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OSCILLATION OF SOLUTIONS OF A CLASS OF NONLINEAR NEUTRAL PARTIAL DIFFERENTIAL EQUATIONS*

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New oscillation criteria are established for a neutral partial functional differential equation

$$\frac{\partial}{\partial t}\left[p(t)\frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{l}\lambda_i(t)\,u(x,t-\tau_i)\right)\right]$$

$$= a(t)\,\Delta u(x,t)+\sum_{k=1}^{s}a_k(t)\,\Delta u(x,t-\rho_k(t))-q(x,t)\,f(u(x,t))$$

$$-\sum_{j=1}^{m}q_j(x,t)\,f_j(u(x,t-\sigma_j)),\quad (x,t)\in\Omega\times R_+\equiv G,$$

where Δ is the Laplacian in Euclidean N-space R^N, $R_+=(0,\infty)$ and Ω is a bounded domain in R^N with a piecewise smooth boundary $\partial\Omega$. Our results are of a high degree of generality and sharper than many previous results.

Key Words : Oscillation; Neutral Type; Partial Differential Equation

1. INTRODUCTION

Recently the qualitative behaviour of solutions of partial differential equations with deviating arguments has received much attention. We mention here the works in [1-9] concerning oscillatory properties of solutions of some parabolic equations and some hyperbolic equations with deviating arguments.

In the present paper we consider the oscillatory behaviour of solutions of the neutral partial functional differential equation of the form

$$\frac{\partial}{\partial t}\left[p(t)\frac{\partial}{\partial t}\left((x,t)+\sum_{i=1}^{l}\lambda_i(t)\,u(x,t-\tau_i)\right)\right]$$

$$= a(t)\,\Delta u(x,t)+\sum_{k=1}^{s}a_k(t)\,\Delta u(x,t-\rho_k(t))-q(x,t)\,f(u(x,t))$$

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$$-\sum_{j=1}^{m} q_j(x,t) f_j(u(x,t-\sigma_j)), \quad (x,t) \in \Omega \times R_+ \equiv G, \qquad \text{... (1.1)}$$

with the boundary condition

$$\frac{\partial u(x,t)}{\partial \nu} + g(x,t)\, u(x,t) = 0, \quad (x,t) \in \partial\Omega \times R_+ \equiv G, \qquad \text{... (1.2)}$$

or

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times R_+ \equiv G. \qquad \text{... (1.3)}$$

Here Δ is the Laplacian in Euclidean N-space R^N, $R_+ = (0, \infty)$ and Ω is a bounded domain in R^N with a piecewise smooth boundary $\partial\Omega$. ν denotes the unit exterior normal vector to $\partial\Omega$, and $g(x, t)$ is a nonnegative continuous function on $\partial\Omega \times R_+$.

Throughout this paper we assume that the following conditions hold :

(A1) $p \in C^1(R_+, R_+)$, $\lim\limits_{t \to \infty} \int_{t_0}^{\infty} \frac{1}{p(s)} ds = \infty$, $\quad t_0 > 0$;

(A2) $\lambda_i \in C^2(R_+, R_+)$, $0 \le \sum_{i=1}^{l} \lambda_i(t) \le 1$, and, the numbers τ_1 are nonnegative real constants, $i \in I_l = \{1, 2, ..., l\}$;

(A3) $q, q_j \in C(\overline{G}, R_+)$, $q(t) = \min\limits_{x \in \overline{\Omega}} q(x,t)$ and $q_j(t) = \min\limits_{x \in \overline{\Omega}} q_j(x,t)$, $j \in I_m = \{1, 2, ..., m\}$;

(A4) $a, a_k \in C(R_+, R_+)$, $\rho_k \in C(R_+, R_+)$, $\lim\limits_{t \to \infty} (t - \rho_k(t)) = \infty$, σ_j are nonnegative constants, $j \in I_m$, $k \in I_s = \{1, 2, ..., s\}$;

(A5) $f, f_j \in C(R, R)$ are convex in R_+ with $f(u)/u \ge \alpha > 0$, and $f_j(u)/u \ge \alpha_j > 0$ for $u \ne 0$ where α, α_j are positive constants for $j \in I_m$.

We refer to these five conditions collectively as conditions (A).

We need the following definitions.

Definition 1.1 — A function $u \in C^2(G) \cup C^1(\overline{G})$ is called a solution of the problem (1.1), (1.2) or (1.1), (1.3)) if it satisfies (1.1) in the domain G and satisfies the corresponding boundary condition.

Definition 1.2 — A solution u of the problem (1.1), (1.2) (or (1.1), (1.3)) is called oscillatory in the domain G if for each positive number b there exist a point $(x_0, t_0) \in \Omega \times [b, \infty)$ such that $u(x_0, t_0) = 0$ holds.

In this paper, by using the generalized Riccati technique and the averaging technique and by considering the function $H(t, s)\, k(s)$, which may not have a nonpositive partial derivative on

D_0 with respect to the second variable, we relax the assumption $(\partial H(t, s)/\partial s) \le 0$ on D_0 in[6, 7], etc., and obtain new oscillation criteria for solutions of the problem (1.1), (1.2) and (1.1), (1.3). By choosing appropriate functions H, k and ϕ, we present a series of explicit oscillation criteria. Thus, the results of this paper extend, improve and unify a number of existing results.

The paper is organized as follows.

In Section 2, we first establish oscillation criteria which extend and improve the results of Li *et al.* [6, Theorems 2.1 and 3.1] and the relevant results of Cui *et al.*[7]

However, there results in Section 2, [2-9] and the references therein as well as most known oscillation criteria involve p and integral of q_{j0} and hence require the information q_{j0} on the entire half-line $[t_0, \infty)$; see [2-9] and the references cited there.

The second purpose of this paper is to obtain interval oscillation criteria of solutions of the problem (1.1), (1.2) and problem (1.1), (1.3) making use of the technique similar to what exploited by Philos[11] and Kong[12] for second order linear differential equation. Some new interval oscillation criteria established in Section 3 for neutral partial differential eq. (1.1) with (1.2) and (1.1) with (1.3) are different from most known ones in the sense that they are based on the information only on a sequence of subintervals of $[t_0, \infty)$, rather than on the whole half-line. Our results (see Theorems 3.1-3.6) complement a number of existing results and handle the case which are not covered by known criteria in [2-12] and others.

In Section 4, several examples that show upon the importance of our results are included.

2. Oscillation Results

Our main results in this section are the following theorems and corollaries.

In the statements and proofs we frequently use the subsets $D_0 = \{(t, s) : t > s \ge t_0\}$ and $D = \{(t, s) : t \ge s \ge t_0\}$.

