Indian J. pure appl. Math., **35**(1) : 23-30, January 2004

REFINEMENTS OF HGA INEQUALITIES AND FAN'S INEQUALITY

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(*Received* 5 *November* 2001; *accepted* 2 *January* 2003)

In this paper, we establish a refinement of HGA inequalities and Fan's inequality

Key Words : Inequality of Ky Fan, HGA Inequalities, Arithmetic Mean; Geometric Mean; Harmonic Mean; Convex Function

1. INTRODUCTION

Throughout, let n be a positive integer and $\alpha_i \geq 0$ $(i = 1, ..., n)$ with $\sum_{i=1}^{n} \alpha_i = 1$. Let

$$A_n = \sum_{i=1}^{n} \alpha_i x_i; \; a_n = \sum_{i=1}^{n} \frac{x_i}{n}, \text{ where } x_i \in R, \; i = 1, ..., n,$$

$$G_n = \prod_{i=1}^{n} x_i^{\alpha_i}; \; g_n = \prod_{i=1}^{n} x_i^{\frac{1}{n}}, \text{ where } x_i \geq 0, \; i = 1, ..., n, \text{ and}$$

$$H_n = \left(\sum_{i=1}^{n} \frac{\alpha_i}{x_i} \right)^{-1}; \; h_n = \left(\sum_{i=1}^{n} \frac{1}{nx_i} \right)^{-1}, \text{ where } x_i > 0, \; i = 1, ..., n$$

be the weighted and unweighted arithmetic mean, geometric mean and harmonic mean of $x_1, ..., x_n$. Also, let

$$A_n' = \sum_{i=1}^{n} \alpha_i (1 - x_i); \; a_n' = \sum_{i=1}^{n} \frac{1 - x_i}{n}, \text{ where } x_i \in R, \; i = 1, ..., n,$$

$$G_n' = \sum_{i=1}^{n} (1 - x_i)^{\alpha_i}; \; g_n' = \sum_{i=1}^{n} (1 - x_i)^{\frac{1}{n}}, \text{ where } x_i \in (-\infty, 1], \; i = 1, ..., n, \text{ and}$$

$$H_n' = \left(\sum_{i=1}^{n} \frac{\alpha_i}{1 - x_i} \right)^{-1}; \; h_n' = \left(\sum_{i=1}^{n} \frac{1}{n(1 - x_i)} \right)^{-1}, \text{ where } x_i \in (0, 1) \; i = 1, ..., n$$

be the weighted and unweighted arithmetic mean, geometric mean and harmonic mean of

$1-x_1, \ldots, 1-x_n$. In recent years many interesting inequalities involving these means have been published, see, for example [1], [2], [6], [7], [8]. The classical HGA (harmonic geometric arithmetic) inequalities are known as the following theorem :

***Theorem* A** — *If* $x_i \in (0, \infty)$, $i = 1, \ldots, n$, *then*

$$H_n \le G_n \le A_n \qquad \ldots (1.1)$$

with equality if and only if $x_1 = \ldots = x_n$.

In[3], Chong gave a refinement of the second inequality of HAG inequality as the following theorem.

***Theorem* B** — *Let* $x_1, \ldots, x_n$ *be positive numbers and*

$$s(t) = \prod_{i=1}^{n} \left[t \sum_{j=1}^{n} \alpha_j x_j + (1-t)\, x_i \right]^{\alpha_i}, \quad t \in [0, 1]. \qquad \ldots (1.2)$$

Then $s(t)$ is strictly increasing on [0, 1], unless $x_1 = \ldots. = x_n$ and

$$G_n = s(0) \le s(t) \le s(1) = A_n. \qquad \ldots (1.3)$$

We note that if x_i is replaced by $1 - x_i\, (i = 1, \ldots, n)$ in (1.2), then we have

$$H_n = \frac{1}{s(1)} \le \frac{1}{s(t)} \le \frac{1}{s(0)} = G_n \qquad \ldots (1.4)$$

which is a refinement of the first inequality of HGA inequalities.

