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POTENTIALLY GENERALIZED BIPARTITE SELF-COMPLEMENTARY BIPARTITIONED SEQUENCES

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In this paper we characterize all graphic bipartitioned sequences for which there is at least one generalized bipartite self-complimentary realization. We also give a step by step procedure to check if a given bipartitioned sequence is pbgipsc or not.

Key Words : Generalized Complement of a Graph; Complementing Permutation; Bipartite Self-Complementary Graph; Degree Sequences

1. INTRODUCTION AND DEFINITIONS

All graphs considered in this paper are finite, undirected graphs without multiple edges or loops. For terminology and notations not defined here the reader is referred to Harary[8].

Let G be graph with $V(G) = \{v_1, v_2, ..., v_p\}$ where $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_p)$. Then the sequence $(d(v_1), d(v_2), ..., d(v_p))$ is called the degree sequence of G. Conversely, a sequence $(d_1, d_2, .., d_p)$ with $d_1 \geq d_2 \geq \ldots \geq d_p$ is called graphic if it is the degree sequence of a graph G. The following result due to Erdos and Gallai[4] on graphic sequence is well known.

Result A — Let $\pi = (d_1, d_2, ..., d_p)$ be a partition of $2q$ into $p > 1$ parts, $d_1 \geq d_2 \geq \ldots \geq d_p$. Then π is graphical if for each $r, 1 \leq r \leq p-1$ $\sum_{i=1}^{r} d_i \leq r(r-1) + \sum_{i=r+1}^{p} \min(r, d_i)$.

A graph is said to be bipartite if $V(G)$ can be bipartitioned into two non empty subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 to a vertex in V_2. Such a partition (V_1, V_2) is called a partition of G and is denoted by $P = \{V_1, V_2\}$. A bipartite graph is denoted by $G(V_1, V_2)$ or $G(P)$. Given a bipartite graph G, its bipartite complement, denoted by $\overline{G}(P)$, is the bipartite graph with $V(G(P)) = V(\overline{G}(P))$ and $E(\overline{G}(P)) = \{uv \mid u \in V_1, v \in V_2$ and $uv \notin E(G(P))\}$. A bipartite graph $G(P)$ is said to be bipartite self-complementary (bipsc)[7] if $G(P)\ \overline{G}(P)$.

Let (G, P) be bipsc. A bipartite complementing permutation (bipcp) of $G(P)$ is an isomorphism between G and $\overline{G}(P)$ i.e. a bijection σ from $V(G)$ to $V(\overline{G}(P)$ such that $\sigma(u)\,\sigma(v)$ is an edge of $\overline{G}(P)$ iff uv is an edge of G. We denote by $C(G, P)$ the class of all bipcp's of the bipsc graph $G(P)$. A cycle of a bipcp is said to be pure if it permutes only vertices belonging to a single set of P and is said to be mixed otherwise. We define a subclass $C_p(G, P)$ of $C(G, P)$ as follows : $C_p(G, P) = \{\sigma \in C(G, P) \mid \text{all cycles of } \sigma \text{ are pure}\}$. Let $G(V_1, V_2)$ be a bipartite graph with $V_1 = \{u_1, u_2, ..., u_m\}$ and $V_2 = \{v_1, v_2, ..., v_n\}$ where $d(u_1) \geq d(u_2) \geq \ldots \geq d(u_m)$ and $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$. Let $d(u_i) = d_i$ and $d(v_j) = e_j$. The bipartitioned sequence $\pi = (d_1, d_2m ..., d_m \mid e_1, e_2, ..., e_n)$ is called the degree sequence of $G(V_1, V_2)$. If $V_1 = \{u_1, u_2, ..., u_m\}$ and $V_2 = \{v_1, v_2, ..., v_n\}$ then we say that $S = \{u_1, u_2, ..., u_m \mid v_1, v_2, ..., v_m\}$ is an ordering of $G(P)$. The bipartite graph $G(P)$ with ordering $(u_1, u_2, .., u_m \mid v_1, v_2, \ldots v_n)$ is said to be a realization of the bipartitioned sequence $\pi = (d_1, d_2 ..., d_m \mid e_1, e_2, ..., e_n)$ if $d(u_i) = d_i$ and $d(v_j) = e_j$ for all i and j. We also say that $G(P)$ is a realization of π if $G(P)$ with some ordering S is a realization of π. A bipartitioned sequence π is said to be bigraphic if there is a realization of π. The following result due to Gale and Ryser[3] and Bhave[2] gives a necessary and sufficient condition for a bipartitioned sequence to be bigraphic.

Result B — let $\pi = (d_1, d_2, ..., d_m \mid e_1, e_2, ..., e_n)$. be a bipartitioned sequence. Then π is bigraphic iff π satisfies the following conditions :

(*i*) $n \geq d_1 \geq d_2 \geq \ldots \geq d_m, m \geq e_1 > e_2 \geq \ldots \geq e_n$ and $\Sigma d_i = \Sigma e_j$.

(*ii*) $\sum_{i=1}^{r} d_i \leq \sum_{j=1}^{n} \min(r, e_j);\ 1 \leq r \leq m$

Condition (*ii*) can be replaced by the equivalent condition

$$\sum_{j=1}^{k} e_j \leq \sum_{i=1}^{m} \min(k, d_i);\quad 1 \leq k \leq n.$$

A bipartitioned sequence π is said to be potentially bipsc (pbipsc) if there exists at least one bipsc realization $G(P)$ of π. The following result of Gangopadhyay[5] will be used in the sequel.

Result C : Let C1 denote the condition :

$$[d_i + d_{m+1-i} = n;\ 1 \leq i \leq m;\ e_j + e_{n+1-j} = m;\ 1 \leq j \leq n]$$

A bipartitioned sequence $\pi = (d_1, d_2, ..., d_m \mid e_1, e_2, ..., e_n)$ is pbipsc iff it satisfies at least one of the following conditions :

(*i*) C1 holds and exactly one of m and n is odd.

(*ii*) C1 holds, both m and n are even and either $d_{m/2} = d_{m/2+1} = \frac{n}{2}$ or $e_{n/2} = e_{n/2+1} = \frac{m}{2}$.

(*iii*) C1 holds, m, n and $\sum_{j=1}^{n/2} e_j - \sum_{i=1}^{m/2} d_i - \frac{1}{4}(mn)$ are all even.

(*iv*) $m = n$ is even, $d_i + e_{m+1-i} = n$ for $1 \leq i \leq m$ and $d_{2i-1} = d_{2i}$ for $i = 1, 2, ..., m/2$.

Furthermore π is the degree sequence of a bipsc graph $G(P)$ with $C_p(G, P) \neq \phi$ iff at least one of (*i*) - (*iii*) holds and π is the degree sequence of a bipsc graph $G(P)$ with a bipcp sending V_1 to V_2 iff (iv) holds.

Let G (V, E) be a graph and $Q = \{W_1, W_2\}$ be a partition of V. We define a generalized bipartite complement G_Q of G as follows : $V(G_Q) = V(G)$ and $E(G_Q) = \{u\,v \mid u \in W_1, v \in W_2$ and $u\,v \notin E\} \cup \{u\,v \mid u, v \in W_i, i = 1, 2$ and $u\,v \in E\}$. G_Q is also called a 2-switched graph with respect to $Q = \{W_1, W_2\}$[1,9]. G is said to be generalized bipsc (gbipsc) if it is isomorphic to G_Q[9,12]. Gangopadhyay and Rao hebbare[6] have investigated graphs G_Q where Q is identical with P.

A sequence $\pi = (d_1, d_2, ..., d_p)$ is said to be potentially gbipsc (pgbipsc) if there exists at least one gbipsc realization of π. A bipartitioned sequence $\pi = (d_1, d_2, ..., d_m \mid e_1, e_2, ..., e_n)$ is said to be pgbipsc if there exists at least one bipartite realization $G(V_1, V_2)$ of π which is gbipsc with respect to a partition $Q = \{W_1, W_2\}$ of $V_1 \cup V_2$.

A bipcp σ of a gbipsc bipartite graph $G(V_1, V_2)$ is said to be pure if $\sigma(W_i) = W_i$, $i = 1,2$. If for a gbipsc bipartite graph $G(V_1, V_2)$ there exists a pure bipcp σ then we call this graph gbipsc (pure). A bipartitioned sequence $\pi = (d_1, d_2, ..., d_m \mid e_1, e_2, ..., e_n)$ is said to be pgbipsc (pure) if there exists at least one bipartite realization of π which is gbipsc (pure). Bhave[2] has characterized bipartitioned sequences which are pgbipsc (pure). In this paper we characterize all bipartitioned sequences that are pbgipsc. This also includes pgbipsc (pure) sequences.