***Theorem* 2.1** — *Let the condition (A) hold. Assume $H \in C(D; R)$ satisfies the conditions*

(i) $H(t, t) = 0$ for $t \ge t_0$; $H(t, s) > 0$ for $t > s \ge t_0$;

(ii) H has a continuous and nonpositive partial derivative on D with respect to the second variable; and

(iii) there exist $h \in C(D, R)$ and $k \in C^1(R_+, R_+)$ such that

$$-\frac{\partial (H(t, s)\, k(s))}{\partial s} = h(t, s) \text{ for all } (t, s) \in D.$$

If there exists a function $\phi \in C^1[t_0, \infty)$ and there exists $j_0 \in I_m$ such that

$$\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[H(t, s)\, k(s)\, \psi(s) - \frac{p(s - \sigma_{j0})\, \Phi(s)}{4H(t, s)\, k(s)} h^2(t, s) \right] ds = \infty, \quad \ldots (2.1)$$

where $$\Phi(s) = \exp\left\{-2\int^{s} \phi(\xi)\,d\xi\right\}$$

and $$\psi(s) = \Phi(s)\left\{\alpha_{j0}\, q_{j0}(s)\left[1 - \sum_{i=1}^{l} \lambda_i(s-\sigma_{j0})\right] + p(s-\sigma_{j0})\,\phi^2(s) - [p(s-\sigma_{j0})\,\phi(s)]'\right\}, \quad \ldots (2.2)$$

then

(i) every solution $u(x, t)$ of the problem (1.1), (1.2) is oscillatory in G; and

(ii) every solution $u(x, t)$ of the problem (1.1), (1.3) is oscillatory in G.

PROOF (I): Suppose to the contrary that there is a solution $u(x, t)$ of the problem (1.1), (1.2) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Without loss of generality we may assume that $u(x, t) > 0, u(x, t-\tau_i) > 0$, $u(x, t-\rho_k(t)) > 0$, and $u(x, t-\sigma_j) > 0$ in $\Omega \times [t_1, \infty), t_1 \geq t_0$, $i \in I_l, k \in I_s, j \in I_m$.

Integrating (1.1) with respect to x over the domain Ω, we have

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\int_{\Omega} u(x,t)\,dx + \sum_{i=1}^{l} \lambda_i(t)\int_{\Omega} u(x,t-\tau_i)\,dx\right)\right]$$

$$= a(t)\int_{\Omega} \Delta u(x,t)\,dx + \sum_{k=1}^{s} a_k(t)\int_{\Omega} \Delta u(x,t-\rho_k(t))\,dx - \int_{\Omega} q(x,t) f(u(x,t))\,dx$$

$$- \sum_{j=1}^{m}\int_{\Omega} q_j(x,t) f_j(u(x,t-\sigma_j))\,dx, \quad t \geq t_1. \quad \ldots (2.3)$$

From Green's formula and boundary condition (1.2), it follows that

$$\int_{\Omega} \Delta u(x,t)\,dx = \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial \nu}\,dS = -\int_{\partial\Omega} g(x,t)\,u(x,t)\,dS \leq 0, t \geq t_1, \quad \ldots (2.4)$$

and $$\int_{\Omega} \Delta u(x,t-\rho_k(t))\,dx = \int_{\partial\Omega} \frac{\partial u(x,t-\rho_k(t))}{\partial \nu}\,dS$$

$$= -\int_{\partial\Omega} g(x,t-\rho_k(t))\,u(x,t-\rho_k(t))\,dS \leq 0, \quad t \geq t_1, k \in I_s \quad \ldots (2.5)$$

where dS is the surface element on $\partial\Omega$. Moreover from (A3) and (A5), and Jensen's inequality, we have

$$\int_{\Omega} q(x,t) f(u(x,t))\,dx \geq q(t)\int_{\Omega} f(u(x,t))\,dx$$

$$\geq q(t)\left(\int_{\Omega} dx\right) f\left[\int_{\Omega} u(x,t)\,dx\left(\int_{\Omega} dx\right)^{-1}\right],\quad t \geq t_1, \qquad \text{... (2.6)}$$

and
$$\int_{\Omega} q_j(x,t) f_j(u(x,t-\sigma_j))\,dx \geq q_j(t)\int_{\Omega} f_j(u(x,t-\sigma_j))\,dx$$

$$\geq q_j(t)\left(\int_{\Omega} dx\right) f_j\left[\int_{\Omega} u(x,t-\sigma_j)\,dx\left(\int_{\Omega} dx\right)^{-1}\right],\quad t \geq t_1. \qquad \text{... (2.7)}$$

Define
$$V(t) = \frac{1}{|\Omega|}\int_{\Omega} u(x,t)\,dx,\quad t \in t_1, \qquad \text{... (2.8)}$$

where
$$|\Omega| = \int_{\Omega} dx.$$

In view of (2.4)-(2:8) and (2.3), we obtain

$$\frac{d}{dt}\left[\left(p(t)\frac{d}{dt}\left(V(t) + \sum_{i=1}^{l} \lambda_i(t)V(t-\tau_i)\right)\right] + q(t)f(V(t)\right)$$

$$+\sum_{j=1}^{m} q_j(t) f_j(V(t-\sigma_j)) \leq 0,\quad t \geq t_1. \qquad \text{... (2.9)}$$

Now let $Z(t) = V(t) + \sum_{i=1}^{l} \lambda_i(t)V(t-\tau_i)$. We have $Z(t) > 0$ and $[p(t)Z'(t)] < 0$ for $t \geq t_1$. hence, $p(t)Z'(t)$ is a decreasing function in the open interval $[t_1, \infty)$. We claim that $p(t)Z'(t) > 0$ for $t \geq t_1$. In fact, if $p(t)Z'(t) < 0$ for $t \geq t_1$, there exist a $T > t_1$ such that $p(T)Z'(T) < 0$. This implies that

$$Z(t) \leq \frac{p(T)Z'(T)}{p(t)} \text{ for } t \geq T,$$

and
$$Z(t) - Z(T) \leq p(T)Z'(T)\int_{T}^{t} \frac{ds}{p(s)},\quad t \geq T.$$

Therefore, $\lim_{t\to\infty} Z(t) = -\infty$, which contradicts the fact that $Z(t) > 0$.

By (A) and (2.9), we have for j_0 in (2.1) that

$$[p(t)z'(t)]' + q_{j_0}(t) f_{j_0}(V(t-\sigma_{j_0}) \leq 0,\quad t \geq t_1. \qquad \text{... (2.10)}$$

or
$$[p(t)\,z'(t)]' + \alpha_{j_0} q_{j_0}(t) f_{j_0}\left[Z(t-\sigma_{j_0}) - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0})\, V(t-\tau_i-\sigma_{j_0}) \right] \le 0, \quad t \ge t_1.$$

Since $Z(t) \ge V(t)$, $Z(t)$ is increasing, it follows that

$$[p(t)\,z'(t)]' + \alpha_{j_0} q_{j_0}(t)\left[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0}) \right] Z(t-\sigma_{j_0}) \le 0, \quad t \ge t_1.$$

Let
$$W(t) = \Phi(t)\left\{ \frac{p(t)\,Z'(t)}{Z(t-\sigma_{j_0})} + p(t-\sigma_{j_0})\,\phi(t) \right\}. \qquad \text{... (2.11)}$$

Then $W'(t) \le -2\,\phi(t)\,W(t) + \Phi(t)$

$$\left\{ -\alpha_{j_0} q_{j_0}(t)\left[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma j_0) \right] - \frac{p(t)\,Z'(t)\,Z'(t-\sigma_{j_0})}{Z^2(t-\sigma_{j_0})} + [p(t-\sigma_{j_0})\,\phi(t)]' \right\}.$$

From the fact that $p(t)\,Z'(t)$ is decreasing, we get

$$p(t)\,Z'(t) \ge p(t-\sigma_{j_0})\,Z'(t-\sigma_{j_0}), \text{ for } t \ge t_1.$$

