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EXPOSED POINTS OF THE UNIT BALLS OF THE SPACES $\mathcal{P}(^2 l_p^2)$ $(p = 1, 2, \infty)$

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Ryan and Turett showed that for finite-dimensional space E, the set of exposed points and the set of extreme points of the closed unit ball of $\hat{E}_\pi^{(n)}$ coincide, where $\hat{E}_\pi^{(n)}$ is the completed n-fold symmetric tensor product of E with the projective s-tensor norm. In this note we show that a similar result is not true for the dual space $\mathcal{P}(^n E)$ and classify the exposed points of the unit balls of the spaces $\mathcal{P}(^2 l_p^2)$ for $p = 1, 2, \infty$.

We recall that a unit vector x in a real Banach space E is *exposed* if there is a unit vector $f \in E^*$ so that $f(x) = 1$ and $f(y) < 1$ for $y \in B_E \setminus \{x\}$, where B_E is the closed unit ball of E. It is easy to see that every exposed point of B_E is an extreme point. We let $\mathcal{L}_s(^n E)$ denote the Banach space of continuous symmetric n-linear mappings of $E^n := E \times \ldots \times E$ into R, endowed with the norm

$$\| A \| = \sup \{ |A(x_1, \ldots, x_n)| : x_j \in B_E, j = 1, \ldots, n \}.$$

A mapping $P : E \to R$ is called a continuous n-homogeneous polynomial if there is $A \in \mathcal{L}_s(^n E)$ such that $P(x) = A(x, \ldots., x)$. We let $\mathcal{P}(^n E)$ denote the Banach space of continuous n-homogeneous polynomials of E into R, endowed with the polynomial norm $\| P \| = \sup_{x \in B_E} | P(x) |$. Note that $\mathcal{P}(^1 E) = \mathcal{L}_s(^1 E)$ is the space of bounded linear forms on E. See[3] for details about polynomials on a Banach space.

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Ryan and Turett[4] showed that for finite-dimensional space E, the set of exposed points and the set of extreme points of the closed unit ball of $\hat{E}_\pi^{(n)}$ coincide and equal the set $\{\pm x^n : \|x\| = 1\}$, where $\hat{E}_\pi^{(n)}$ is the completed n-fold symmetric tensor product of E with the projective s-tensor norm. Note that the dual space $\hat{E}_\pi^{(n)}$ is the space $\mathcal{P}(^nE)$. In this note we show that a similar result is not true for the dual space $\mathcal{P}(^nE)$ and classify the exposed points of the unit balls of the spaces $\mathcal{P}(^2l_p^2)$ for $p = 1, 2, \infty$.

We denote by $expB_E$ and $extB_E$ the sets of exposed points and extreme points of B_E, respectively. It was showed in[1,2] that

$$P(x, y) = ax^2 + by^2 + cxy \in ext\, B_{\mathcal{P}(^2l_1^2)}$$

if and only if

$$(|a| = |b| = 1, |c| = 2) \text{ or } (a = -b, 2 < |c| \le 4, 4a^2 = 4|c| - c^2,$$

$$extB_{\mathcal{P}(^2l_\infty^2)} = \left\{ \pm x^2, \pm y^2, \pm (ax^2 - ay^2 \pm 2\sqrt{a(1-a)}\, xy) : \frac{1}{2} \le a \le 1 \right\},$$

and
$$P(x, y) = ax^2 + by^2 + cxy \in ext\, B_{\mathcal{P}(^2l_2^2)}$$

if and only if

$$(|a| = |b| = 1, c = 0) \text{ or } (a = -b, 0 < |c| \le 2, 4a^2 = 4 - c^2).$$

***Theorem* 1** — $expB_{\mathcal{P}(^2l_\infty^2)} = ext\, B_{\mathcal{P}(^2l_\infty^2)} \setminus \{1/2\, x^2 - 1/2y^2 \pm xy, -1/2x^2 + 1/2y^2 \pm xy\}$.

