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ON SOME NEW SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

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In this paper we introduce some new sequence spaces using Orlicz function. We also examine some properties of these sequence spaces.

Key Words : Sequence Space; Orlicz Function; Invariant Mean

1. Lindenstrauss and Tzafriri[1] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ x : \sum_k M\left(\frac{|x_k|}{r}\right) < \infty \text{ for some } r > 0 \right\}.$$

The space l_M with the norm

$$\|x\| = \operatorname{Inf}\left\{ r > 0 : \sum_k M\left(\frac{|x_k|}{r}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

In the present note we introduce and examine some properties of three sequence spaces defined using Orlicz function M, which generalize the well-known sequence spaces l_∞, c and c_0 the Banach spaces of bounded and convergent and null sequences $x = (x_k)$ normed by $\|x\| = \sup_k |x_k|$ respectively.

Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$, $m = 1, 2, 3, \ldots$ A continuous linear functional φ on l_∞ is said to be an invariant mean or a σ-mean if and only if

1. $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
2. $\varphi(e) = 1$, where $e = (1, 1, \ldots)$ and
3. $\varphi(\{x_{\sigma(n)}\}) = \varphi(\{x_n\})$ for all $x \in l_\infty$.

For certain kinds of mappings σ, every invariant mean φ extends the limit functional on the space c, in the sense that $\varphi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$, where V_σ is the set of bounded sequences all of whose σ-means are equal.

If $x=(x_n)$, set $Tx=(Tx_n)=(x_{\sigma(n)})$. it can be shown that

$$V_\sigma = \{x \in l_\infty : \lim_m t_{mn}(x) = Le, \text{ uniformly in } n, L = \sigma\text{-}\lim x\} \quad .. (1.1)$$

where $t_{mn}(x) = (x_n + Tx_n + \ldots + T^m x_n)/m+1$ (See Schafer[2]).

The special case of (1.1) in which $\sigma(n)=n+1$ was given by Lorentz[3] (Theorem 1) and the general result can be proved in a similar way.

An Orlicz function is a function $M:[0,\infty)\to[0,\infty)$, which is continuous, nondecreasing and convex with $M(0)=0, M(x)>0$ for $x>0$ and $M(x)\to\infty$ as $x\to\infty$. If convexity of Orlicz function M is replaced by $M(x+y)\le M(x)+M(y)$ then this function is called modulus function, defined and discussed by Ruckle[4] and Maddox[5].

Let $p=(p_k)$ be a sequence of real numbers such that $p_k>0$ for all k and $\sup_k p_k = H<\infty$. We define the following sequence spaces.

$$l_\infty(M,p,\sigma,s)=\left\{x: \sup_{n,k} k^{-s}\left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho}\right)\right]^{p_k} \prec \infty, \text{for some } \rho \succ 0, s\ge 0\right\},$$

$$c(M,p\,\sigma,s)=\left\{x: \lim_{k\to\infty} k^{-s}\left[M\left(\frac{|x_{\sigma^k(n)}-L|}{\rho}\right)\right]^{p_k} =0,\right.$$

uniformly in n, for some $\rho, L \succ 0, s\ge 0\}$,

$$c_0(M,p,\sigma,s)=\left\{x: \lim_{k\to\infty} k^{-s}\left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho}\right)\right]^{p_k} =0,\right.$$

uniformly in n, for some $\rho \succ 0, s\ge 0\}$.

When $M(x) = x$, then we have $l_\infty(p,\sigma,s), c(p,\sigma,s)$ and $c_0(p,\sigma,s)$. For instance,

$$c(p,\sigma,s)=\left\{x: \lim_{k\to\infty} k^{-s}\left(\frac{|x_{\sigma^k(n)}-L|}{\rho}\right)^{p_k} =0,\right.$$

uniformly in n, for some $\rho, L>0, s\ge 0\}$.

When $M(x) = x$, $\sigma(n)=n$ for all n and $s = 0$ and then the family of sequences defined above become $l_\infty(p), c(p)$ and $c_0(p)$ which were defined by Maddox[6]. When $\sigma(n)=n$ for all n, and Orlicz function M is replaced by Modulus function f, the family of sequences defined above become $l_\infty(p,f,s), c(p,f,s)$ and $c_0(p,f,s)$ which were defined by Esi[7]. When $M(x)=x$, $\sigma(n)=n$ for all n, $s = 0$ and $p_k=1$ for all k, we have l_∞, c and c_0.

2. In this section we shall establish some basic properties of three sequence spaces defined by Orlicz function M. In order to discuss the properties of these spaces we assume that $p=(p_k)$ is bounded.