Thus

$$W'(t) \le -2\,\phi(t)\,W(t) + \Phi(t)$$

$$\left\{ -\alpha_{j_0} q_{j_0}(t)\left[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0}) \right] - \frac{1}{p(t-\sigma_{j_0})}\left(\frac{p(t)\,Z'(t)}{Z^2(t-\sigma_{j_0})} \right)^2 + [p(t-\sigma_{j_0})\,\phi(t)]' \right\}$$

$$= -2\,\phi(t)\,W(t) + \Phi(t)$$

$$\left\{ -\alpha_{j_0} q_{j_0}(t)\left[1 - \sum_{i=1}^{l} \lambda_i(t-\sigma_{j_0}) \right] - \frac{1}{p(t-\sigma_{j_0})}\left(\frac{W(t)}{\Phi(t)} - p(t-\sigma_{j_0})\,\phi(t) \right)^2 + [p(t-\sigma_{j_0})\,\phi(t)]' \right\}$$

$$= -\psi(t) - \frac{W^2(t)}{p(t-\sigma_{j_0})\,\Phi(t)}, \qquad \text{... (2.12)}$$

where

$$\psi(t) = \Phi(t)\left\{ \alpha_{j_0} q_{j_0}(t)\left[1 - \sum_{i=1}^{i} \lambda_i(t-\sigma_{j_0}) \right] + p(t-\sigma_{j_0})\,\phi^2(t) - [p(t-\sigma_{j_0})\,\phi(t)]' \right\}.$$

Multiplying (2.12), with t replaced by s, by $H(t,s)\,k(s)$ and integrating from t_2 to t $(t \ge t_2 \ge t_1)$, after simple computation, we have

$$\int_{t_2}^{t} H(t,s)\,k(s)\,\psi(s)\,ds \le H(t,t_2)\,k(t_2)\,W(t_2) + \int_{t_2}^{t} \frac{\partial}{\partial s}\,(H(t,s)\;(s))\,W(s)\,ds$$

$$-\int_{t_2}^{t} H(t,s)\,k(s)\,\frac{W^2(s)}{p(s-\sigma_{j_0})\,\Phi(s)}\,ds$$

$$= H(t,t_2)\,k(t_2)\,W(t_2) - \int_{t_2}^{t} h(t,s)\,W(s)\,ds - \int_{t_2}^{t} H(t,s)\,k(s)\,\frac{W^2(s)}{p(s-\sigma_{j_0})\,\Phi(s)}\,ds$$

$$= H(t,t_2)\,k(t_2)\,W(t_2) + \frac{1}{4}\int_{t_2}^{t} \frac{p(s-\sigma_{j_0})\,\Phi(s)}{H\,t,s)\,k(s)}\,h^2(t,s)\,ds$$

$$-\int_{t_2}^{t}\left[\sqrt{\frac{H(t,s)\,k(s)}{p(s-\sigma_{j_0})\,\Phi(s)}}\,W(s) + \frac{1}{2}\sqrt{\frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(t,s)\,k(s)}}\,h(t,s)\right]^2 ds.$$

That is,

$$\int_{t_2}^{t}\left[H(t,s)\,k(s)\,\psi(s) - \frac{1}{4}\,\frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(t,s)\,k(s)}\,h^2(t,s)\right]ds \le H(t,t_2)\,k(t_2)\,W(t_2)$$

$$-\int_{t_2}^{t}\left[\sqrt{\frac{H(t,s)\,k(s)}{p(s-\sigma_{j_0})\,\Phi(s)}}\,W(s) + \frac{1}{2}\sqrt{\frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(t,s)\,k(s)}}\,h(t,s)\right]^2 ds. \qquad \text{... (2.13)}$$

Using property (*ii*), we deduce from (2.13) that for every $t \ge t_1$

$$\int_{t_1}^{t}\left[H(t,s)\,k(s)\,\psi(s) - \frac{1}{4}\,\frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(t,s)\,k(s)}\,h^2(t,s)\right]ds$$

$$\le H(t,t_1)\,k(t_1)\,W(t_1) \le H(t,t_1)\,k(t_1)\,|\,W(t_1)\,| \le h(t,t_0)\,k(t_1)\,|\,W(t_1)\,|.$$

Again applying property (*ii*), it follows that

$$\int_{t_0}^{t}\left[H(t,s)\,k(s)\,\psi(s) - \frac{1}{4}\,\frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(t,s)\,k(s)}\,h^2(t,s)\right]ds$$

$$= \int_{t_0}^{t_1}\left[H(t,s)\,k(s)\,\psi(s) - \frac{1}{4}\,\frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(t,s)\,k(s)}\,h^2(t,s)\right]ds$$

$$+\int_{t_1}^{t}\left[H(t,s)\,k(s)\,\psi(s)-\frac{1}{4}\frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(t,s)\,k(s)}h^2(t,s)\right]ds$$

$$\le H(t,t_0)\left[\int_{t_0}^{t_1}|\,k(s)\,\psi(s)\,|\,ds+k(t_1)\,|\,W(t_1)\,|\right],$$

for every $t \ge t_1$. It follows that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\,k(s)\,\psi(s)-\frac{1}{4}\frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(t,s)\,k(s)}h^2(t,s)\right]ds$$

$$\le H(t,t_0)\left[\int_{t_0}^{t_1}|\,k(s)\,\psi(s)\,|\,ds+k(t_1)\,|\,W(t_1)\,|\right]<\infty.$$

This contradicts (2.1). The proof of (I) is complete.

(II) To prove (II), the following fact will be used, see[1].

The smallest eigenvalue η_0 of the Dirichlet problem

$$\left.\begin{array}{ll}\Delta u(x)+\eta\,u(x)=0 & in\ \ \Omega\\ u(x)=0 & on\ \ \partial\Omega,\end{array}\right\} \qquad \ldots (2.14)$$

is positive and the corresponding eigenfunction $\varphi(x)$ is positive in Ω.

Now we prove part (II). Suppose to the contrary that there is a solution $u(x, t)$ of the problem (1.1), (1.3) which has no zero in $\Omega\times[t_0,\infty)$ for some $t_0>0$. Without loss of generality we may assume that $u(x,t)>0, u(x,t-\tau_i)>0$, $u(x,t-\rho_k(t))>0$, and $u(x,t-\sigma_j)>0$ in $\Omega\times[t_1,\infty)$, $t_1\ge t_0$, $i\in I_l$, $k\in I_s$, $j\in I_m$.

