.r.t 'r', and from (41) we have

$$8\,\pi\, G\, p \exp\left\{ 2\,\alpha - 2\,(n-1)\,\beta \right\} = \left(\frac{2 - n^2 w}{4} \right) mC_1^2\, [\cos 2mt\, \{2mJ_0'\,(mr)$$

$$+\ J_0\,(mr)\, J_0''\,(mr) + 3rm^2\, J_0\,(mr)\, J_0''\,(mr)\ +\ 7m^2\, J_0\,(mr)\, J_0''\,(mr)$$

$$-\, 4m^2\, r\, J_0\,(mr)\, J_0'\,(mr)\}\ +\ 2mJ_0'^2\,(mr) \cos^2\,(mt)\ -\ 2mJ_0^2\,(mr) \sin^2\,(mt)] \qquad \text{... (48)}$$

DISCUSSION

It has been observed that the metric reduces the magneto-fluid to Zel'dovich fluid where the rest mass density is equal to the proper pressure even in Sen-Dunn scalar tensor theory of gravitation like our previous paper[11] in Einstein theory of gravitation.

An examination regarding the regularity in the sense of Bonner[13] of the gravitational field, characterized by the value of α and β for both cases, has revealed that the solutions are not everywhere regular. The solution in fact, is singular along the z-axis and at spatial infinity.

It is to be noted that the product symmetry PT will be preserved only for ultraweak interaction for solution II and if we choose $C_1 = 0$ then for solution (1). Moreover, the scalar-field given as by solution II is free from singularity for any r.

The component of the energy-momentum tensor vanishes on the Z-axis initially at $t = 0$ as well as spatial infinity.

The curvature invariant is regular on the Z-axis but vanishes at spatial infinity.

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