***Theorem* 1** — *$l_\infty(M, p, \sigma, s)$, $c(M, p, \sigma, s)$ and $c_0(M, p, \sigma, s)$ are linear spaces over the set of complex numbers C.*

PROOF : We consider only $c_0(M, p, \sigma, s)$. The others can be treated similarly. Let $x, y \in c_0(M, p, \sigma, s)$ and $\alpha, \beta \in C$. In order to prove the result we need to find some ρ_3 such that

$$k^{-s}[M(|\alpha x^k_{\sigma(n)} + \beta y^k_{\sigma(n)}|/\rho_3)]^p_k \to 0 \text{ as } k \to \infty, \text{ uniformly in } n, \ s \geq 0.$$

Since $x, y \in c_0(M, p, \sigma, s)$, therefore there exist some ρ_1 and ρ_2 such that

$$k^{-s}[M(|x^k_{\sigma(n)}|/\rho_1)]^p_k \to 0 \text{ as } k \to \infty, \text{ uniformly in } n, \ s \geq 0.$$

and $k^{-s}[M(|y^k_{\sigma(n)}|/\rho_2)]^p_k \to 0$ as $k \to \infty$, uniformly in n, $s \geq 0$.

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing and convex

$$\begin{aligned} k^{-s}[M(|\alpha x^k_{\sigma(n)} + \beta y^k_{\sigma(n)}|/\rho_3)]^p_k &\leq k^{-s}[M(|\alpha x^k_{\sigma(n)}|/\rho_3 + |\beta y^k_{\sigma(n)}|/\rho_3)]^p_k \\ &\leq k^{-s}\frac{1}{2^{p_k}}[M(|x^k_{\sigma(n)}|/\rho_1)] + M(|y^k_{\sigma(n)}|/\rho_2)]^p_k \\ &\leq Ck^{-s}[M(|x^k_{\sigma(n)}|/\rho_1)]^p_k + Ck^{-s}[M(|y^k_{\sigma(n)}|/\rho_2)]^p_k \\ &\to 0 \text{ as } k \to \infty, \text{ uniformly in } n, \ s \geq 0 \end{aligned}$$

where $C = \max(1, 2^{H-1})$. This proves that $c_0(M, p, \sigma, s)$ is linear.

***Theorem* 2** — *Let $H = \max(1, \sup_k p_k)$. Then $c_0(M, p, \sigma, s)$ is a linear topological space paranormed by*

$$g(x) = \text{Inf}\,\{r^{p_n/H} : (k^{-s}[M(|x^k_{\sigma(n)}|/\rho)]^p_k)^{1/H} \leq 1, \ s \geq 0, n = 1, 2, 3, \ldots\}$$

PROOF : Clearly $g(x) = g(-x)$. Thus subaddivity of g follows from Theorem 1. Since $M(0) = 0$, we get $\text{Inf}\,\{r^{p_n/H}\} = 0$ for $x = 0$. Conversely, suppose that $g(x) = 0$, then it is easy to see that $x = 0$. Finally using the same technique of Theorem 2 of Nuray and Gülcü[8], it can be easily seen that scalar multiplication is continuous. This completes the proof.

In order to discuss further result we need the following definition.

Definition (Krasnoselskii and Rutitsky[9]) — An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists, constant $K > 0$, such that $M(2u) < KM(u)$ $(u > 0)$. The Δ_2-condition is equivalent to the satisfaction of inequalty $M(Lu) \leq KLM(u)$ for all values of u and for $L > 1$.

***Theorem* 3** — *Let M be an Orlicz function which satisfies Δ_2-condition. Then*

(*i*) $l_\infty(p, \sigma, s) \subset l_\infty(M, p, \sigma, s)$, (*ii*) $c(p, \sigma, s) \subset c(M, p, \sigma, s)$

(*iii*) $c_0(p, \sigma, s) \subset c_0(M, p, \sigma, s)$.

PROOF : We give the proof only for (ii). Let $x \in c(p, \sigma, s)$, then

$$S_k = k^{-s} | x_{\sigma(n)^k} - L |_{k^p} \to 0 \text{ as } k \to \infty, \text{ uniformly in } n. \quad \dots (2.1)$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \le t \le \delta$.

Write $\overset{k}{z}_{\sigma(n)} = | \overset{k}{x}_{\sigma(n)} - L |$ and consider

$$I_1 = \{(k \in N : z_{\sigma(n)^k} > \delta\} \text{ and } I_2 = \left\{ k \in N : z_{\sigma(n)^k} \le \delta \right\}.$$

For $\quad z_{\sigma(n)^k} > \delta$,

$$z_{\sigma(n)^k} < z_{\sigma(n)^k}/\delta < 1 + z_{\sigma(n)^k}/\delta$$

where $k \in I_2$. Since M is nondecreasing and convex, it follows that

$$M(z_{\sigma(n)^k}) < M(1 + z_{\sigma(n)^k}/\delta) < \frac{1}{2} M(2) + \frac{1}{2} M(2 z_{\sigma(n)^k}/\delta).$$