Multiplying (1.1) by $\varphi(x)>0$ and integrating with respect to x over the domain Ω, we obtain for $t \ge t_1$

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(\int_{\Omega}u(x,t)\,\varphi(x)\,dx+\sum_{i=1}^{l}\lambda_i(t)\int_{\Omega}u(x,t-\tau_i)\,\varphi(x)\,dx\right)\right]$$

$$=a(t)\int_{\Omega}\Delta u(x,t)\,\varphi(x)\,dx+\sum_{k=1}^{s}a_k(t)\int_{\Omega}\Delta u(x,t-\rho_k(t))\,\varphi(x)\,dx$$

$$-\int_{\Omega}q(x,t)f(u(x,t))\,\varphi(x)\,dx-\sum_{j=1}^{m}\int_{\Omega}q_j(x,t)f_j(u(x\,t-\sigma_j))\,\varphi(x)\,dx \qquad \ldots (2.15)$$

From Green's formula and boundary condition (1.3), it follows that

$$\int_{\Omega} \Delta u(x,t)\,\varphi(x)\,dx = \int_{\Omega} u(x,t)\,\Delta\varphi(x)\,dx = -\eta_0 \int_{\Omega} u(x,t)\,\varphi(x)\,dx \le 0,\ t \ge t_1. \quad (2.16)$$

and
$$\int_{\Omega} \Delta u(x, t-\rho_k(t))\,\varphi(x)\,dx = \int_{\Omega} u(x, t-\rho_k(k))\,\Delta\varphi(x)dx$$

$$= -\eta_0 \int_{\Omega} u(x, t-\rho_k(k))\,\varphi(x)dx \le 0, \quad t \ge t_1,\ k \in I_s \quad \dots (2.17)$$

Moreover from (A3), (A5), and Jensen's inequality, we have

$$\int_{\Omega} q(x,t) f(u(x,t))\,\varphi(x)\,dx \ge q(t) \int_{\Omega} f(u(x,t))\,\varphi(x)\,dx$$

$$\ge q(t)\left(\int_{\Omega} \varphi(x)\,dx\right) f\left[\int_{\Omega} u(x,t)\,\varphi(x)\,dx \left(\int_{\Omega} \varphi(x)\,dx\right)^{-1}\right],\ t \ge t_1, \quad \dots (2.18)$$

and
$$\int_{\Omega} q_j(x,t) f_j(u(x, t-\sigma_j))\,\varphi(x)\,dx \ge q_j(t) \int_{\Omega} f_j(u(x, t-\sigma_j))\,\varphi(x)\,dx$$

$$\ge q_j(t)\left(\int_{\Omega} \varphi(x)\,dx\right) f_j\left[\int_{\Omega} u(x, t-\sigma_j)\,\varphi(x)\,dx \left(\int_{\Omega} \varphi(x)\,dx\right)^{-1}\right],\ t \ge t_1, \dots \quad (2.19)$$

Define
$$V(t) = \int_{\Omega} u(x,t)\,\varphi(x)\,dx \left(\int_{\Omega} \varphi(x)\,dx\right)^{-1},\quad t \in t_1. \quad \dots (2.20)$$

In view of (2.15)-(2.20), we have

$$\frac{d}{dt}\left[p(t)\frac{d}{dt}\left(V(t) + \sum_{i=1}^{l} \lambda_i(t)\,V(t-\tau_i)\right)\right] + q(t) f(V(t))$$

$$+ \sum_{j=1}^{m} q_j(t) f_j(V(t-\sigma_j)) \le 0,\quad t \ge t_1. \quad \dots (2.21)$$

The remainder of the proof is similar to the proof of part (I), so we omit the details. The proof of Theorem 2.1 is complete.

Corollary 2.1 — Let the condition (A) hold. Assume the inequality (2.9) (resp. (2.21)) has no eventually positive solution, then every solution $u(x, t)$ of the problem (1.1), (1.2) (resp. (1.1), (1.3)) is oscillatory in G.

If $h(t, s)$ is replaced by $h(t,s)\sqrt{H(t,s)\,k(s)}$ in Theorem 2.1-2.3, we have the following theorems. The proofs are similar, so we omit the details.

***Theorem* 2.2** — *Let the conditions (A) hold. Assume* $H \in C(D; R)$ *satisfy conditions (i) and (ii) in Theorem 2.1 and also satisfy*

(iii) there exist $h \in C(D; R)$ *and* $k \in C^1(R_+, R_+)$ *such that*

$$-\frac{\partial (H(t,s)\ (s))}{\partial s} = h(t,s)\sqrt{H(t,s)\,k(s)}, \text{ for all } (t,s) \in D.$$

Also assume there exists a function $\phi \in C^1[t_0, \infty)$ *and there exists* $j_0 \in I_m$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\,k(s)\,\psi(s) - \frac{1}{4} p(s-\sigma_{j_0})\,\Phi(s)\,h^2(t,s) \right] ds = \infty, \quad \dots (2.2)$$

where $\Phi(s) = \exp\left\{-2\int^{s} \phi(\xi)\,d\xi\right\}$ *and* ψ *is defined as (2.2). Then*

(I) every solution $u(x, t)$ *of the problem (1.1), (1.2) is oscillatory in G; and*

(II) every solution $u(x, t)$ *of the problem (1.1), (1.3) is oscillatory in G.*

Remark 2.1 : If $f(u) = u$, the Theorem 2.1 and 2.2 extend and improve Theorems 2.1 and 3.1 of Li and Cui[6]. We note that conditions in Theorems 2.1 and 2.2 also improve and extend the relevant works of Cui *et al.*[7] for (1.1) with $p(t) = 1$, $\lambda_i(t) = 0$ and $f(u) = f_j(u) = u$.

3. Intrerval Oscillation Criterion

In this section, we establish new oscillation criteria for solutions of the problem (1.1) with (1.2) and (1.1) with (1.3) are different from most known ones in the sense that they are based on the information only on a sequence of subintervals of $[t_0, \infty)$ rather than on the whole half-line.

***Theorem* 3.1** — *Suppose that conditions (A) hold. Let functions* $H \in C(D; R)$; h_1, h_2, $\phi \in C(D_0; R)$; *and* $k \in C^1([t_0, \infty); R_+)$ *satisfy the following conditions :*

(i) $H(t, t) = 0$ *for* $t \geq t_0$, $H(t, s) > 0$ *on* D_0;

(ii) $\frac{\partial}{\partial t}(H(t,s)\,k(t)) - h_1(t,s)$, $\frac{\partial}{\partial s}(H(t,s)\,k(s)) = -h_2(t,s)$ $\forall (t,s) \in D_0$.

Assume also that for sufficiently large $T_0 \geq t_0$, *there exist* $a, b, c \in R$ *with* $T_0 \leq a < c < b$ *and* $j_0 \in I_m$ *such that*

$$\frac{1}{H(c,a)} \int_a^c H(s,a)\,k(s)\,\psi(s)\,ds + \frac{1}{H(b,c)} \int_c^b H(b,s)\,k(s)\,\psi(s)\,ds$$

$$> \frac{1}{4} \left[\frac{1}{H(c,a)} \int_a^c \frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(s,a)\,k(s)} h_1^2(s,a)\,ds \right.$$

$$+\frac{1}{H(b,c)}\int_c^b \frac{p\, s-\sigma_{j_0})\,\Phi(s)}{H(b,s)\,k(s)}\,h_2^2(b,s)\,ds\Bigg], \qquad \text{... (3.1)}$$

where $\Phi(s)=\exp\left\{-2\int^s \phi(\xi)\,d\xi\right\}$ *and* ψ *is defined in (2.2). Then*

(I) every solution u (x, t) of the problem (1.1), (1.2) is oscillatory in G; and

(II) every solution u (x, t) of the problem (1.1), (1.3) is oscillatory in G.