A bipartite graph $G(V_1, V_2)$ is said to be symmetric if there exists an automorphism σ of G such that $\sigma(V_1) = V_2$ and $\sigma(V_2) = V_1$. A bipartitioned sequence is said to be potentially symmetric bigraphic if it has a symmetric realization. The following result of Bhave[2] will be used in the sequel.

Result D — A bipartitioned sequence $\pi = (d_1, d_2, ..., d_m \mid e_1, e_2, ..., e_m)$ is potentially symmetric bigraphic with a realization in which a vertex is adjacent to its image iff π is bigraphic and $d_i = e_i$, $1 \leq i \leq m$.

Definition 1 — Let $G(V_1, V_2)$ be a bipartite graph which is gbipsc with respect to a partition $Q = \{W_1, W_2\}$. G is said to be gbipsc of type I if at least one of $W_i \cap V_j$, $i, j = 1, 2$ is empty and of type II otherwise.

Definition 2 — A bipartitioned sequence $\pi = (d_1, d_2, ..., d_m \mid e_1, e_2, ..., e_n)$ is said to be pgbipsc of type I (type II) if there exists at least one bipartite realization of π that is gbipsc of type I (type II)

We now state the two main theorems of this paper which characterize pgbipsc type I and type II sequences.

***Theorem* 1** — *A bipartitioned sequence $\pi = (a_1, a_2, ..., a_m \mid b_1, b_2, ..., b_n)$ is pgbipsc of type I if and only if it is one of the following types with suitable ordering of a_i's and b_i's*

1^0 — π satisfies one of the conditions of (*i*) - (*iv*) of result *C*.

2^0 — $\pi = (m_2 + (d_1, d_2, ..., d_{m_1}), (f_1, f_2, ..., f_{m_2}) \mid (e_1, e_2, ..., e_{n-m_2}), m_1 + (f_1, f_2, ..., f_{m_2}))$ where $(d_1, d_2, ..., d_{m_1} \mid e_1, e_2, ..., e_{n-m_2})$ satisfies one of the conditions (*i*)-(*iii*) of result *C* and $(f_1, f_2, ..., f_{m_2} \mid f_1, f_2, ..., f_{m_2})$ is the degree sequence of any symmetric graph.

3^0 — $\pi = (m_1 - m_2 + (d_1, d_2, ..., d_{m_1 - m_2}), 2m_1 - m_2 - (e_1, e_2, ..., e_{m_2}), m_1 - m_2 - (e_1, e_2, ..., e_{m_2}) \mid m_1 + m_2 - (d_1, d_2, ..., d_{m_1 - m_2}), m_2 + (e_1, e_2, ..., e_{m_2}), m_2 - (d_1, d_2, ..., d_{m_1 - m_2}))$ where $(d_1, d_2, ..., d_{m_1 - m_2} \mid e_1, e_2, ..., e_{m_2})$ is any bigraphic sequence.

Here $m = m_1 + m_2$.

***Theorem* 2** — *A bipartitioned sequence $\pi = (a_1, a_2, ..., a_m \mid b_1, b_2, ..., b_n)$ is pgbipsc of type II iff it is of the following type 4^0 with suitable ordering of a_i's and b_i's.*

4^0 — $(m_2 + (d_1, d_2, ..., d_{m_1}) n_1 + (f_1, f_2, ..., f_{m_2}) \mid m_1 + (f_1, f_2, ..., f_{m_2}), m_2 + (e_1, e_2, ..., e_{n_1}))$ where $(d_1, d_2, ..., d_{m_1} \mid e_1, e_2, ..., e_{n_1})$ is any bigraphic sequence and $(f_1, f_2, ..., f_{m_2} \mid f_1, f_2, ..., f_{m_2})$ is the degree sequence of any symmetric graph. Here $m = m_1 + m_2$ and $n = n_1 + m_2$.

In sections 2 and 3 we prove Theorem 1 and 2 respectively and in Section 4 we give a step by step procedure to check if a given bipartitioned sequence is pgbipsc or not.

In figures used in this paper small circles represent vertices and big circles (ovals) represent set of vertices. A single line between two vertices represents an edge whereas a single line between sets of vertices means a complete bipartite graph between these sets. No line between sets of vertices mean no edge between those vertices i.e. a null graph. A line with an arrow between sets of vertices indicates a specific graph between these sets. $G(A \mid B)$ denotes the induced bipartite subgraph of $G(V_1, V_2)$ between A and B. While writing the degree sequences, $r + (d_1, d_2, ..., d_m)$ means $(r + d_1, r + d_2, ..., r + d_m)$ and $(r)^t$ means the t-tuple $(r, r, ..., r)$.

2. PROOF OF THEOREM 1

Let $G(V_1, V_2)$ be a gbipsc type I realization of π with a bipcp σ from G to G_Q where $Q = \{W_1, W_2\}$. For definiteness we assume $W_1 \cap V_2 = \phi$. Thus $V_1 = \{u_1, u_2, ..., u_{m_1}, u_{m_1+1}, u_{m_1+2}, ..., u_{m_1+m_2}\}$, $V_2 = \{v_1, v_2, ..., v_n\}$ and $W_1 = \{u_1, u_2, ..., u_{m_1}\}$, $W_2 = \{u_{m_1+1}, u_{m_1+2}, ..., u_{m_1+m_2}, v_1, v_2, ..., v_n\}$. Clearly m_1 and n cannot be zero. If $m_2 = 0$ then $W_1 = V_1$ and $W_2 = V_2$. So G_Q is $\overline{G}(P)$. Hence π satisfies one of the conditions (*i*)-(*iv*) of result C and thus in this case π is of type 1^0 of Theorem 1.

For convenience let us denote W_1 by A and $V_1 - W_1$ by B. Since there are no internal edges in V_1, the induced bipartite graph $G(A \mid B)$ is null, N_{m_1, m_2}, and since G_Q is the 2-switched graph with $Q = \{W_1, W_2\}$, $G_Q(A \mid B)$ is full, $K_{m_1 m_2}$. Clearly this K_{m_1, m_2} must have its m_1 and m_2 parts in different parts of bipartition of G_Q viz. $\sigma(V_1)$ and $\sigma(V_2)$. We have two cases to consider, $W_1 \subset \sigma(V_1)$ or $W_1 \subset \sigma(V_2)$.

Case 2.1 — $W_1 \subset \sigma(V_1)$ i.e. $A \subset \sigma(V_1)$ then $B \subset \sigma(V_2)$. Let $\sigma(V_1) = A \cup C$ and $\sigma(V_2) = B \cup D$ so that $V_2 = C \cup D$. Thus $\sigma(V_1) = \sigma(A \cup B) = A \cup C$ and $\sigma(V_2) = \sigma(C \cup D) = B \cup D$. Also $W_1 = A$ and $W_2 = B \cup C \cup D$. Since $Q = \{W_1, W_2\}$, there is switching between A and C in G_Q while there is no switching between B and D in G_Q. Since there are no edges in V_1 in G, there are no edges in $\sigma(V_1)$ in G_Q i.e. $G_Q(A \mid C)$ is null and due to switching $G(A \mid C)$ is full. Similarly, $G_Q(B \mid D)$ is null and due to no switching $G(B \mid D)$ is also null. Also $G(C \mid D)$ and $G_Q(C \mid D)$ are null. This type of argument which uses bipartition, isomorphism and switching will be called the Bipartition Isomorphism Switching Technique (BIST). Now the only