Since M satisfies Δ_2-condition, therefore

$$M(z_{\sigma(n)^k} < \frac{1}{2} K z_{\sigma(n)^k}/\delta M(2) + \frac{1}{2} K z_{\sigma(n)^k}/\delta M(2)$$

$$= K z_{\sigma(n)^k}/\delta M(2).$$

For $\quad z_{\sigma(n)^k} \le \delta$,

$$M(z_{\sigma(n)^k}) < \varepsilon,$$

where $k \in I_1$. Hence

$$k^{-s} [M(| z_{\sigma(n)^k} |)]^{k_p} = k^{-s} [M(| z_{\sigma(n)^k} |)]^{k_p} + k^{-s} [M(| z_{\sigma(n)^k} |)]^{k_p}$$

where the first term in the right side over $k \in I_1$ and the second over $k \in I_2$.

$$k^{-s} [M(| z_{\sigma(n)^k} |)]^{k_p} \le k^{-s} \varepsilon^H + \max(1, K \delta^{-1} M(2))^H \; S_k \to 0 \text{ as } k \to \infty,$$

uniformly in n.

This and from (2.1) $c(p, \sigma, s) \subset c(M, p, \sigma, s)$. Following similar arguments we can prove that $c_0(p, \sigma, s) \subset c_0(M, p, \sigma, s)$ and $l_\infty(p, \sigma, s) \subset l_\infty(M, p, \sigma, s)$.

***Theorem* 4** — *(i) Let* $0 < \inf_k p_k \le p_k \le 1$. *Then* $c(M, p, \sigma, s) \subset c(p, \sigma, s)$,

(ii) Let $1 \le p_k \le \sup_k p_k < \infty$. *Then* $c(p, \sigma, s) \subset c(M, p, \sigma, s)$,

(iii) Let $0 < p_k \le q_k$ *and* (q_k/p_k) *be bounded. Then* $c(M, q, \sigma, s) \subset c(M, p, \sigma, s)$,

(iii) Let $0 < p_k \le q_k$ and (q_k/p_k) be bounded. Then $c(M, q, \sigma, s) \subset c(M, p, \sigma, s)$,

(iv) $s_1 \le s_2$ implies $c(M, p, \sigma, s_1) \subset c(M, p, \sigma, s_2)$.

PROOF : (*i*) Let $x \in c(M, p, \sigma, s)$, since $0 < \inf_k p_k \le 1$, we get

$$k^{-s}[M_k(|x_{\sigma(n)}{}^k - L|/\rho)] \le k^{-s}[M_k(|x_{\sigma(n)}{}^k - L|/\rho)]^{p_k}$$

for each n and hence get $x \in c(p, \sigma, s)$.

(*ii*) Let $p_k \ge 1$ for each k, $\sup_k p_k < \infty$ and $x \in c(p, \sigma, s)$. Then for each $0 < \varepsilon < 1$ there exists a positive integer N such that

$$k^{-s}[M_k(|x_{\sigma(n)}{}^k - L|/\rho)] \le \varepsilon < 1 \text{ for each } n \text{ and for all } k \ge N.$$

This implies that

$$k^{-s}[M_k(|x_{\sigma(n)}{}^k - L|/\rho)]^{p}_{k} \le k^{-s}[M_k(|x_{\sigma(n)}{}^k - L|/\rho)]$$

Thus we get $x \in c(M, p, \sigma, s)$.

(*iii*) If we take $w_k = k^{-s}[M(|x^{k}_{\sigma(n)} - L|/\rho)]^{q}_{k}$ for all k, then using the same technique of Theorem 2 of Nanda[10], it is easy to prove (iii).

(*iv*) Let $s_1 \le s_2$. Then $k_2^{-s} < k_1^{-s}$ for all $k \in \mathbb{N}$. Since

$$k_2^{-s}[M(|x_{\sigma(n)}{}^k - L|/\rho)]^{p_k} \le k_1^{-s}[M(|x_{\sigma(n)}{}^k - L|/\rho)]^{p_k}$$

for all k and n, then we have $c(M, p, \sigma, s_1) \subset c(M, p, \sigma, s_2)$.

For $r > 0$, a nonempty subset Δ of a linear space X is said to be absolutely r-convex if $x, y \in \Delta$ and $|\lambda|^r + |\mu|^r \le 1$ together imply that $\lambda x + \mu y \in \Delta$. A linear topological space X is said to be r-convex (see Maddox and Roles[11]) if every neighbuorhood of $0 \in X$ contains as absolutely r-convex neighbourhood of $0 \in X$.

We have :

Theorem 5 — *$c_0(M, p, \sigma, s)$ and $c(M, p, \sigma, s)$ are 1-convex.*

PROOF : Let $x \in (M, p, \sigma, s)$. We define

$$h(x) = \sup_{k,n}(k^{-s}[M(|x_{\sigma(n)}{}^k|/\rho)]^{p}_{k})^{1/H}$$

If $0 \prec \delta \prec 1$, then $\Delta = \{x : h(x) \le \delta\}$ is an absolutely 1-convex set, for $x, y \in \Delta$ and $|\lambda| + |\mu| \le 1$, then using by convexity of M,

$$h(\lambda x + \mu y) \le (|\lambda| + |\mu|)^{p_k/H}\,\delta \le \delta.$$

This completes the proof.

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