In[9], Wang and Yang established the inequalities (1.3) and (1.4) for unweighted mean.

***Theorem* C** — *Assume* $x_i \in (0, \infty)$ $(i = 1, \ldots, n)$ *which do not all coincide. For* $t \in \left[0, \frac{1}{n}\right]$ *let*

$$\alpha(t) = \prod_{i=1}^{n} \left[\frac{1}{x_i} + t \sum_{j=1}^{n} \left(\frac{1}{x_j} - \frac{1}{x_i} \right) \right]^{\frac{1}{n}} \qquad \ldots (1.5)$$

and

$$\beta(t) = \prod_{i=1}^{n} \left[x_i + t \sum_{j=1}^{n} (x_j - x_i) \right]^{\frac{1}{n}} \qquad \ldots (1.6)$$

Then $\alpha(t)$ and $\beta(t)$ are continuous strictly monotonic functions on $\left[0, \frac{1}{n}\right]$ such that

$$h_n = \alpha\left(\frac{1}{n}\right) \le \alpha(t) \le \alpha(0) = g_n = \beta(0) \le \beta(t) \le \beta\left(\frac{1}{n}\right) = a_n. \qquad \ldots (1.7)$$

In 1961, Beckenbach and Bellman [4, p. 25] published a remarkable counterpart of the classical AG (arithmetic-geometric) inequality due to Ky Fan :

$$\frac{g_n}{g'_n} \le \frac{a_n}{a'_n}, \quad \text{where } x_i \in \left(0, \frac{1}{2}\right], i = 1, ..., n, \qquad \text{... (1.8)}$$

with equality if and only if $x_1 = \ldots = x_n$.

Many authors have verified that Fan's inequality holds for weighted mean too, i.e.

$$\frac{G_n}{G'_n} \le \frac{A_n}{A'_n}, \quad \text{where } x_i \in \left(0, \frac{1}{2}\right], i = 1, ..., n, \qquad \text{... (1.9)}$$

with equality if and only if $x_1 = \ldots = x_n$. (see for example[5])

In[9], Wang and Yang gave a refinement of Fan's inequality for unweighted mean as the following theorem :

***Theorem* D** — *Assume* $x_i \in \left(0, \frac{1}{2}\right]$ $(i = 1, ..., n)$ *which do not all coincide. For* $t \in \left[0, \frac{1}{n}\right]$, *let*

$$\gamma(t) = \prod_{i=1}^{n} \left[\frac{1}{x_i} + t \sum_{j=1}^{n} \left(\frac{1}{x_j} - \frac{1}{x_i}\right) - 1\right]^{\frac{-1}{n}} \qquad \text{... (1.10)}$$

and

$$\rho(t) = \frac{\prod_{i=1}^{n} \left[x_i + t \sum_{j=1}^{n} (x_j - x_i)\right]^{\frac{1}{n}}}{\prod_{i=1}^{n} \left[1 - x_i - t \sum_{j=1}^{n} (x_j - x_i)\right]^{\frac{1}{n}}} \qquad \text{... (1.11)}$$

Then $\gamma(t)$ and $\rho(t)$ are continuous strictly monotonic functions on $\left[0, \frac{1}{n}\right]$ such that

$$\frac{h_n}{1 - h_n} = \gamma\left(\frac{1}{n}\right) \le \gamma(t) \le \gamma(0) = \frac{g_n}{g'_n} = \rho(0) \le \rho(t) \le \rho\left(\frac{1}{n}\right) = \frac{a_n}{a'_n}. \qquad \text{... (1.12)}$$

We remark that (1.9) can not be extended to

$$\frac{G_n^{\alpha}}{G_n'^{\beta}} \le \frac{A_n^{\alpha}}{A_n'^{\beta}}, \qquad \text{... (1.13)}$$

where $\alpha, \beta > 0, x_i \in \left(0, \frac{1}{2}\right]$, $i = 1, ..., n$; for example, let $n = 2$, $x_1 = \frac{5}{11}, x_2 = \frac{1}{2}$, $\alpha_1 = \alpha_2 = \frac{1}{2}$, $\alpha = 1, \beta = 2$, then

$$\frac{G_n}{G_n'^2} > \frac{A_n}{A_n'^2}.$$

In next section, we shall prove that the inequality (1.13) holds under some condition.