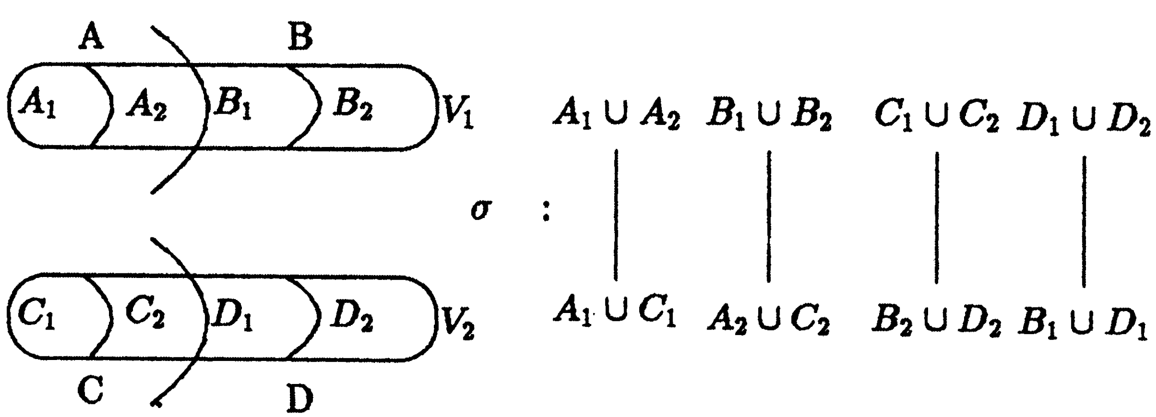


FIG. 2.1.

undetermined graphs in G are $G(A \mid D)$ and $G(B \mid C)$. In this pursuit let $\sigma(A) = A_1 \bigcup C_1$ where $A_1 \subset A$ and $C_1 \subset C, \sigma(C) = B_2 \bigcup D_2$ where $B_2 \subset B$ and $D_2 \subset D, \sigma(B) = A_2 \bigcup C_2$ where $A_2 = A - A_1, C_2 = C - C_1$ and $\sigma(D) = B_1 \bigcup D_1$ where $B_1 = B - B_2$ and $D_1 = D - D_2$. We present this information in the following Fig. 2.1.

$G(A \mid C)$ is full. By Isomorphism, $G_Q(\sigma(A) \mid \sigma(C))$ is full i.e. $G_Q(A_1, C_1 \mid B_2, D_2)$ is full, a contradiction unless either $C_1 = \phi$ or $D_2 = \phi$ (since $G_Q(C \mid D)$ is null).

Subcase (a) — Let $C_1 = \phi$ in Fig.2.1. Therefore $A_2 = \phi$ and we get the following σ. $\sigma(A_1) = A_1, \sigma(B_1 \bigcup B_2) = C_2$, $\sigma(C_2) = B_2 \bigcup D_2$ and $\sigma(D_1 \bigcup D_2) = B_1 \bigcup D_1$. We know, $G(A_1 \mid C_2)$ is full (since $G(A \mid C)$ is full). By Isomorphism, $G_Q(A_1 \mid B_2, D_2)$ is full and because of Switching, $G(A_1 \mid D_2)$ is null. By Isomorphism, $G_Q(A_1 \mid \sigma(D_2))$ is null. Thus, $\sigma(D_2) \subset D_1$. Because of Switching, $G(A_1 \mid \sigma(D_2))$ is full and by Isomorphism, $G_Q(A_1 \mid \sigma^2(D_2))$ is full which by Switching gives, $G(A_1 \mid \sigma^2(D_2) \bigcap D_1)$ is null. Let $\sigma^2(D_2) \bigcap D_1 = D_{11}$. Thus $G_Q(A_1 \mid \sigma(D_{11})$ is null and so $\sigma(D_{11}) \subset D_1$. Because of Switching, $G(A_1 \mid \sigma(D_{11}))$ is full and by Isomorphism, $G_Q(A_1 \mid \sigma^2(D_{11}))$ is full. By Switching, $G(A_1 \mid \sigma^2(D_{11}) \bigcap D_{1})$ is null. Let $\sigma^2(D_{11}) \bigcap D_1 = D_{12}$. Continuing thus we have a subset X of D_1 with $\sigma(X) = X$ and $\sigma^2(D_{1i}) \bigcap D_1 = D_{1i+1}$, i = 1, 2, ... such that $G(A_1 \mid X)$ is a bipsc graph satisfying one of the conditions (*i*) - (*iii*) of result C, $G(A_1 \mid D_2), G(A_1 \mid D_{1i})$ are null and $G(A_1 \mid \sigma(D_2)), G(A_1 \mid \sigma(D_{1i}))$ are full.

Thus $G(A_1 \mid D_1, D_2)$ is bipsc since it is the union of null, full and a bipsc graph H. Hence it satisfies one of the conditions (i) - (iii) of result C. Also we know, $G(B_1, B_2 \mid D_1, D_2)$ is null. By Isomorphism, $G_Q(C_2 \mid B_1, D_1)$ is null and because of no Switching, $G(B_1 \mid C_2)$ is null. By Isomorphism, $(G_Q(\sigma(B_1) \mid \sigma(C_2))$ is null i.e. $G_Q(\sigma(B_1) \mid B_2, D_2)$ s null and because of no Switching, $G(B_2 \mid \sigma(B_1))$ is null. By Isomorphism, $G_Q(\sigma(B_2) \mid \sigma^2(B_1))$ is null and because of no Switching, $G(\sigma^2(B_1) \bigcap B_2 \mid \sigma(B_2))$ is null. Continuing thus, either $G(B_2 \mid C_2)$ is null or there exists $Y \subset B_2$ such that $\sigma^2(Y) = Y, G(Y \mid \sigma(Y))$ is symmetric and the remaining subgraph of $G(B_2 \mid C_2)$ is null. As null graph is also symmetric $G(B_1, B_2 \mid C_2)$ is symmetric. Now the complete description of G is as below. $G(A_1 \mid C_1)$ is full, $G(A_1 \mid D_1, D_2) = H_1$, a bipsc graph $G(B_1, B_2 \mid C_2) = H_2$, a symmetric graph and $G(B_1, B_2 \mid D_1, D_2)$ is null. Since H_1 is bipsc, the degree sequence $(d_1, d_2, ..., d_{m_1} \mid e_1, e_2, ..., e_{n-m_2})$ of H_1 satisfies one of the conditions (*i*) - (*iii*) of result C and since H_2 is symmetric, by result D, it has the degree sequence $(f_1, f_2, ..., f_{m_2} \mid f_1, f_2, ..., f_{m_2})$ and hence π satisfies 2^0 of Theorem 1.

Subcase (b) — Let $D_2 = \phi$ in Fig. 2.1. Therefore $B_1 = \phi$ and we get the following σ. $\sigma(A_1 \cup A_2) = A_1 \cup C_1$, $\sigma(B_2) = A_2 \cup C_2$, $\sigma(C_1 \cup C_2) = B_2$ and $\sigma(D_1) = D_1$.

Applying BIST as described in subcase (a), the complete description of G is as below : $G(A_1, A_2 \mid C_1, C_2)$ is full, $G(A_1, A_2 \mid D_1) = H_1$, a bipsc graph, $G(B_2 \mid C_1, C_2) = H_2$, a symmetric graph and $G(B_2 \mid D_1)$ is null. Since H_1 is bipsc, the degree sequence $(d_1, d_2, ..., d_{m_1} \mid e_1, e_2, ..., e_{n-m_2})$ of H_1 satisfies one of the conditions (*i*) - (*iii*) of result C and since H_2 is symmetric, by result D, its degree sequence is of the type $(f_1, f_2, ..., f_{m_2} \mid f_1, f_2, ..., f_{m_2})$. Thus the degree sequence of G i.e. π satisfies 2^0 of Theorem 1.

Case 2.2 — $W_1 \subset \sigma(V_2)$ i.e. $A \subset \sigma(V_2)$. Therefore $B \subset \sigma(V_1)$. As in case 2.1, let $\sigma(V_2) = A \cup C$ and $\sigma(V_1) = B \cup D$ where $V_2 = C \cup D$. As in case 2.1 we breakup A, B, C, D to get the following Fig. 2.2.