PROOF : (I) Suppose to the contrary that there is a solution $u(x, t)$ of the problem (1.1), (1.2) which has no zero in $\Omega\times[t_0,\infty)$ for some $t_0>0$. Without loss of generality we may assume that $u(x,t)>0, u(x,t-\tau_i)>0, u(x,t-\rho_k(t))>0$, and $u(x,t-\sigma_j)>0$ in $\Omega\times[t_1,\infty), t_1\ge t_0, i\in I_l$, $k\in I_s, j\in I_m$.

As in the proofs part (I) of Theorem 2.1, we can obtain (2.8), (2.9) and (2.12). Moreover (2.13) holds for $t\ge t_2\ge t_1$. With t_2 and $h(t, s)$ replaced by c and $h_2(t,s)$, respectively, it follows that

$$\int_c^t H(t,s)\,k(s)\,\psi(s)\,ds\le H(t,c)\,k(c)\,W(c)$$

$$+\frac{1}{4}\int_c^t \frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(t,s)\,k(s)}\,h_2^2(t,s)\,ds \qquad \text{... (3.2)}$$

where $t\in[c,b)$. Letting $t\to b^-$ in (3.2) and dividing both sides by $H(b, c)$ we get

$$\frac{1}{H(b,c)}\int_c^b H(b,s)\,k(s)\,\psi(s)\,ds\le k(c)\,\upsilon(c)$$

$$+\frac{1}{4H(b,c)}\int_c^b \frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(b,s)\,k(s)}\,h_2^2(b,s)\,ds. \qquad \text{... (3.3)}$$

Similar to the proof of Theorem 2.1, multiplying (2.12), with t replaced by s, by $H(s,t)\,k(s)$ and then integrating with respect to s from t to c for $t\in(a,c]$ we obtain

$$\int_t^c H(s,t)\,k(s)\,\psi(s)\,ds\le -H(c,t)\,k(c)\,W(c)+\int_t^c \frac{\partial}{\partial s}(H(s,t)\,k(s))\,W(s)\,ds$$

$$-\int_t^c H(s,t)\,k(s)\frac{W^2(s)}{p(s-\sigma_{j_0})\,\Phi(s)}\,ds$$

$$= -H(c,t)\,k(c)\,W(c) + \int_{t_2}^{t} h_1(s,t)\,W(s)\,ds - \int_{t}^{c} H(s,t)\,k(s)\,\frac{W^2(s)}{p(s-\sigma_{j_0})\,\Phi(s)}\,ds$$

$$= -H(c,t)\,k(c)\,W(c) + \frac{1}{4}\int_{t}^{c} \frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(t,s)k(s)}\,h_1^2(s,t)\,ds$$

$$-\int_{t}^{c}\left[\sqrt{\frac{H(s,t)\,k(s)}{p(s-\sigma_{j_0})\,\Phi(s)}}\,W(s) - \frac{1}{2}\sqrt{\frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(s,t)\,k(s)}}\,h_1(s,t)\right]^2 ds.$$

Thus

$$\int_{t}^{c} H(s,t)\,k(s)\,\psi(s)\,ds \le -H(c,t)\,k(c)\,W(c)$$

$$+\frac{1}{4}\int_{t}^{c} \frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(s,t)\,k(s)}\,h_1^2(s,t)\,ds. \qquad \ldots (3.4)$$

Letting $t \to a^+$ in (3.4) and dividing both sides by $H(c, a)$ we get

$$\frac{1}{H(c,a)}\int_{a}^{c} H(s,a)\,k(s)\,\psi(s)\,ds \le -k(c)\,W(c)$$

$$+\frac{1}{4H(c,a)}\int_{a}^{c} \frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(s,a)\,k(s)}\,h_1^2(s,a)\,ds. \qquad \ldots (3.5)$$

Now we claim that every nontrivial solution of differential inequality (2.9) has at least one zero in (a, b).

Suppose the contrary. By Corollary 2.1, without loss of generality, we may assume that there is a solution of (2.9) such that $V(t) > 0$ for $t \in (a, b)$. Adding (3.3) and (3.5), we get the inequality which contradicts the assumption (3.1)., Thus, the claim holds.

Pick up a sequence $\{T_i\} \subset [t_0, \infty)$ such that $T_i \to \infty$ as $i \to \infty$. By he assumptions of Theorem 3.1, for each $i \in N$, there exist $a_i, b_i, c_i \in R$ such that $T_i \le a_i < c_i < b_i$, and (3.1) holds with a, b, c replaced by a_i, b_i, c_i respectively. From that, every nontrivial solution $V(t)$ of (2.9) has at least one zero $t_i \in (a_i, b_i)$. Noting that $t_i > a_i \ge T_i$, $i \in N$, we see that every solution $V(t)$ has arbitrarily large zero. This contradicts fact hat $V(t)$ is nonoscillatory by (2.8) and the assumption $u(x, t) > 0$ in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Hence, every solution of problem (1.1), (1.2) is oscillatory in G.

(II) Similar to part (I). This completes the proof of Theorem 3.1.

As an immediate consequence of Theorem 3.1 we get the following oscillation criteria for every solution of the problem (1.1), (1.2) (resp. (1.1), (1.3)) in G.

***Theorem* 3.2** — *Let conditions (3.1) in Theorem 3.1 be replaced by*

$$\limsup_{t\to\infty} \int_l^t \left[H(s,l)\,k(s)\,\psi(s) - \frac{1}{4}\frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(s,l)\,k(s)} h_1^2(s,l) \right] ds > 0 \qquad \text{... (3.6)}$$

and

$$\limsup_{t\to\infty} \int_l^t \left[H(s,l)\,k(s)\,\psi(s) - \frac{1}{4}\frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(s,l)\,k(s)} h_2^2(t,s) \right] ds > 0, \qquad \text{... (3.7)}$$

for each sufficient large $l \geq t_0$. Then

(I) every solution $u(x, t)$ of the problem (1.1), (1.2) is oscillatory in G; and

(II) every solution $u(x, t)$ of the problem (1.1), (1.3) is oscillatory in G.

PROOF : For any $T \geq T_0 \geq t_0$, let $a = T$. In (3.6) choose $l = a$. Then there exists $c > a$ such that

$$\int_a^c \left[H(s,a)\,k(s)\,\psi(s) - \frac{1}{4}\frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(s,a)\,k(s)} h_1^2(s,a) \right] ds > 0. \qquad \text{... (3.8)}$$

In (3.7) choose $l = c$. Then there exists $b > c$ such that

$$\int_c^b \left[H(b,s)\,k(s)\,\psi(s) - \frac{1}{4}\frac{p(s-\sigma_{j_0})\,\Phi(s)}{H(b,s)\,k(s)} h_2^2(b,s) \right] ds > 0. \qquad \text{... (3.9)}$$

Combining (3.8) and (3.9) we obtain (3.1). The conclusion thus come from Theorem 3.1.

If $h_1(t,s)$ and $h_2(t,s)$ are replaced by $h_1(t,s)\sqrt{H(t,s)\,k(s)}$ and $h_2(t,s)\sqrt{H(t,s)\,k(s)}$ in Theorem 3.1 and 3.2, respectively, we obtain the following results. The proofs are similar to those of theorems 3.1 and 3.2, so we omit the details.