2. Main Results

***Theorem* 1** — *If I, J are two intervals in $R, f: I \to R$ is a decreasing function with $f(I) \subset J, g: J \to R$ is a continuous strictly increasing function such that $g \circ f$ is convex, $x_i \in I\, (i = 1, ..., n)$ with $\sum_{i=1}^{n} \alpha_i x_i \le z$ where $z \in I$, and if G is defined on [0, 1] by*

$$G(t) = g^{-1}\left[\sum_{i=1}^{n} \alpha_i\, (g \circ f)\, ((1-t)\, x_i + tz)\right], \qquad \text{... (2.1)}$$

then G is decreasing on [0, 1] and

$$f(z) = G(1) \le G(t) \le G(0) = g^{-1}\left[\sum_{i=1}^{n} \alpha_i\, (g \circ f)\, (x_i)\right], 0 \le t \le 1. \qquad \text{... (2.2)}$$

Proof : Since g is strictly increasing and f is decreasing, so that $g \circ f$ is decreasing. Now, using the convexity of $g \circ f$ and the assumption that $\sum_{i=1} \alpha_i x_i \le z$, we have

$$\begin{aligned}(g \circ G)\,(t) &= \sum_{i=1}^{n} \alpha_i\, (g \circ f)\, ((1-t)\, x_i + tz) \\ &\ge (g \circ f)\left[\sum_{i=1}^{n} \alpha_i\, ((1-t)\, x_i + tz)\right] \\ &= (g \circ f)\left[(1-t) \sum_{i=1}^{n} \alpha_i x_i + tz\right] \\ &\ge (g \circ f)\,(z) = (g \circ G)\,(1),\end{aligned} \qquad \text{... (2.3)}$$

for all $t \in [0, 1]$.

We note that the composition of a convex function and a linear function is convex and that a positive constant mulitple of convex function and a sum of convex functions are convex, hence $g \circ G$ is convex on [0, 1]. If $0 \le s < t < 1$, then it follows from the convexity of $g \circ G$ and (2.3) that

$$\frac{(g \circ G)\,(t) - (g \circ G)\,(s)}{t-s} \le \frac{(g \circ G)\,(1) - (g \circ G)\,(t)}{1-t} \le 0 \qquad \text{... (2.4)}$$

which shows that $g \circ G$ is decreasing on [0, 1]. Since g is strictly increasing, $g \circ G$ is decreasing on [0, 1] so that $G = g^{-1} \circ (g \circ G)$ is decreasing on [0, 1]. Hence (2.2) holds. This completes the proof.

***Theorem* 2** — *Let* $\alpha > 0, \beta > 0, c > 0, I = \left(0, \frac{c\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}\right], x_i \in I\ (i = 1, ..., n)$ *and let* $A_n \le z \le \frac{c\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}$, $0 < z' \le H_n$. *Let M (t) and N (t) be defined on [0, 1] by*

$$M(t) = \prod_{i=1}^{n} \frac{((1-t)\,x_i + tz)^{\alpha\,\alpha_i}}{(c-(1-t)\,x_i - tz)^{\beta\,\alpha_i}} \qquad \text{... (2.5)}$$

and

$$N(t) = \prod_{i=1}^{n} \frac{\left(\frac{(1-t)}{x_i} + \frac{t}{z'}\right)^{\alpha\,\alpha_i}}{\left(\frac{(1-t)\,c}{x_i} + \frac{tc}{z'} - 1\right)^{\beta\,\alpha_i}} \qquad \text{... (2.6)}$$

