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# GENERALIZED FUNCTIONS AND AN EXTENDED GAP THEOREM

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N. Levinson proved a gap theorem which roughly states that if $f$ is an integrable function such that a subsequence of the Fourier coefficients of $f$ decreases rapidly to zero, then $f$ cannot vanish almost everywhere on an interval of length $L$ (where $L$ depends on the subsequence) unless $f$ vanishes almost everywhere. We will extend this theorem to a space of generalized functions known as Boehmians. Using Levinson's theorem we also obtain necessary conditions for a sequence of complex numbers to be the Fourier coefficients for a Boehmian.

**Key Words : Boehmian; Fourier Coefficients; Gap Theorem; Generalized Functions**

## 1. INTRODUCTION

In the mid 1930's N. Levinson[2] proved the following gap theorem.

***Theorem* 1.1** — *Suppose* $f \in \mathcal{L}(-\pi, \pi)$ *and* $f(t) \sim \sum_{-\infty}^{\infty} a_n e^{int}$. *Let* $\{\lambda_n\}$ *be an increasing sequence of positive integers such that* $\lim_{n \to \infty} \frac{n}{\lambda_n} = D$. *Let* $a_{-\lambda_n} = O(e^{-\theta(\lambda_n)})$, *where* $\theta(t)$ *is monotone increasing and* $\int_1^{\infty} \frac{\theta(t)}{t^2} dt = \infty$. *If* $f(t)$ *vanishes almost everywhere over an interval exceeding* $2\pi(1-D)$ *in length, then* $f(t)$ *is equivalent to zero.*

The growth condition for the subsequence of negative Fourier coefficients can be replaced by an equivalent condition for a subsequence of the positive .Fourier coefficients.

We will extend this theorem to a space of generalized functions called Boehmians which contains $\mathcal{L}(-\pi, \pi)$, periodic measures, as well as all Schwartz distributions on the unit circle. Also, using Levinson's theorem, we will improve upon Theorem 4.4 in [4] by obtaining necessary conditions for a sequence of complex numbers to be the Fourier coefficients for a Boehmian.

## 2. PRELIMINARIES

Let $\mathcal{L}(T)$ denote the space of all Lebesgue integrable functions on the unit circle $T$. Let $C(T)$ denote the subspace of $\mathcal{L}(T)$ consisting of all complex valued continuous functions, and $C^N(T)$ denote the collection of sequence in $C(T)$. We make no distinction between a function on $T$ and a $2\pi$-periodic function on $\mathbb{R}$.

A sequence $\{\phi_n\}$ of nonnegative functions in $C(T)$ is called a delta sequence if

(*i*) $\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_n(t)\, dt = 1$ for all $n \in \mathbb{N}$, and (*ii*) $\phi_n(t) = 0$ for $0 < \varepsilon_n < |t| < \pi$, where $\varepsilon_n \to 0$.

The collection of delta sequences will be denoted by $\Delta$.

Let $f, g \in C(T)$. The *convolution* of two functions $f$ and $g$ is given by

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t - \sigma)\, g(\sigma)\, d\sigma.$$

Let $\mathcal{A} = \{((f_n), (\phi_n)) \in C^N(T) \times \Delta : f_n * \phi_k = f_k * \phi_n$ for all $n, k \in I\!N\}$.

$((f_n), (\phi_n)) \sim ((g_n), (\delta_n))$ if $f_n * \delta_k = g_k * \phi_n$ for all $n, k \in I\!N$. "~" is an equivalence relation on $\mathcal{A}$. The collection of equivalence classes will be denoted by $\beta$. Elements of $\beta$ are called *Boehmians*, and a typical element of $\beta$ is written as $\frac{f_n}{\phi_n}$.

Addition, multiplication, and scalar multiplication are defined in the natural way, and $\beta$ with these operations is an algebra with identity $\delta = \frac{\phi_n}{\phi_n}$.

$$\frac{f_n}{\phi_n} + \frac{g_n}{\delta_n} = \frac{f_n * \delta_n + g_n * \phi_n}{\phi_n * \delta_n}$$

$$\frac{f_n}{\phi_n} * \frac{g_n}{\delta_n} = \frac{f_n * g_n}{\phi_n * \delta_n},$$

and

$$\alpha \frac{f_n}{\phi_n} = \frac{\alpha f_n}{\phi_n}$$

$\mathcal{L}(T)$ can be identified with a subspace of $\beta$ by

$$f \leftrightarrow \frac{f * \phi_n}{\phi_n}.$$

Similarly, $D'(T)$, the space of Schwartz distributions on the unit circle[1], can be identified with a subspace of $\beta$.

For $f \in C(T)$, the $k$th Fourier coefficient is given by

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt.$$

*Definition* 2.1 — For $F = \frac{f_n}{\phi_n} \in \beta$, the $k$th *Fourier coefficient* is defined by

$$\hat{F}(k) = \lim_{n \to \infty} \hat{f}_n(k).$$

The limit in the above definition is independent of the representative of $F$.