***Theorem* 3.3** — *Assume that conditions (A) hold. Let function $h \in C(D, R)$ satisfy condition (i) of Theorem 3.1. Assume there exist functions $h_1, h_2 \in C(D_0, R)$ and $k \in C^1([t_0, \infty), R_+)$ such that for all $(t, s) \in D_0$*

$$(ii)\quad \frac{\partial}{\partial t}(H(t,s)\,k(t) = h_1(t,s)\sqrt{H(t,s)\,k(t)},\ \frac{\partial}{\partial s}(H(t,s)\,k(s)) = -h_2(t,s)\sqrt{H(t,s)\,k(s)}.$$

Assume also that for sufficiently large each $T_0 \geq t_0$, there exist $a, b, c \in R$ with $T_0 \leq a < c < b$ such that

$$\frac{1}{H(c,a)}\int_a^c H(s,a)\,k(s)\,\psi(s)\,ds+\frac{1}{H(b,c)}\int_c^b H(b,s)\,k(s)\,\psi(s)\,ds$$

$$>\frac{1}{4}\left[\frac{1}{H(c,a)}\int_a^c p(s-\sigma_{j_0})\,\Phi(s)\,h_1^2(s,a)\,ds\right.$$

$$\left.+\frac{1}{H(b,c)}\int_c^b p(s-\sigma_{j_0})\,\Phi(s)\,h_2^2(b,s)\,ds\right], \quad \ldots (3.10)$$

where $\Phi(s)=\exp\left\{-2\int^s \phi(\xi)\,d\xi\right\}$ *and* ψ *is defined in (2.2). Then*

(I) every solution u (x, t) of the problem (1.1), (1.2) is oscillatory in G; and

(II) every solution u (x, t) of the problem (1.1), (1.3) is oscillator in G.

***Theorem* 3.4** — *Let conditions (3.10) in Theorem 3.3 be replaced by*

$$\limsup_{t\to\infty}\int_l^t \left[H(s,l)\,k(s)\,\psi(s)-\frac{1}{4}p(s-\sigma_{j_0})\,\Phi(s)\,h_1^2(s,l)\right]ds>0$$

and
$$\limsup_{t\to\infty}\int_l^t \left[H(t,s)\,k(s)\,\psi(s)-\frac{1}{4}p(s-\sigma_{j_0})\,\Phi(s)\,h_2^2(t,s)\right]ds>0,$$

for each sufficient large $r\ge t_0$. *Then*

(I) every solution u (x, t) of the problem (1.1), (1.2) is oscillatory in G; and

(II) every solution u (x, t) of the problem (1.1), (1.3) is oscillatory in G.

For the case where $H := H(t-s)$ and $k(t) = 1$ satisfy (i) and (ii) in Theorem 3.3, let $h\,(t-s)$ denote $h_1(t-s)=h_2(t-s)$. Applying Theorem 3.3, we obtain

***Theorem* 3.5** — *Assume that conditions (A) hold. Let function* $H(t-s)\in C(D,R)$ *satisfy condition (i) in Theorem 3.1. Assume there exists a functions* $h\in C(D_0,R)$ *sch that for all* $(t,s)\in D_0$

$$(ii)\ \frac{\partial}{\partial t}(H(t-s))=h(t-s)\sqrt{H(t-s)},\ \frac{\partial}{\partial s}(H(t-s)\,k(s))=-h(t-s)\sqrt{H(t-s)}.$$

Assume also that for sufficiently large each $T_0\ge t_0$, there exist $a, c\ R$ with $T_0\le a<c$ such that

$$\int_a^c H(s-a)\,[\psi(s)+\psi(2c-s)]\,ds$$

$$> \frac{1}{4} \int_a^c [p(s-\sigma_{j_0})\, \Phi(s) + p(2c+\sigma_{j_0}-s)\, \Phi(2c-s)]\, h^2(s-a)\, ds, \qquad \text{... (3.11)}$$

where $\Phi(s) = \exp\left\{-2 \int^s \phi(\xi)\, d\xi\right\}$ *and* ψ *is defined as (2.2). Then*

(I) every solution $u(x, t)$ *of the problem (1.1), (1.2) is oscillatory in* G; *and*

(II) every solution $u(x, t)$ *of the problem (1.1), (1.3) is oscillatory in* G.

PROOF : Let $b = 2c - a$. Then $H(b-c) = H(c-a) = H\left(\frac{b-a}{2}\right)$, and for any $g \in L[a, b]$, we have

$$\int_c^b g(s)\, ds = \int_a^c g(2c-s)\, ds.$$

Hence
$$\int_c^b H(b-s)\, \psi(s)\, ds = \int_a^c H(s-a)\, \psi(2c-s)\, ds,$$

and
$$\int_c^b p(s-\sigma_{j_0})\, h^2(b-s)\, ds = \int_a^c p(2c+\sigma_{j_0}-s)\, h^2(s-a)\, ds.$$

Thus if (3.11) holds, then implies that (3.10) holds. Therefore the conclusion of Theorem 3.5 holds by Theorem 3.3. The proof of Theorem 3.5 is complete.

Define
$$R(t) = \int_l^t \frac{1}{p(s-\sigma_{j_0})}\, ds, \quad t \geq l \geq t_0, \qquad \text{... (3.12)}$$

and let
$$H(t, s) = [R(t) - R(s)]^{\lambda}, \; t \geq t_0, \qquad \text{... (3.13)}$$

where $\lambda > 1$ is a constant.

***Theorem* 3.6** — *Assume conditions (A) and* $\lim_{t \to \infty} R(t) = \infty$ *hold. Also assume that for each* $l > t_0$ *and for some* $\lambda > 1$ *the two inequalities*

$$\limsup_{t \to \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t [R(s) - R(l)]^{\lambda}\, \alpha_{j_0} q_{j_0}(s)$$

$$\left[1 - \sum_{i=1}^{l} \lambda_i (s - \sigma_{j_0})\right] ds > \frac{\lambda^2}{4(\lambda-1)} \qquad \text{... (3.14)}$$

and
$$\limsup_{t \to \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t [R(t) - R(s)]^{\lambda}\, \alpha_{j_0} q_{j_0}(s)$$

$$\left[1-\sum_{i=1}^{l}\lambda_i(s-\sigma_{j_0})\right]ds>\frac{\lambda^2}{4(\lambda-1)}. \qquad \ldots (3.15)$$

hold. Then

(I) every solution u (x, t) of the problem (1.1), (1.2) is oscillatory in G; and

(II) every solution u (x, t) of the problem (1.1), (1.3) is oscillatory in G.