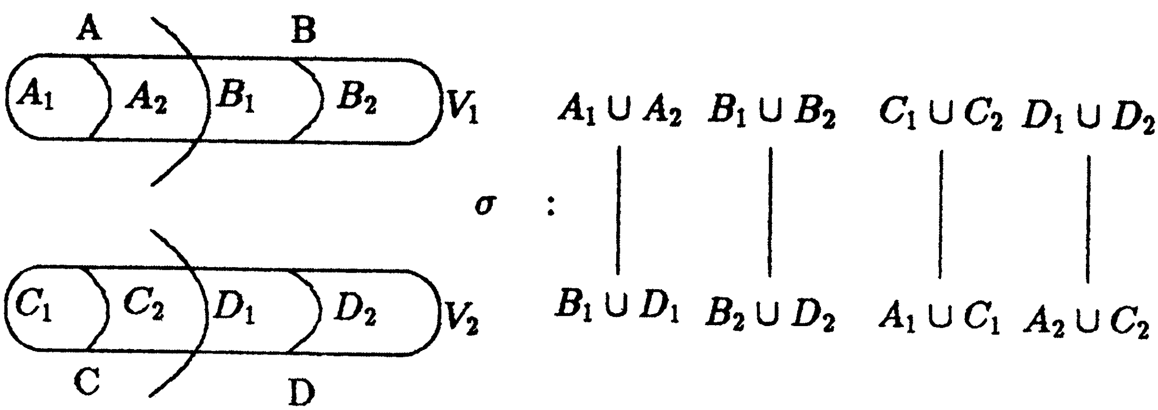


FIG. 2.2

As argued earlier, $G(B \mid D)$ as well as $G_Q(B \mid D)$ are null, $G(C \mid D)$ as well as $G_Q(C \mid D)$ are also null whereas $G_Q(A \mid C)$ ($A \cup C$ being $\sigma(V_2)$) is null and due to switching $G(A \mid C)$ is full. BIST gives four subcases viz. $A_2 = \phi$ and $C_1 = \phi$, $A_2 = \phi$ and $D_1 = \phi$, $B_2 = \phi$ and $C_1 = \phi$, $B_2 = \phi$ and $D_1 = \phi$. We consider them one by one.

Subcase (*a*) — Let $A_2 = \phi$ and $C_1 = \phi$ (see Fig. 2.2). Then σ becomes, $\sigma(A_1) = B_1 \cup D_1$, $\sigma(B_1 \cup B_2) = B_2 \cup D_2$, $\sigma(C_2) = A_1$ and $\sigma(D_1 \cup D_2) = C_2$

We know $G(A_1 \mid C_2)$ is full. By Isomorphism, $G_Q(B_1, D_1 \mid A_1)$ is full and because of

Switching, $G(A_1 \mid D_1)$ is null. Hence, $G_Q(B_1 D_1 \mid \sigma(D_1))$ is null and because of no Switching, $G(B_1 \mid \sigma(D_1))$ is null. Hence, $G_Q(\sigma(B_1) \mid \sigma^2(D_1))$ is null which gives that $\sigma(B_1) \subset D_2$ (since $G_Q(A_1 \mid B_2)$ is full) $\Rightarrow$ $\sigma(B_2) \supset B_2$.

Hence $\sigma(B_2) = B_2$ and $\sigma(B_1) = D_2$. Also, $G(B_1, B_2 \mid D_1, D_2)$ is null. By Isomorphism, $G_Q(B_2, D_2 \mid C_2)$ is null and because of no Switching, $G(B_2 \mid C_2)$ is null. Hence, $G_Q(B_2 \mid A_1)$ is null which gives a contradiction unless $B_2 = \phi$. If $B_2 = \phi$ then applying BIST we have the following σ, $A_1 = \sigma^2(D_1) \cup \sigma^6(D_1) \ldots \cup \sigma^{4k+6}(D_1)$, $B_1 = \sigma^3(D_1) \cup \sigma^7(D_1) \ldots \cup \sigma^{4k+3}(D_1)$, $C_2 = \sigma\ (D_1) \cup\ \sigma^5(D_1)\ \ldots \cup \sigma^{4k+5}(D_1)$, $D_1 = \sigma^{4k+7}(D_1)$ and $D_2 = \sigma^4(D_1) \cup \sigma^8(D_1) \ldots \cup\ \sigma^{4k+4}(D_1)$ where $k = 0, 1, \ldots$ and the degree sequence of G is $((2k+3)\,t)^t, ((2k+2)\,t)^t, \ldots, (2t)^t, (t)^t \mid ((2k+3)\,t)^t, ((2k+2)\,t)^t, \ldots, (t)^t, (0)^t), k \geq 0, t \geq 1$ with $m_1 = (k+2)\,t, m_2 = (k+1)\,t$ and $n = (2k+4)\,t$.This sequence satisfies condition (*i*) or (*ii*) of result C and hence π is of type 1^0 of Theorem 1.

Subcase (b) — Let $D_1 = \phi$ and $B_2 = \phi$ (see Fig. 2.2). Then σ becomes, $\sigma(A_1 \cup A_2) = B_1$, $\sigma(B_1) = D_2$, $\sigma(C_1 \cup C_2) = A_1 \cup C_1$ and $\sigma(D_2) = A_2 \cup C_2$. Applying BIST as in subcase (a) we get the degree sequence of G as $(((2k+3)\,t)^t, ((2k+2)\,t)^t, \ldots, (t)^t, (0)^t \mid ((2k+3)\,t)^t, (2k+2)\,t)^t, \ldots, (2t)^t, (t)^t); k \geq 0, t \geq 1$ with $m_1 = (k+2)\,t = m_2$ and $n = (2k+3)\,t$. This sequence satisfies conditions (*i*) or (*iii*) of result C and hence π is of type 1^0 of Theorem 1.

Subcase (c) — Let $A_2 = \phi$ and $D_1 = \phi$ (see Fig. 2.2). Then σ becomes,

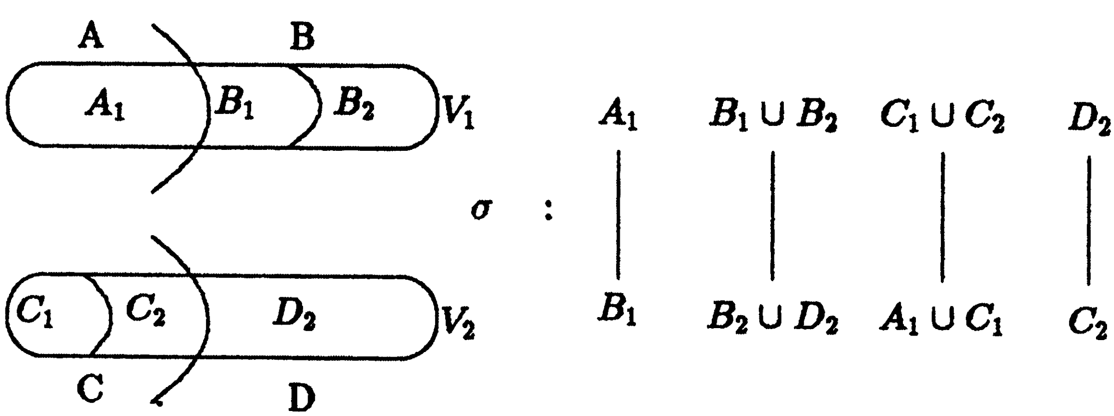


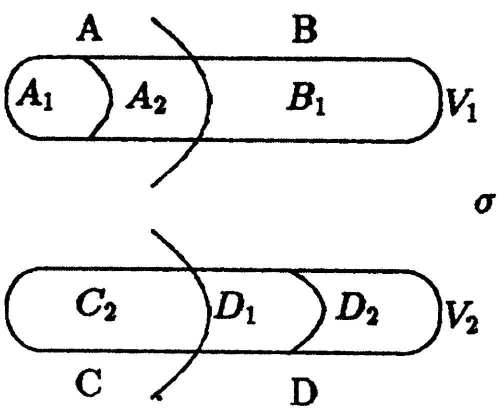
FIG. 2.3

We know, $G(A_1 \mid C_1, C_2)$ is full. Hence, $G_Q(B_1 \mid A_1, C_1)$ is full and because of no Switching, $G(B_1 \mid C_1)$ is full. By Isomorphism, $G_Q(\sigma(B_1) \mid \sigma(C_1)$ is full. Also, we know, $G(B_1, B_2 \mid D_2)$ is null. Hence, $G_Q(B_2, D_2 \mid C_2)$ is null and because of no Switching, $G(B_2 \mid C_2)$ is null. By Isomorphism, $G_Q(\sigma(B_2) \mid \sigma(C_2))$ is null.

Breaking up A_1, B_2, C_1 and D_2 we have four cases which in turn give another four cases and ultimately at some stage we must have $\sigma(B_2^k) = B_2^k$ or D_2^k and $\sigma(C_1^k) = A_1^k$ or C_1^k where A_1^k, B_2^k, C_1^k and D_2^k represent the remaining parts of A_1, B_2, C_1 and D_2 respectively at the kth stage. Thus without loss of generality we consider the following four subcases of the subcase (c).