A sequence $\{F_n\}$ of Boehmians is said to be *Δ-convergent* to $F$ if there exists a delta sequence $\{\phi_n\}$ such that $(F_n - F) * \phi_n \in C(T)$ for all $n$ and $(F_n - F) * \phi_n \to 0$ uniformly on $T$ as $n \to \infty$. $\beta$ with $\Delta$-convergence is an $F$-space (see [3]), that is, a complete topological vector space in which the topology is given by an invariant metric. Moreover, $F = \Delta - \lim_{n \to \infty} \sum_{k=-n}^{n} \hat{F}(k)\, e^{ikt}$, for all $F \in \beta$.

***Theorem* 2.2** (see[5]) — *Let $\omega$ be a real-valued even function defined on the integers $\mathbb{Z}$ such that $0 = \omega(0) \le \omega(n+m) \le \omega(n) + \omega(m)$ for all $m, m \in \mathbb{Z}$ and $\sum_{n=1}^{\infty} \frac{\omega(n)}{n^2} < \infty$. Suppose that the set of positive integers is partitioned into two disjoint sets $\{t_n\}$ and $\{s_n\}$ such that $\sum_{n=1}^{\infty} \frac{1}{t_n} < \infty$. If $\{\xi_n\}$ is a sequence of complex numbers such that $\xi_{\pm s_n} = O(e^{\omega(s_n)})$ as $n \to \infty$, then $\{\xi_n\}$ is the sequence of Fourier coefficients for some Boehmian.*

***Theorem* 2.3** (see 4) — *Let $\omega : \mathbb{Z} \to \mathbb{R}$ be an increasing function for $n = 0, 1, 2, \ldots.$ and $\sum_{n=1}^{\infty} \frac{\omega(n)}{n^2} = \infty$. Suppose $\{\xi_n\}$ is a sequence of complex numbers such that there exist positive $A$, $M$ and $\varepsilon$ such that $|\xi_n| \ge A\, e^{\varepsilon\, \omega(n)}$ for all $n \ge M$. Then $\{\xi_n\}$ is not the Fourier coefficients of a Boehmian.*

Theorem 2.2 gives sufficient conditions for a sequence of complex numbers to be the Fourier coefficients of a Boehmian, while Theorem 2.3 gives necessary conditions for a sequence of complex numbers to be the Fourier coefficients of a Boehmian.

By using Theorem 2.2 it is clear that $\beta$ contains a proper subspace which can be identified with the space of periodic Schwartz distributions. Theorem 2.2 also shows that there are Boehmians which are not hyperfunctions. Conversely, by using Theorem 2.3, we see that there are hyperfunctions which are not Boehmians.

## 3. The Main Results

In this section we extend Theorem 1.1 to include all Boehmians. Also, by using Theorem 1.1, we will generalize Theorem 2.3. But first we give :

*Definition* 3.1 — Let $F = \frac{f_n}{\phi_n} \in \beta$ and $\Omega$ be an open arc. Then $F = 0$ on $\Omega$ if $f_n \to 0$ uniformly on compact subsets of $\Omega$ as $n \to \infty$.

The limit in the above definition is independent of the representatitive used for $F$.

If $f$ is a function (distribution), then Definition 3.1 is consistent with the usual definition of a function (distribution) vanishing on an open interval.

*Example* 3.2 — Let $\delta_{2\pi}$ be the periodic Dirac delta measure (distributiuon), i.e., point mass at $2\pi n, n \in \mathbb{Z}$. Then $\delta_{2\pi} = \frac{\delta_{2\pi} * \varphi_n}{\varphi_n}$, where $\{\varphi_n\}$ is an infinitely differentiable delta sequence and $\delta_{2\pi} * \varphi_n$ denotes the convolution of $\delta_{2\pi}$ and $\varphi_n$ as distributions. Since $\delta_{2\pi} * \varphi_n = \varphi_n \to 0$ uniformly for $0 < \varepsilon \le |t| \le \pi$, $\delta_{2\pi} = 0$ on $0 < |t| < \pi$.

**Theorem 3.3** — *Let $\{\lambda_n\}$ be an increasing sequence of positive integers such that* $\lim_{n\to\infty} \frac{n}{\lambda_n} = D$. *Let $F \in \beta$ such that $\hat{F}(-\lambda_n) = O(e^{-\theta(\lambda_n)})$, where $\theta(t)$ is monotone increasing and $\int_1^\infty \frac{\theta(t)}{t^2}\, dt = \infty$. If $F$ vanishes over an interval exceeding $2\pi(1-D)$ in length, then $F$ is equivalent to zero.*

PROOF : Let $F = \frac{f_n}{\varphi_n} \in \beta$ such that

$$\hat{F}(-\lambda_n) = O(e^{-\theta\lambda_n}). \qquad \text{... (3.1)}$$

Suppose that $F = 0$ on $(a, b)$, where $b - a > 2\pi(1-D)$. Pick $\eta > 0$ so that $b - a - 2\eta > 2\pi(1-D)$. Now,

$$f_k = f_k - (f_k * \varphi_j) + (f_k * \varphi_j), \quad k, j \in \mathbb{N}. \qquad \text{... (3.2)}$$

Since $\{\varphi_j\}$ is a delta sequence, for each $k$

$$f_k * \varphi_j \to f_k \text{ uniformly on } T \text{ as } j \to \infty. \qquad \text{... (3.3)}$$

Pick $k_0 \in \mathbb{N}$ such that supp $\varphi_k \subset \left(\frac{-\eta}{2}, \frac{\eta}{2}\right)$, for all $k \ge k_0$.