PROOF : (i) It is easy to see that

$$h_1(t,s)=\lambda\,[R(t)-R(s)]^{\frac{\lambda}{2}-1}\frac{1}{p(t-\sigma_{j_0})}$$

and
$$h_2(t,s)=\lambda\,[R(t)-R(s)]^{\frac{\lambda}{2}-1}\frac{1}{p(s-\sigma_{j_0})}.$$

Noting that

$$\int_l^t p(s-\sigma_{j_0})\,h_1^2(s,l)\,ds$$

$$=\int_l^t p(s-\sigma_{j_0})\,\lambda^2\,[R(s)-R(l)]^{\lambda-2}\frac{1}{p^2(s-\sigma_{j_0})}\,ds=\frac{\lambda^2}{(\lambda-1)}[R(t)-R(l)]^{\lambda-1},$$

and
$$\int_l^t p(s-\sigma_{j_0})\,h_2^2(t,s)\,ds$$

$$\int_l^t p(s-\sigma_{j_0})\,\lambda^2\,[R(t)-R(s)]^{\lambda-2}\frac{1}{p^2(s-\sigma_{j_0})}\,ds=\frac{\lambda^2}{(\lambda-1)}[R(t)-R(l)]^{\lambda-1}$$

It follows that

$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-1}(t)}\int_l^t$$

$$\left\{H(s,l)\,\alpha_{j_0}q_{j_0}(s)\left[1-\sum_{i=1}^{i}\lambda_i(s-\sigma_{j_0})\right]-\frac{1}{4}p(s-\sigma_{j_0})\,h_1^2(s,l)\right\}ds$$

$$=\limsup_{t\to\infty}\frac{1}{R^{\lambda-1}(t)}\int_l^t$$

$$[R(s)-R(l)]^{\lambda}\,\alpha_{j_0}\,q_{j_0}(s)\left[1-\sum_{i=1}^{i}\lambda_i(s-\sigma_{j_0})\right]ds-\frac{\lambda^2}{4(\lambda-1)}>0$$

and

$$\limsup_{t\to\infty}\frac{1}{R^{\lambda-1}(t)}\int_l^t$$

$$\left\{H(t,s)\,\alpha_{j_0}\,q_{j_0}(s)\left[1-\sum_{i=1}^{i}\lambda_i(s-\sigma_{j_0})\right]-\frac{1}{4}p(s-\sigma_{j_0})\,h_2^2(t,s)\right\}ds$$

$$=\limsup_{t\to\infty}\frac{1}{R^{\lambda-1}(t)}\int_l^t[R(t)-R(s)]^{\lambda}\,\alpha_{j_0}\,q_{j_0}(s)\left[1-\sum_{i=1}^{i}\lambda_i(s-\sigma_{j_0})\right]ds-\frac{\lambda^2}{4(\lambda-1)}>0,$$

by Theorem 3.4, the conclusions are true.

Remark 3.1 : From Theorems 2.1-2.2 and Corollary 2.1, Theorems 3.1-3.6, we can present different explicit sufficient conditions for the oscillation of the solutions to problem (1.1), (1.2) (res. (1.1), (1.3)) by appropriate choice of $H(t,s), k(s)$ and $\psi(s)$. For instance, if we choose $H(t,s)=(t-s)^{\alpha}, H(t,s)=[R(t)-R(s)]^{\lambda}$, or $H(t,s)=[\log Q(t)/Q(s)]^{\lambda}$, or $H(t,s)=\left[\int_s^t\frac{1}{w(z)}dz\right]^{\lambda}$, or $H(t,s)=\rho(t-s)$ etc., for $t\geq s\geq t_0$; $k(s), \psi(s)$ may choose 1, s etc., where $\alpha>1$ is a constant, $R(t)=\int_{t_0}^t ds/u(s), Q(t)=\int_t^{\infty}ds/u(s)<\infty$, for $t\geq t_0$, $w\in C([t_0,\infty),(0,\infty))$ satisfying $\int_{t_0}^{\infty}\frac{1}{w(z)}dz=\infty, \rho(0)>0, \rho(u)>0$ and $\rho'(u)\geq 0$ for $u>0$.

4. EXAMPLES

The conditions in our paper are sharper than the conditions in [2-9]. We will see that the oscillations cannot be demonstrated by most other known criteria in the following examples.

Example 4.1 — Let constant $b\geq 0$. Consider the differential equation

$$\frac{\partial}{\partial t}\left[\frac{1}{t+\pi}\frac{\partial}{\partial t}\left(u(x,t)+\frac{6}{t+\pi}u(x,t-2\pi)\right)\right]$$

$$=\left(\frac{1}{t+\pi}+\frac{6}{(t+\pi)^2}-\frac{18}{(t+\pi)^4}\right)\Delta u(x,t)$$

$$+\left(\frac{1}{(t+\pi)^2}+\frac{18}{(t+\pi)^3}\right)\Delta u\left(x,t-\frac{3}{2}\pi\right)+\left(\cos^2 t+8t+\frac{11}{t^3}\right)\Delta u(x,t-\pi)$$

$$-\left(\cos^2 t+8t+\frac{16}{t^3}\right)u(x,t)\left[1+\frac{b}{1+u^2(x,t)}\right]-\frac{5}{t^3}u(x,t-\pi),$$

$$(x,t)\in(0,\pi)\times R_+\equiv G, \qquad \dots (4.1)$$

with the boundary condition

$$u(0,t)=u(\pi,t)=0,\quad t\geq 0 \qquad \dots (4.2)$$

Here $N = 1$, $L = 1$, $S = 2$, $m = 1$, $p(t)=\frac{1}{t+\pi}$, $\lambda_1(t)=\frac{6}{t+\pi}$, $\tau_1=2\pi$, $a(t)=\frac{1}{t+\pi}+\frac{6}{(t+\pi)^2}-\frac{18}{(t+\pi)^4}$, $a_1(t)=\frac{1}{(t+\pi)^2}+\frac{18}{(t+\pi)^3}$, $\rho_1(t)=\frac{3}{2}\pi$, $a_2(t)=\cos^2 t+8t+\frac{11}{t^3}$, $\rho_2(t)=\pi$, $q(x,t)=\cos^2 t+8t+\frac{16}{t^3}$, $q_1(x,t)=\frac{5}{t^5}$, $f(u)\doteq u(x,t)\left[1+\frac{b}{1+u^2(x+t)}\right]$, $f_1(u)=u$, and $\sigma_1=\pi$. A straightforward verification shows that the functions $q_1(t)=\frac{5}{t^3}$, $\lambda_1(t-\sigma_1)=\lambda_1(t-\pi)=6/t$ and $p(t-\sigma_1)=p(t-\pi)=1/t$. Let $\phi(t)=-1/t$. Then $\Phi(t)=t^2$ and

$$\psi(s)=\Phi(s)\left\{q_1(s)\left[1-\lambda_1(s-\pi)\right]+p(s-\pi)\right]+p(s-\pi)\,\phi^2(s)-\left[p(s-\pi)\,\phi(s)\right]'\right\}$$

$$=t^2\left[\frac{5}{t^3}\left(1-\frac{6}{t}\right)-\frac{1}{t^3}\right]=\frac{4}{t}-\frac{30}{t^2}.$$

Choose $t_0=2\pi$, $k(s)=1$ and $H(t,s)=(t-s)^2$. Then $h(t-s)=2$ and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\,k(s)\,\psi(s)-\frac{1}{4}p(s-\sigma_{j_0})\,\Phi(s)\,h^2(t,s)\right]ds$$

$$=\limsup_{t\to\infty}\frac{1}{(t-2\pi)^2}\int_{2\pi}^{t}\left[(t-s)^2\,\psi(s)-\frac{1}{4}p(s-\pi)\,\Phi(s)\times 4\right]ds$$

$$=\limsup_{t\to\infty}\frac{1}{t^2}\int_{2\pi}^{t}\left[(t-s)^2\left(\frac{4}{s}-\frac{30}{s^2}\right)-s^2\times\frac{1}{s}\right]ds=\infty.$$

This shows that all conditions of Theorem 22 are satisfied. Therefore, every solution $u(x, t)$ of the problem (4.1), (4.2) is oscillatory in G. For example, if $b = 0$, $u(x, t) = \sin x \cos t$ is such a solution. However, criteria in [1-12] fail to imply tis fact. In addition, those criteria are quite difficult to apply to get oscillation of all solutions of problem (4.1), (4.2) for $b > 0$.