Subcace (c) (i) — Let $\sigma(B_1) = B_2$ and $\sigma(C_1) = A_1$ (see Fig. 2.3). Therefore $\sigma(B_2) = D_2$ and $\sigma(C_2) = C_1$ and $\sigma = (A_1\, B_1\, B_2\, D_2\, C_2\, C_1)$. Hence we get the following adjacencies in G. $G(A_1 \mid C_1, C_2)$ is full, $G(B_1 \mid C_1)$ is full, $G(B_1 \mid D_2)$ is null and $G(B_2 \mid C_2, D_2)$ is null. If $G(A_1 \mid D_2) = H$ is any bipartite graph then applying BIST $G(D_2 \mid A_1)\, \overline{H}$ which means that H must be a bipsc graph satisfying condition (*iv*) of result C. In fact we observe that the degree sequence of G also satisfies condition (*iv*) of result C and thus π is of type 1^0 of Theorem 1. Applying BIST to the remaining subcases viz. $\sigma(B_1) = B_2$ and $\sigma(C_1) = C_1$; $\sigma(B_1) = D_2$ and $\sigma(C_1) = A_1$; $\sigma(B_1) = D_2$ and $\sigma(C_1) = C_1$ gives a contradiction.

Subcase (*d*) — Let $B_2 = \phi$ and $C_1 = \phi$ (see Fig. 2.2). Then we get the following σ



$$\sigma : \begin{array}{cccc} A_1 \cup A_2 & B_1 & C_2 & D_1 \cup D_2 \\ | & | & | & | \\ B_1 \cup D_1 & D_2 & A_1 & A_2 \cup C_2 \end{array}$$

FIG. 2.4

We know, $G(A_1, A_2 \mid C_2)$ is full. Hence $G_Q(B_1, D_1 \mid A_1)$ is full and due to Switching, $G(A_1 \mid D_1)$ is null. By Isomorphism, $G_Q(\sigma(A_1) \mid \sigma(D_1))$ is null. Also, we know, $G(B_1 \mid D_1, D_2)$ is null. Hence, $G_Q(D_2 \mid A_2, C_2)$ is null and due to Switching, $G(A_2 \mid D_2)$ is full. By Isomorphism, $G_Q(\sigma(A_2) \mid \sigma(D_2))$ is full. Breaking up A_2, B_1, C_2 and D_1 we have four cases which in turn give another four cases and ultimately at some stage we must have either $\sigma(A_1^k) = B_1^k$ or D_1^k and

$\sigma(D_1^k) = A_2^k$ or C_2^k where A_2^k, B_1^k, C_2^k and D_1^k represent the remaining parts of A_2, B_1, C_2 and D_1 respectively at the kth stage. Thus without loss of generality we consider the following four subcases of the subcase (d).

Subcase (d) (*i*) — Let $\sigma(A_1) = D_1$ and $\sigma(D_1) = C_2$ (see Fig. 2.4). Therefore $\sigma(A_2) = B_1$ and $\sigma(D_2) = A_2$ and $\sigma = (A_1, D_1, C_2)(A_2, B_1, D_2)$. The adjacencies in G are as follows; $G(A_1 \mid C_2)$ is full, $G(A_1 \mid D_1)$ is null, $G(A_2 \mid C_2 D_2)$ is full and $G(B_1 \mid D_1, D_2)$ is null. If $G(A_1 \mid D_2) = H$ is any bipartite graph then applying BIST we observe that it is just a bipartite graph and if it has the degree seqeunce $(d_1, d_2, ..., d_{m_1 - m_2} \mid e_1, e_2, ..., e_{m_2})$, then the degree sequence of $\overline{H}$ is $(m_2 - (d_1, d_2, ..., d_{m_1 - m_2}) \mid m_1 - m_2 - (e_1, e_2, ..., e_{m_2}))$. Hence the degree sequence of G is $(m_1 - m_2 + (d_1, d_2, ..., d_{m_1 - m_2}), 2m_1 - m_2 - (e_1, e_2, \dots, e_{m_2}), m_1 - m_2 - (e_1, e_2, ..., e_{m_2}) \mid m_1 + m_2 - (d_1, d_2, ..., d_{m_1 - m_2}), m_2 + (e_1, e_2 ..., e_{m_2}), m_2 - (d_1, d_2, ..., d_{m_1 - m_2}))$. Thus π is of type 3^0 of Theorem 1.

Subcase (*d*) (*ii*) — Let $\sigma(A_1) = B_1$ and $\sigma(D_1) = C_2$ (see Fig. 2.4). Therefore $\sigma(A_2) = D_1$ and $\sigma(D_2) = A_2$ and $\sigma(A_1 B_1 D_2 A_2 D_1 C_2)$. This gives the following adjacencies in $G: G(A_1 \mid C_2, D_2)$ is full, $G(A_1 \mid D_1)$ is null,. $G(A_2 \mid C_2, D_2)$ is full, $G(A_2 \mid D_1)$ is null, $G(B_1 \mid C_2, D_1, D_2)$ is null. This is the same as subscase (d) (i) with H = full and $m_1 = 2m_2$.

Subcase (*d*) (*iii*) — Let $\sigma(A_1) = D_1$ and $\sigma(D_1) = A_2$ (see Fig. 2.4). Therefore $\sigma(A_2) = B_1$ and $\sigma(D_2) = C_2$ and $\sigma(A_1 D_1 A_2 B_1 D_2 C_2)$ and we get the same adjacencies as in subcase (d) (i) with H = null and $m_1 = 2m_2$.

Subcase (*d*) (*iv*) — Let $\sigma(A_1) = B_1$ and $\sigma(D_1) = A_2$ (See Fig. 2.4). Therefore $\sigma(A_2) = D_1$ and $\sigma(D_2) = C_2$ and $\sigma = (A_1 B_1 D_2 C_2)(A_2 D_1)$. BIST gives a contradiction. This completes the proof of Necessity of Theorem 1.

PROOF OF SUFFICIENCY

We take the degree sequences mentioned in Theorem 1 one by one, give a method of constructing a gbipsc realization of each of them, define a map σ from G to G_Q and prove that it is a bipcp.

1^0 — Let π be of type 1^0 of Theorem 1. By result *C*, π is pbipsc. Hence there exists a bipsc realization $G(V_1, V_2)$ of π and a bipcp σ of G onto its bipartite complement $\overline{G}$. With $W_1 = V_1$ and $W_2 = V_2$, $G_Q = \overline{G}\,\overline{G}$, σ is also a bipcp from G to G_Q.

2^0 — Let π be of type 2^0 of Theorem 1. Since $(d_1, d_2, .., d_{m_1} \mid e_1, e_2, ..., e_{n - m_2})$ satisfies one of the conditions (*i*) - (*iii*) of result *C*, there exists a bipsc realization $H_1(A \mid D)$ of $(d_1, d_2, ..., d_{m_1} \mid e_1, e_2, ..., e_{n - m_2})$ and a bipcp σ_1 with $\sigma_1(A) = A$ and $\sigma_1(D) = D$. Again as