PROOF : First we show that $P(x, y) = 1/2\, x^2 - 1/2\, y^2 + xy$ is not in the set $\exp B_{\mathcal{P}(^2l_\infty^2)}$. Let $f \in \mathcal{P}(^2l_\infty^2)^*$ be such that $f(P) = 1 = \|f\|$. Let $\alpha = f(x^2), \beta = f(y^2)$ and $\gamma = f(xy)$. Note that $|\alpha| \le 1, |\beta| \le 1$ and $|\gamma| \le 1$. Clearly,

$$1 = 1/2\,(\alpha - \beta) + \gamma, \qquad \text{... (1)}$$

so $\alpha \ge \beta$. Since f doesn't expose P if $\alpha = 1$ or $\beta = -1$, it is enough to consider the case where $-1 < \beta \le \alpha < 1/$ By Theorem 4.2 in[2], we have

$$1 = \|f\| = 1/2\,(|\alpha - \beta| + \sqrt{(\alpha - \beta)^2 + 4\gamma^2}). \qquad \text{... (2)}$$

It follows from (1) and (2) that $f(xy) = \gamma = 1$, which implies that f doesn't expose P. Similarly, we can prove that the polynomials $\pm(1/2\, x^2 - 1/2\, y^2 \pm xy)$ are not in the set $expB_P(^2l_\infty^2)$.

Now we show that the other extreme points are exposed points.

1° Let $P(x, y) = x^2$ and let $f \in \mathcal{P}({}^2 l_\infty^2)^*$ be such that $f(x^2) = 1, f(y^2) = 1/2)$ and $f(xy) = 0$. Then $\|f\| = 1 = f(P)$. We claim that f exposes P. Suppose that $\|Q\| = 1 = f(Q)$ for $Q(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}({}^2 l_\infty^2)$. Clearly $|a| \leq 1, |b| \leq 1$ and $|c| \leq 1$ and $a + (1/2)b = 1$, hence $a \geq 0$ and $b \geq 0$. In this case $\|Q\| = a + b + |c| = 1$, which implies $c = 0$, $a = 1$ and $b = 0$, that is, $Q(x, y) = x^2$. Similarly, we can show that the polynomials $-x^2$ and $\pm y^2$ are exposed points.

2° — Let $P(x, y) = wx^2 - wy^2 + 2\sqrt{w(1-w)}\,xy$ for $1/2 < w \leq 1$ and let $f \in \mathcal{P}({}^2 l_\infty^2)^*$ be such that $f(x^2) = 1 - \frac{1}{2w}, f(y^2) = -1 + \frac{1}{2w}$ and $f(xy) = \sqrt{\frac{1-w}{w}}$. Clearly $1 = \|f\| = f(P)$. We claim that f exposes P. Suppose that $\|Q\| = 1 = f(Q)$ for $Q(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}({}^2 l_\infty^2)$. Clearly $|a| \leq 1$, $|b| \leq 1$ and $|c| \leq 1$ and

$$a - b + \frac{2w}{2w-1}\sqrt{\frac{1-w}{w}}\,c = \frac{2w}{2w-1}. \qquad \text{... (3)}$$

Let $M = \max\{|a|, |b|\}$ and $m = \min\{|a|, |b|\}$. If $ab \geq 0$ or if $a < 0$ and $b > 0$, it follows from (3) that

$$\frac{2w}{2w-1} \leq + \frac{2w}{2w-1}\sqrt{\frac{1-w}{w}} < \frac{2w}{2w-1}$$

for $1/2 < w \leq 1$, hence $a > 0$ and $b < 0$. In this case the eq. (3) is equivalent to the equation

$$m + M + \frac{2w}{2w-1}\sqrt{\frac{1-w}{w}}\,c = \frac{2w}{2w-1}. \qquad \text{... (4)}$$

If $|c| > 2m$, by Theorem 5.1 in[2] we have

$$M - m + |c| = 1. \qquad \text{... (5)}$$

It follows from (4) and (5) that

$$m = \frac{1}{4w-2} - \frac{w}{2w-1}\sqrt{\frac{1-w}{w}}\,c + \frac{|c|}{2},$$

which implies that

$$\frac{-1}{2w-1} + \frac{2w}{2w-1}\sqrt{\frac{1-w}{w}}\,c > 0.$$

Hence $w \neq 1$ and $c > \frac{1}{2\sqrt{w(1-w)}} > 1$ for $\frac{1}{2} < w < 1$, which is a contradiction.

Therefore $|c| \leq 2m$ and in this case, applying Theorem 5.1 in [2] again, we have

$$M = 1 - \frac{c^2}{4m}. \qquad \text{... (6)}$$

Put $t=\frac{1-2c\sqrt{w(1-w)}}{2w-1}$. It follows from (4) and (6) that we get : $m^2-tm-\frac{c^2}{4}=0$, so $\frac{c^2}{4}=m^2-tm$. Solving this equation we get :

$$m=\frac{t+\sqrt{t^2+c^2}}{2}.$$

Note that $\frac{c^2}{4m}=\frac{1}{m}(m^2-tm)=m-t$.