Example 4.2 — Let $b\geq 0$. Consider the differential equation

$$\frac{\partial}{\partial t}\left[\frac{1}{t+\pi}\frac{\partial}{\partial t}\left(u(x,t)+\frac{6}{t+\pi}u(x,t-2\pi)\right)\right]$$

$$= \left(\frac{1}{t+\pi} + \frac{6}{(t+\pi)^2} - \frac{21}{(t+\pi)^4} \right) \Delta u\,(x, t)$$

$$+ \left(\frac{1}{(t+\pi)^2} + \frac{18}{(t+\pi)^3} \right) \Delta u \left(x, t - \frac{3}{2}\pi \right) + \left(\sin t + 3t + \frac{4\pi}{t^3} \right) \Delta u\,(x, t-\pi)$$

$$- \left(\sin t + 3t + \frac{9\pi}{t^2} + \frac{3}{(t+\pi)^4} \right) u\,(x, t)\,[1 + b \sin^2 u\,(x, t)] - \frac{5\pi}{t^3} u\,(x, t-\pi),$$

$$(x, t) \in (0, \pi) \times R_+ \equiv G, \qquad \text{... (4.3)}$$

with the boundary condition

$$u_x\,(0, t) = u_x\,(\pi, t) = 0, \quad t \geq 0, \qquad \text{... (4.4)}$$

It is easy to see that all conditions of Theorem 2.2 are satisfied. Thus, every solution $u(x, t)$ of the problem (4.3), (4.4) is oscillatory in G. For example, if $b = 0$, $u\,(x, t) = \cos x \sin t$ is such a solution. However, criteria in [1-12] fail to reveal this fact. Again, the sufficient conditions for oscillation in [1-12] are relatively difficult to apply to problem (4.3), (4.4) if $b > 0$.

Example 4.3 — Let $n \in N_0 = \{1, 2, 3, ...\}$, $b \geq 0$ and

$$q\,(t) = \begin{cases} \sin \frac{\pi}{3} t, & 0 \leq t < 3 \\ \frac{5t\,(t-3n)}{t-1}, & 3n \leq t \leq 3n+1 \\ \frac{5(-t+3n+2)}{t-1}, & 3n+1 < t \leq 3n+2 \\ |\sin \pi t|, & 3n+2 < t \leq 3n+3. \end{cases}$$

Consider the differential equation

$$\frac{\partial}{\partial t}\left[\frac{1}{t+\pi} \frac{\partial}{\partial t} \left(u\,(x, t) + \frac{5}{t+\pi} u\,(x, t-2\pi) \right) \right]$$

$$= \left(\frac{1}{t+\pi} - \frac{15}{(t+\pi)^4} \right) \Delta u\,(x, t)$$

$$\left(\frac{1}{(t+\pi)^2} + \frac{15}{t+\pi)^3} \right) \Delta u \left(x, t - \frac{3}{2}\pi \right) + \left(\frac{5}{(t+\pi)^2} + 6q\,(t) \right) \Delta u\,(x, t-\pi)$$

$$- \left(\frac{10}{(t+\pi)^2} + 7q\,(t) \right) u\,(x, t)\,[1 + b \cos^2 u\,(x, t)] - q\,(t)\, u\,(x, t-\pi),$$

$$(x, t) \in (0, \pi) \times R_+ \equiv G, \qquad \text{... (4.5)}$$

with the boundary condition (4.4).

A straightforward verification shows that the functions $q_1(t) = q(t)$, $\lambda_1(t-\sigma_1) = l + \lambda_1(t-\pi) = 1/t$ and $p(t-\sigma_1) = p(t-\pi) = 1/t$. For any $T > 1$ there exists $n \in N_0$ such that $3n > T$. Let a $3n$, $c = 3n + 1$, $H(t-s) = (t-s)^2$ and $\phi(s) = 0$. Then $h(t-s) = 2$, $\Phi(s) = 1$, $\psi(s) = q(s)[1 - \lambda_1(s-\pi) = (s-1)q(s)/s$ and

$$\int_a^c \{H(s-a)[\psi(s) + \psi(2c-s)]$$

$$-\frac{1}{4}[p(s-\pi)\Phi(s) + p(2c+\pi-s)\Phi(2c-s)]h^2(s-a)\} \, ds$$

$$= \int_{3n}^{3n+1} \left\{(s-3n)^2\left[\frac{(s-1)q(s)}{s} + \frac{(2(3n+1)-s-1)q(2(3n+1)-s)}{2(3n+1)-s}\right]\right.$$

$$\left.-\frac{1}{4}\left[\frac{1}{s} + \frac{1}{6n+2+2\pi-s)}\right] \times 4\right\} ds$$

$$= \int_{3n}^{3n+1} (s-3n)^2 \times [5(s-3n) + 5(s-3n)] \, ds - \left[\ln\left(1+\frac{1}{3n}\right) + \ln\left(1+\frac{1}{3n+1+2\pi}\right)\right]$$

$$> 10/4 - \ln 4 > 0. \qquad \dots (4.6)$$

Therefore, (4.6) holds implies (3.11) holds. It is clear that all conditions of Theorem 3.5 are satisfied. Thus, every solution $u(x, t)$ of the problem (4.5), (4.4) is oscillatory in G. For example, if $b = 0$, $u(x, t) = \cos x \sin t$ is such a solution. However, oscillation of solutions to problem (4.5), (4.4) cannot be demonstrated by other known criteria if $b > 0$.

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REFERENCES

1. D. P. Mishev and D. D. Bainov, *Appl. Math. Comput.*, **28** (1988), 97-111.
2. L. H. Erbe, Q. Kong, B. G. Zhang, *Utilitas Math.*, **43** (1993), 129-39.
3. B. S. Lalli, Y. H. Yu and B. T. Cui, *Appl. Math. Comput.*, **53** (1993), 97-110.
4. B. S. Lalli, Y. H. Yu and B. T. Cui, *Bull. Austral. Math. Soc.*, **46** (1992), 373-80.
5. B. T. Cui, *Demonstratio Math.*, **29** (1996), 61-68.
6. W. N. Li, B. T. Cui, *J. Math. Anal. Appl.*, **234** (1999), 123-46.
7. B. T. Cui, Y. H. Yu and S. Z. Lin, *Acta Math. Appl. Sinica*, **19** (1996), 80-88 [in Chinese].
8. D. D. Bainov, B. T. Cui and D. P. Mishev, *J. Comput. Appl. Math.*, **72** (1996), 309-18.
9. Y. H. Yu, B. Lin and Z. R. Liu, *Acta Math. Sinica (New Series)*, **13** (1997), 563-70.
10. Yu. V. Rogovchenko, *Nonl. Anal.*, **41** (2000), 1005-28.
11. Ch. G. Philos, *Arch. Math.*, **53** (1989), 483-92.
12. Q. Kong, *J. Math. Anal. Appl.*, **229** (1999), 258-70.