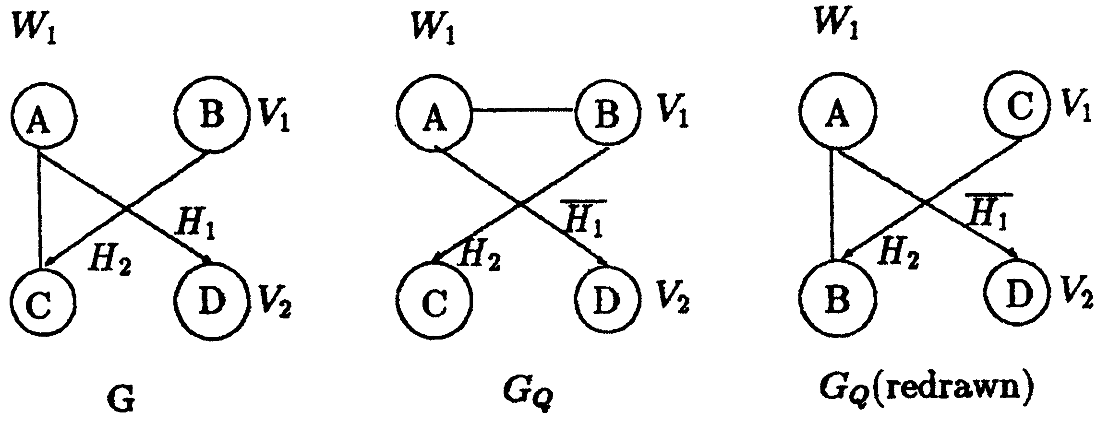


FIG. 2.5

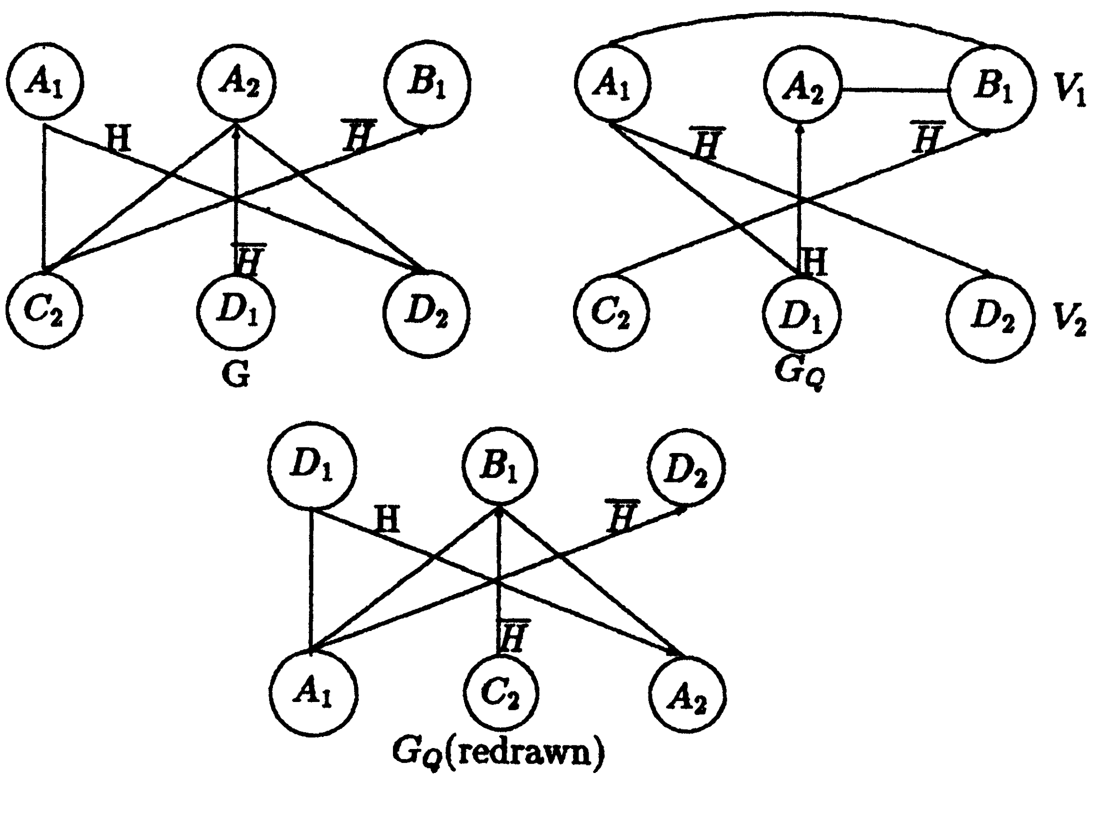


FIG. 2.6

$(f_1, f_2, \ldots, f_{m_2} \mid f_1, f_2, \ldots, f_{m_2})$ satisfies conditions of result D, there exists a corresponding symmetric realization $H_2\,(B \mid C)$ and a permutation σ_2 with $\sigma_2\,(B) = C$ and $\sigma_2\,(C) = B$. A realization G of π, its generalized complement G_Q and a bipcp σ are shown in Fig. 2.5 :

$\sigma = \sigma_1 \cup \sigma_2$. In the figure, H_1 is a bipsc graph and H_2 is a symmetric graph,

3^0 — Let π be of type 3^0 of Theorem 1. A realization G of π, its generalized complement and a bipcp are shown in the Fig. 2.6.

Here H is any bipartite graph and $W_1 = \{A_1, A_2\}$, $W_2 = \{B_1, C_2, D_1, D_2\}$ with $\sigma = (A_1\, D_1\, C_2)\,(B_1\, D_2\, A_2)$. □

3. Proof of Theorem 2

Let $G(V_1, V_2)$ be a gbipsc type II realization of π and $Q = \{W_1, W_2\}$ be a partition of $V_1 \cup V_2$ such that $G \simeq G_Q$. As G is gbipsc of type II, $W_i \cap V_j \neq \phi$ for all $i, j = 1, 2$. For definiteness let $W_1 = \{u_1, u_2, ..., u_{m_1}, v_{n_2+1}, v_{n_2+2}, ..., v_{n_2+n_1}\}$ and $W_2 = \{u_{m_1+1}, u_{m_1+2}, ..., u_{m_1+m_2}, v_1, v_2, ..., v_{n_2}\}$ where $m = m_1 + m_2$ and $n = n_1 + n_2$. For convenience let us denote $V_1 \cap W_1$ by A, $V_1 \cap W_2$ by B, $V_2 \cap W_1$ by C and $V_2 \cap W_2$ by D so that $V_1 = A \cup B$ and $V_2 = C \cup D$, $W_1 = A \cup C$ and $W_2 = B \cup D$. Thus let

$$A = \{u_1, u_2, ..., u_{m_1}\} \quad B = \{u_{m_1+1}, u_{m_1+2}, ..., u_{m_1+m_2}\}$$

$$C = \{v_{n_2+1}, v_{n_2+2}, ..., v_{n_2+n_1}\} \quad D = \{v_1, v_2, ..., v_{n_2}\}$$

By applying BIST, the only possibilities for $K_{m_1 m_2}$ and $K_{n_1 n_2}$ in G are given in the following cases.

Case 3.1 — $G(A \mid D) = K_{m_1 m_2}$ and $G(C \mid B) = K_{n_1 n_2}$. Since $G_Q(A \mid B) = K_{m_1 m_2}$ and $G_Q(C \mid D) = K_{n_1 n_2}$ we get $\sigma = (A)\,(BD)\,(C)$ and since $|B| = |D|$, $m_2 = n_2$. Now the only undetermined graphs in G are $G(A \mid C)$ and $G(B \mid D)$. Let $G(A \mid C) = H_1$. Isomorphism gives $G_Q(A \mid C) \simeq H_1$ and because of no Switching, $G(A \mid C) \sim H_1$. Thus H_1 is a bipartite graph. Let $G(B \mid D) = H_2$. By Isomorphism, $G_Q(D \mid B) \simeq H_2$ and due to no Switching, $G_Q(D \mid B) \simeq H_2$. Thus H_2 is any symmetric graph. Let H_1 have the degree sequence $(d_1, d_2, ..., d_{m_1} \mid e_1, e_2, ..., e_{n_1})$ and H_2 have the degree sequence $(f_1, f_2, ..., f_{m_2} \mid f_1, f_2, ..., f_{m_2})$. Then the degree sequence of G is $(m_2 + (d_1, d_2, ..., d_{m_1}), n_1 + (f_1, f_2, ..., f_{m_2}) \mid m_2 + (e_1, e_2, ..., e_{n_1}), m_1 + (f_1, f_2, ..., f_{m_2}))$. Thus π is of type 4^0 of Theorem 2.

Case 3.2 — $G(D \mid A) = K_{m_1 m_2}$ and $G(B \mid C) = K_{n_1 n_2}$. Since $G_Q(A \mid B) = K_{m_1 m_2}$ and $G_Q(C \mid D) = K_{n_1 n_2}$ we get $\sigma = (A\, B\, C\, D)$ and since $|A| = |B| = |C| = |D|$, $m_1 = m_2 = n_1 = n_2$. Now the only undetermined graphs in G are $G(A \mid C)$ and $G(B \mid D)$. If $G(A \mid C) = H$ then BIST shows that it is any symmetric graph and the degree sequence of G is as in case 3.1 with $H_1 = H_2 = H$, a symmetric graph and $m_1 = m_2 = n_1 = n_2$.