It follows that

$$M-m=1-\left(\frac{c^2}{4m}+m\right)=1-(2m-t)=1-\sqrt{t^2+c^2}$$

$$=1-\sqrt{c^2+\left(\frac{2w}{2w-1}\sqrt{\frac{1-w}{w}}\,c-\frac{1}{2w-1}\right)^2}\geq 0.$$

Note that $\sqrt{c^2+\left(\frac{2w}{2w-1}\sqrt{\frac{1-w}{w}}\,c-\frac{1}{2w-1}\right)^2}\leq 1$

$$\Leftrightarrow\ c^2+\frac{1}{(2w-1)^2}(2\sqrt{w(1-w)}\,c-1)^2-1\leq 0$$

$$\Leftrightarrow\ \frac{1}{(2w-1)^2}(c-2\sqrt{w(1-w)})^2\leq 0 \text{ (by factorizing in terms of } c\text{)},$$

hence $c=2\sqrt{w(1-w)}$. Replacing c in (4) by $2\sqrt{w(1-w)}$ gives $M+m=2w$. It follows from (6) and $M+m=2w$ that $M=m=w$, hence $a=w$, $b=-w$ and $Q=P$. Similarly, we can show that $-wx^2+wy^2\pm 2\sqrt{w(1-w)}\,xy\in \exp B_{\mathcal{P}(^2l_\infty^2)}$ for $1/2<w\leq 1$. ■

The mapping $T(x,y)=(x-y,x+y)$ is a linear isometry of l_1^2 with l_∞^2. Applying this linear isometry, we can also classify the exposed points of the unit ball of $\mathcal{P}(^2l_1^2)$, that is,

$$\exp B_{\mathcal{P}(^2l_1^2)}=ext\,B_{\mathcal{P}(^2l_1^2)}\setminus\{x-y^2\pm 2xy,-x^2+y^2\pm 2xy\}.$$

***Theorem* 2** — $\exp B_{\mathcal{P}(^2l_1^2)}=ext\,B_{\mathcal{P}(^2l_1^2)}$.

PROOF : First we show that $x^2-y^2\in\exp B_{\mathcal{P}(^2l_2^2)}$. Indeed, let $P(x,y)=x^2-y^2$ and let $f\in\mathcal{P}(^2l_2^2)^*$ be such that $f(x^2)=1/2, f(y^2)=-1/2$ and $f(xy)=0$. Clearly $1=\|f\|=f(P)$ and f exposes P. Indeed, suppose that $\|Q\|=1=f(Q)$ for $Q(x,y)=ax^2+by^2+cxy\in\mathcal{P}(^2l_2^2)$. Clearly $|a|\leq 1, |b|\leq 1$ and $|c|\leq 2$ and $a-b=2$, hence we have $a=1$ and $b=-1$. By Lemma 2, 1 in [2] we have $c=0$, hence $Q(x,y)=x^2-y^2$.

We make two observations : Firstly, if x_0 is an extreme (resp. exposed) point of the unit ball of a Banach space E and T is an isometry of E, then $T(x_0)$ is an extreme (resp. exposed) point of the unit ball of E.

Secondly, note that if T is an isometry of E, then $P \to P \circ T$ is an isometry of $\mathcal{P}(^nE)$ for every positive integer n

Note that the ismetries of l_2^2 are all of the form $T_\theta(x, y) = (x \cos \theta + y \sin \theta - x \sin \theta + y \cos \theta)$ for $\theta \in [0, 2\pi]$. Therefore $P(x, y) = x^2 - y^2$ is an exposed point of the unit ball of $\mathcal{P}(^2l_2^2)$ if and only if

$$P \circ T_\theta(x, y) = \cos 2\theta x^2 - \cos 2\theta y^2 - 2 \sin \theta xy$$

is exposed for some (and hence any) $\theta \in [0, 2\pi]$. Since $\mathcal{P}(^2l_2^2)$ is reflexive, the closed convex hull of its exposed points is equal to its unit ball. As dim $\mathcal{P}(^2l_2^2) = 3$ and $\{\cos 2\theta x^2 - \cos 2\theta y^2 - 2 \sin 2\theta xy : \theta \in [0, 2\pi]\}$ spans a two dimensional subspace, we see that exposed points of the unit ball of $\mathcal{P}(^2l_2^2)$ are as given in Theorem 2. ■

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