Then $M(t)$ is increasing on [0, 1], $N(t)$ is decreasing on [0, 1]

$$\frac{(z')^{\alpha}}{(c-z')^{\beta}} = N(1) \le N(t) \le N(0) = \frac{\left(\prod_{i=1}^{n} x_i^{\alpha_i}\right)^{\alpha}}{\left(\prod_{i=1}^{n} (c-x_i)^{\alpha_i}\right)^{\beta}}$$

$$= M(0) \le M(t) \le M(1) = \frac{z^{\alpha}}{(c-z)^{\beta}}, \quad t \in [0, 1] \qquad \text{... (2.7)}$$

PROOF : (1) Let $f(x) = \frac{(c-x)^{\beta}}{x^{\alpha}}, x \in I$ and $g(x) = \ln x, x \in (0, \infty)$, $x_i, z \in \left(0, \frac{c\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}\right]$, $(i = 1, ..., n)$ with $\sum_{i=1}^{n} \alpha_i x_i \le z$. Then g is continuous strictly increasing on $(0, \infty)$ and

$$(g \circ f)\,(x) = \beta \ln (c-x) - \alpha \ln x,$$

so that

$$\frac{d^2}{dx^2}(g \circ f)\,(x) = \frac{[\sqrt{\alpha}\,(c-x) - \sqrt{\beta}\,x]\,[\sqrt{\alpha}\,(c-x) + \sqrt{\beta}\,x]}{x^2\,(c-x)^2} > 0$$

and

$$\frac{d}{dx} f(x) = \frac{-\beta x\,(c-x)^{\beta-1} - \alpha\,(c-x)^{\beta}}{x^{\alpha+1}} < 0$$

for $x \in \left(0, \frac{c\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}\right)$. Hence $g \circ f$ is convex on $\left(0, \frac{c\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}\right]$ and f is decreasing on

$\left(0, \frac{c\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}\right]$. By Theorem 1, $G(t)$ is increasing on [0, 1]. Now

$$G(t) = \prod_{i=1}^{n} \frac{(c-(1-t)x_i - tz)^{\beta\alpha_i}}{((1-t)x_i + tz)^{\alpha\alpha_i}} = \frac{1}{M(t)}$$

Hence $M(t)$ is increasing on [0, 1], and

$$\frac{\left(\prod_{i=1}^{n} x_i^{\alpha_i}\right)^{\alpha}}{\left(\prod_{i=1}^{n} (c-x_i)^{\alpha_i}\right)^{\beta}} = M(0) \le M(t) \le M(1) = \frac{z^{\alpha}}{(c-z)^{\beta}}. \qquad \text{... (2.8)}$$

(2) Let $f(x) = x^{\beta-\alpha}(cx-1), x \in \left[\frac{\sqrt{\alpha}+\sqrt{\beta}}{c\sqrt{\alpha}}, \infty\right)$, and $g(x) = \ln x, x \in (0, \infty)$, $y_i, z' \in \left[\frac{\sqrt{\alpha}+\sqrt{\beta}}{c\sqrt{\alpha}}, \infty\right)$ $(i = 1, ..., n)$ with $\sum_{i=1}^{n} \alpha_i y_i \le z'$. Then g is continuous strictly increasing on $(0, \infty)$ and

$$(g \circ f)(x) = (\beta-\alpha)\ln x + \ln(cx-1),$$

so that
$$\frac{d^2}{dx^2}(g \circ f)(x) = \frac{\alpha(cx-1)^2 + \beta(2cx-1)}{x^2(cx-1)^2} > 0$$

and
$$\frac{d}{dx} f(x) = x^{\beta-\alpha-1}(cx-1)^{-\beta-1}[-\beta-\alpha(x-1)] < 0$$

for $x \in \left(\frac{\sqrt{\alpha}+\sqrt{\beta}}{c\sqrt{\alpha}}, \infty\right)$. Hence $g \circ f$ is convex on $\left[\frac{\sqrt{\alpha}+\sqrt{\beta}}{c\sqrt{\alpha}}, \infty\right)$ and f is decreasing on $\left[\frac{\sqrt{\alpha}+\sqrt{\beta}}{c\sqrt{\alpha}}, \infty\right)$. By Theorem 1,

$$G(t) = \prod_{i=1}^{n} \frac{((1-t)y_i + tz')^{(\beta-\alpha)\alpha_i}}{((1-t)cy_i + tcz' - 1)^{\beta\alpha_i}}$$

is decreasing on [0, 1].