Case 3.3 — $G(A \mid C) = K_{m_1 m_2}$ and $G(D \mid B) = K_{n_1 n_2}$. Since $G_Q(A \mid B) = K_{m_1 m_2}$ and $G_Q(C \mid D) = K_{n_1 n_2}$ we get $\sigma = (A)(B\,D\,C)$ and since $|B| = |D| = |C|$, $m_2 = n_1 = n_2$. BIST shows that the degree sequence of G is as in case 3.1 with H_1 = full, H_2 = full and $m_2 = n_1 = n_2$.

Case 3.4 — $G(C \mid A) = K_{m_1 m_2}$ and $G(B \mid D) = K_{n_1 n_2}$. Since $G_Q(A \mid B) = K_{m_1 m_2}$ $|A| = |B_1| = |C|$ and $G_Q(C \mid D) = K_{n_1 n_2}$ we get $\sigma = (A\,B\,C)(D)$ and since $|A| = |B_1| = |C|$, $m_1 = m_2 = n_1$. BIST shows that the degree sequence of G is as in case 3.1 with H_1 = full, H_2 = full and $m_1 = m_2 = n_1$.

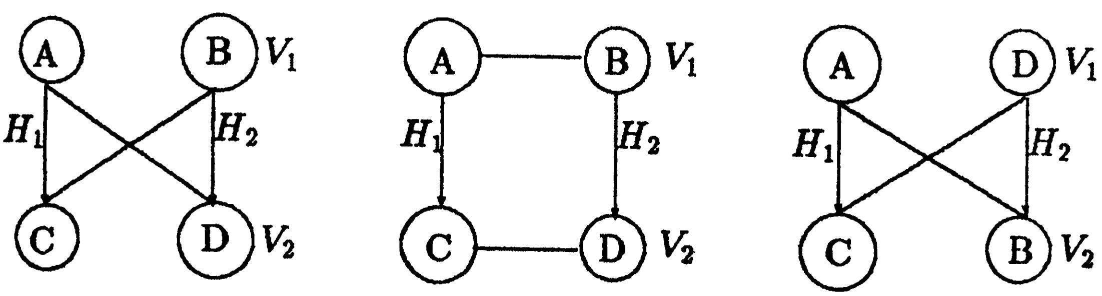
Case 3.5 — $G(A \mid D) = K_{m_1 m_2}$ and $G(B \mid C) = K_{n_1 n_2}$. Since $G_Q(A \mid B) = K_{m_1 m_2}$ and $G_Q(C \mid D) = K_{n_1 n_2}$ we get $\sigma = (A)(B\,C\,D)$ and since $|B| = |C| = |D|$, $m_2 = n_1 = n_2$. BIST shows that the degree sequence of G as in case 3.1 with H_1 = null and H_2 = null and $m_2 = n_1 = n_2$.

Case 3.6 — $G(D \mid A) = K_{m_1 m_2}$ and $G(C \mid B) = K_{n_1 n_2}$. Since $G_Q(A \mid B) = K_{m_1 m_2}$ and $G_Q(C \mid D) = K_{n_1 n_2}$, we get $\sigma = (A\,B\,D)(C)$ and since $|A| = |B| = |D|$, $m_1 = m_2 = n_2$. BIST shows that the degree sequence for G is as in case 3.1 with H_1 = null and H_2 = null and $m_1 = m_2 = n_2$.

The remaining two cases viz. $G(A \mid C) = K_{m_1 m_2}$; $G(B \mid D) = K_{n_1 n_2}$ and $G(C \mid A) = K_{m_1 m_2}$; $G(D \mid B) = K_{n_1 n_2}$ give a contradiction which can be seen easily by applying BIST. Interchanging W_1 and W_2 we get 8 similar cases with no new degree sequences.

This completes the necessity of Theorem 2.

PROOF OF SUFFICIENCY : Let π be of type 4^0 of Theorem 2. The graph G given below is a realization of π and is gbipsc with $W_1 = \{A, C\}$, $W_2 = \{B, D\}$ and $\sigma = (A)(B\,D)(C)$, thus proving that π is pgbipsc of type II.



G $\quad$ G_Q $\quad$ G_Q(redrawn)

H_1 is a bipartite graph whose degree sequence is $(d_1, d_2, ..., d_{m_1} \mid e_1, e_2, ..., e_{n_1})$ and H_2 is a symmetric graph whose degree sequence is $(f_1, f_2, ..., f_{m_2} \mid f_1, f_2, ..., f_{m_2})$. □

4. Procedure to Check if a Given Bipartitioned Sequence is pgbipsc

In section 2, from the proof of sufficiency of Theorem 1, we get a systematic method to construct bipsc type I graphs starting from any bigraphic sequence. Similarly in section 3, from the proof of sufficiency of Theorem 2, we get a systematic method to construct gbipsc type II graphs starting from any bigraphic sequence and the degree sequence of any symmetric graph. All we need to do is construct a graph whose degree sequence is the given bigraphic sequence and/or a symmetric sequence and by adding some more vertices and edges properly we get a supergraph which is gbipsc of type I or II.

To get a realization of the given bigraphic sequence we can use the following procedure which is similar to Havel - Hakimi procedure[8]. Let

$$\pi = (d_1, d_2, ..., d_m \mid e_1, e_2, ..., e_n)$$

where $$n \geq d_1 \geq d_2 \geq \ldots \geq d_m \text{ and } m \geq e_1 \geq e_2 \geq \ldots \geq e_n.$$

(*i*) Construct a new bipartitioned sequence

$$\pi_1' = (d_2 - 1, d_3 - 1, ..., d_{e_1} - 1, d_{e_1+1}, d_{e_1+2}, ..., d_m \mid e_2 - 1, e_3 - 1, ..., e_{d_1} - 1,$$

$$e_{d_1+1}, e_{d_1+2}, ..., e_n).$$

(Note that π is bigraphic iff π_1' is so).

(*ii*) Reorder the terms of π_1' so that they are non increasing and call the resulting bipartition π_1.

(*iii*) Determine the modified bipartition π_2' of π_1 as in step (i) and the reordered partition π_2.

(*iv*) Continue the process as long as some bigraphic sequence is obtained whose realization is known.

To get a realization of a symmetric sequence we use the following construction :

Let $\pi = (d_1, d_2, ..., d_n \mid e_1, e_2, ..., e_n)$ where $n \geq d_1 \geq d_2 \geq \ldots \geq d_n$ and $n \geq e_1 \geq e_2 \geq \ldots \geq e_n$ be a symmetric degree sequence. Let $d_i = d(u_i)$ and $e_i = d(v_i)$. We know that $d_i = e_i$.

Join u_1 to $v_1, v_2, ..., v_{d_1}$. Thus $d(u_1) = d_1$.

Join v_1 to $u_1, u_2, ..., u_{d_1}$. Thus $d(v_1) = d_1$.

Join u_2 to $v_2, v_3, ..., v_{d_2 - 1}$. Thus $d(u_2) = d_2$.

Join v_2 to $u_2, u_3, ..., u_{d_2 - 1}$. Thus $d(v_2) = d_2$. Continuing thus we get a required realization.

To see whether the given sequence is pgbipsc or not we have to check if the given bipartitioned sequence fits into one of the types 1^0, 2^0, 3^0 or 4^0. To do this we give below a step by step procedure and illustrate the same by an example.

Let $\pi = (d_1, d_2, ..., d_m \mid e_1, e_2, ..., e_n)$ be a given bipartitioned sequence.

Step 1 — Check if π is bigraphic or not by checking the conditions of result B. If π is not bigraphic then it is not pgbipsc.

Step 2 — Check if π satisfies the conditions of result C. If it does then π is of type 1^0 of Theorem 1 and hence is pgbipsc of type I.

Step 3 — If $2n - m = 3r$ then choose $d_{i_1}, d_{i_2}, ..., d_{i_r}$ and $e_{i_1}, e_{i_2}, ..., e_{i_{n-2r}}$ such that $(d_{i_1}, d_{i_2}, ..., d_{i_r}) = r + (f_1, f_2, ..., f_r)$ and $(e_{i_1}, e_{i_2}, ..., e_{i_{n-2r}}) = n - 2r + (h_1, h_2, ..., h_{n-2r})$ where $(f_1, f_2, ..., f_2 \mid h_1, h_2, ..., h_{n-2r})$ should be bigraphic. (If such a choice is not possible then π cannot be of type 3^0 of Theorem 1).Once such a choice is made, check if $(d_{i_{r+1}}, d_{i_{r+2}}, ..., d_{i_{3+n-2r}} = r - (h_1, h_2, ..., h_{n-2r})$, $(e_{i_{n-2r+1}}, e_{i_{n-2r+2}}, ..., e_{i_{n-2r+r}}) = n - 2r - (f_1, f_2, ..., f_r)$ and $d_{i_{n-r+1}}, d_{i_{n-r+2}}, ..., d_{i_m}) + n - r + r - (h_1, h_2, ..., h_{n-2r})$, $(e_{i_{n-r+1}}, ..., e_{i_n}) = n - r + n - 2r - (f_1, f_2, ..., f_r)$.

If this is so then π is pgbipsc of type I with

$$W_1 = \{u_{i_1}, u_{i_2}, ..., u_{i_r}, u_{i_{n-r+1}}, u_{i_{n-r+2}}, ..., u_{i_m}\} \text{ where } d(u_{i_k}) = d_{i_k}.$$

Step 4 — If r and s are such that $m - r = n - s$ for smallest r and s then choose $(d_{i_1}, d_{i_2}, ..., d_{i_r}) = m - r + (f_1, f_2, ..., f_r)$ and $(e_{i_1}, e_{i_2}, ..., e_{i_s}) = (h_1, h_2, ..., h_s)$ such that the sequence $(f_1, f_2, ..., f_r \mid h_1, h_2, ..., h_s)$ satisfies conditions of result C. If such a choice of r and s is not possible then check for the next possible values of r and s. If for any values of r and s such a choice of d_i's and e_i,s is not possible then π cannot be of type 2^0 of Theorem 1. If a choice of d_i's and e_i's is made as above and if $(e_{i_{s+1}}, e_{i_{s+2}}, ..., e_{i_n}) = r + (d_{i_{r+1}}, d_{i_{r+2}}, ..., d_{i_m})$ then π is of type 2^0 of Theorem 1 and hence π is pgbisc of type I with $W_1 = \{u_{i_1}, u_{i_2}, .., u_{i_r}\}$ where $d(u_{i_k}) = d_{i_k}$, $k = 1, 2, ..., r$.

Step 5 — If any of $d_1, d_2, ..., d_m, e_1, e_2, ..., e_n$ is 0 then π cannot be of type 4^0 of Theorem 2.

If r and s are such that $m - r = n - s$ for smallest r and s then choose $(d_{i_1}, d_{i_2}, ..., d_{i_r}) = m - r + (f_1, f_2, ..., f_r)$ and $(e_{i_1}, e_{i_2}, ..., e_{i_s}) = m - r + (h_1, h_2, ..., h_s)$ such that $(f_1, f_2, ..., f_r \mid h_1, h_2, ..., h_s)$ is bigraphic and then check if $(d_{i_{r+1}}, d_{i_{r+2}}, ..., d_{i_m}) = s + (k_1, k_2, ..., k_{m-r})$ and $(e_{i_{s+1}}, e_{i_{s+2}}, ..., e_{i_n}) = r + (k_1, k_2, .., k_{n-r})$. If this is so then π is of type 4^0 of Theorem 2 and π is pgibsc of type II with

$W_1 = \{u_{i_1}, u_{i_2}, ..., u_{i_r}, v_{i_1}, v_{i_2}, ..., v_{i_s}\}$ where $d(u_{i_k}) = d_{i_k}$ and $d(v_{i_l}) = e_{i_l}$, $k = 1, 2, ..., r$, $l = 1, 2, ..., s$.

If not repeat the above procedure for different possible choice of r and s. If the procedure fails for all choices of r and s then π cannot be pgbipsc of type II.

Step 6 — If π :s not of any of the above types then π is not pgbipsc.

Illustration — Int he following illustration, for the sake of non triviality we consider a bigraphic sequence which is not pbipsc. Thus step 1 and 2 of the above procedure can be skipped.

$$\pi = (8, 7, 6, 5^3, 2, 1 \mid 7^3, 6, 5, 4, 1^3, 0) \text{ Here } m = 8, n = 10.$$

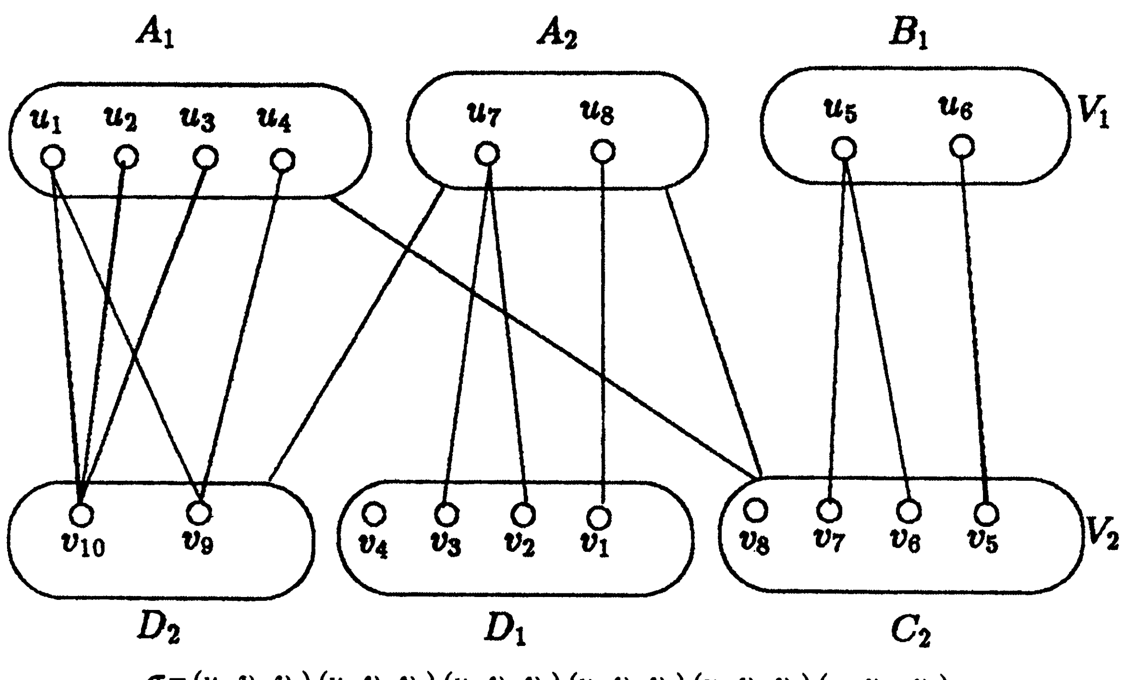
Step 3 — $2n - m = 12 = 3(4)$. Hence $r = 4$. Therefore choose $(d_1, d_2, d_3, d_4) = 4 + (f_1, f_2, f_3, f_4)$ and $(e_1, e_2) = 2 + (h_1, h_2)$ such that $(f_i \mid h_j)$ is bigraphic.

The only choices are (2, 1, 1, 1 | 4, 1) and (2, 1, 1, 1 | 3, 2).

If $(f_i \mid h_j) = (2, 1, 1, 1, 1 \mid 4, 1)$ then there should be 2 of the d_i's such that $(d_5, d_6) = 4 - (h_1, h_2)$ which are not there. Thus $(f_i \mid h_j) = (2, 1, 1, 1 \mid 3, 2)$. Therefore $(d_1, d_2, d_3, d_4) = (6, 5^3)$ and $(e_1, e_2) = (5, 4)$. Now $(d_5, d_6) = 4 - (3, 2) = (1, 2)$ and $(e_3, e_4, e_5, e_6) = 2 - (2, 1, 1, 1) = (0, 1^3)$. Also $(d_7, d_8) = 10 - (h_1, h_2) = (7, 8)$ and $(e_7, e_8, e_9, e_{10}) = 8 - (e_1, e_2, e_3, e_4) = (6, 7^3)$.

Thus π is of type 3^0 of Theorem 1 and hence π is pgbipsc of type I with $W_1 = \{u_1, u_2, u_3, u_4, u_7, u_8\}$ where $d(u_i) = d_i$.

Following is a realization of π.



$$\sigma = (u_1\, v_4\, v_8)\,(u_2\, v_3\, v_7)\,(u_3\, v_2\, v_6)\,(u_4\, v_1\, v_5)\,(u_5\, v_9\, v_7)\,(u_6\, v_{10}\, v_8)$$

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