If $x_i \in \left(0, \frac{c\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}\right]$ $(i = 1, ..., n)$ and $0 < z = \frac{1}{z'} \le \left(\sum_{i=1}^{n} \frac{\alpha_i}{x_i}\right)^{-1}$ then

$$y_i = \frac{1}{x_i},\ z' = \frac{1}{z} \in \left[\frac{\sqrt{\alpha}+\sqrt{\beta}}{c\sqrt{\alpha}}, \infty\right) \ (i = 1, ..., n)$$

and

$$N(t) = \prod_{i=1}^{n} \frac{\left(\frac{(1-t)}{x_i} + \frac{t}{z'}\right)^{(\beta-\alpha)\,\alpha_i}}{\left(\frac{(1-t)\,c}{x_i} + \frac{tc}{z'} - 1\right)^{\beta\,\alpha_i}} = G\,(t)$$

is decreasing on [0, 1], so that

$$\frac{(z')^{\alpha}}{(c-z')^{\beta}} = N(1) \le N(t) \le N(0) = \frac{\left(\prod_{i=1}^{n} x_i^{\alpha_i}\right)^{\alpha}}{\left(\prod_{i=1}^{n} (c-x_i)^{\alpha_i}\right)^{\beta}}\,. \qquad \text{... (2.9)}$$

This completes the proof.

Remark 2.1 : In Theorem 2, let $c = 1$. Then

$$M(t) = \prod_{i=1}^{n} \frac{((1-t)x_i + tz)^{\alpha\alpha_i}}{(1-(1-t)x_i - tz)^{\beta\alpha_i}}$$

is increasing on [0, 1], and

$$N(t) = \prod_{i=1}^{n} \frac{\left(\frac{(1-t)}{x_i} + \frac{1}{z'}\right)^{(\beta-\alpha)\alpha_i}}{\left(\frac{(1-t)}{x_i} + \frac{1}{z'} - 1\right)^{\beta\,\alpha_i}}$$

is decreasing on [0, 1], so that

$$\frac{(z')^{\alpha}}{(c-z')^{\beta}} = N\,(1) \le N\,(t) \le N\,(0) = \frac{G_n^{\alpha}}{G_n'^{\beta}} = M\,(0) \le M\,(t) \le M\,(1) = \frac{z^{\alpha}}{(1-z)^{\beta}}. \qquad \text{... (2.10)}$$

If we choose $z' = H_n$ and $z = A_n$, then

$$\frac{H_n^{\alpha}}{(1-H_n)^{\beta}} = N\,(1) \le N\,(t) \le N\,(0) = \frac{G_n^{\alpha}}{G_n'^{\beta}} = M\,(0) \le M\,(t) \le M\,(1)\,\frac{A_n^{\alpha}}{A_n'^{\beta}} \qquad \text{... (2.11)}$$

where $x_i \in \left(0, \frac{\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\beta}}\right]$, $i = 1, \ldots, n$. We note that (1.9) is a special case of (2.11) when $\alpha = \beta = 1$.

Remark 2.2 : In Theoem 2, if we choose $\alpha=\beta=c=1$, $\alpha_i=\frac{1}{n}$, $x_i \in \left(0, \frac{1}{2}\right]$ not all coincide $(i = 1, ..., n)$, and let $z=\sum_{i=1}^{n} \frac{x_i}{n}$ and $z'=\left(\sum_{i=1}^{n} \frac{1}{nx_i}\right)^{-1}$. Then $N(nt)=\gamma(t)$ and $M(nt)=\rho(t)$, $t \in \left[0, \frac{1}{n}\right]$ where $\gamma(t)$ and $\rho(t)$ are defined as in (1.10) and (1.11). Hence, Theorem *D* is a special case of Theorem 2.

Remark 2.3 : The inequalities (1.3) and (1.4) can be deduced from (2.7) by taking $\alpha=1, z'=H_n', z=A_n$ and $\beta \to 0$.

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