Let $\varepsilon > 0$.

Since $F = 0$ on $(a, b)$, $f_j \to 0$ uniformly on $\left[a + \frac{\eta}{2}, b - \frac{\eta}{2}\right]$ as $j \to \infty$.

Thus, there exists a $j_0 \in \mathbb{N}$ such that for all $j \ge j_0$,

$$|f_j(t)| < \varepsilon, \text{ for all } t \in \left[a + \frac{\eta}{2}, b - \frac{\eta}{2}\right].$$

Let $k$ be any fixed integer greater than $k_0$. Then for all $j \ge j_0$,

$$|(f_k * \varphi_j)(t)| = |(f_j * \varphi_k)(t)| \leq \frac{1}{2\pi} \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} |f_j(t-\sigma)|\, \varphi_k(\sigma)\, d\sigma$$

$$< \frac{\varepsilon}{2\pi} \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} \varphi_k(\sigma d\sigma) = \varepsilon,$$

for all $t \in (a+\eta, b-\eta)$. That is, for each $k \geq k_0$,

$$f_k * \varphi_j \to 0 \text{ uniformly on } (a+\eta, b-\eta) \text{ as } j \to \infty \quad \text{... (3.4)}$$

By combining (3.2), (3.3) and (3.4) we see that for each $k \geq k_0$

$$f_k = 0 \text{ on } (a+\eta, b-\eta). \quad \text{... (3.5)}$$

Now, $\hat{f}_k(-\lambda_n) = \hat{f}_k(-\lambda_n) \lim_{j\to\infty} \hat{\varphi}_j(-\lambda_n) = \lim_{j\to\infty} (f_k * \varphi_j)^\wedge(-\lambda_n)$

$$= \lim_{j\to\infty} (f_j * \varphi_k)^\wedge(-\lambda_n) = \lim_{j\to\infty} \hat{f}_j(-\lambda_n)\, \hat{\varphi}_k(-\lambda_n) = \hat{F}(-\lambda_n)\, \hat{\varphi}_k(-\lambda_n).$$

That is, $\hat{f}_k(-\lambda_n) = \hat{F}(-\lambda_n)\, \hat{\varphi}_k(-\lambda_n)$, $k, n \in I\!N$. ... (3.6)

By (3.1) and (3.6)

$$\hat{f}_k(-\lambda_n) = O(e^{-\theta(\lambda_n)}), k \in I\!N. \quad \text{... (3.7)}$$

Now using (3.5), (3.7), and Theorem 1.1, we see that $f_k$ is equivalent to zero for $k \geq k_0$, and hence, $F$ is equivalent to zero. This establishes the theorem. □

*Example* 3.4 — Let $\{\xi_n\}$ be any sequence of complex numbers. Then, by Theorem 2.2, there exists a Boehmian $F$ such that $F = \sum_{n=1}^{\infty} \xi_n \exp(i\, 2^n t)$. By Theorem 3.3, $F$ does not vanish on any interval.

The next theorem strengthens the result of Theorem 2.3.

***Theorem* 3.5** — *Let $\theta(t)$ be a monotone increasing function such that $\int_1^\infty \frac{\theta(t)}{t^2} dt = \infty$. Let $\{\lambda_n\}$ be an increasing sequence of positive integers such that $\lim_{n\to\infty} \frac{n}{\lambda_n} = D > 0$. Then*

$$sup_{F \in \beta}\ \inf_{n \in N} \left\{ e^{-\theta(\lambda_n)} |\hat{F}(-\lambda_n)| \right\} = 0.$$

PROOF : Let $\lim_{n\to\infty} \frac{n}{\lambda_n} = D > 0.$

Suppose for some $F = \frac{f_n}{\varphi_n} \in \beta$ there exists an $M > 0$ such that

$$e^{-\theta(\lambda_n)} |\hat{F}(-\lambda_n)| \geq M, \text{ for all } n \in I\!N.$$

Then for each $k \in I\!N$ there exists $M_k > 0$ such that

$$|\hat{\varphi}_k(-\lambda_n)| = \frac{|\hat{f}_k(-\lambda_n)|}{|\hat{F}(-\lambda_n)|} \leq M_k e^{-\theta(\lambda_n)}, \text{ for all } n \in I\!N.$$

For sufficiently large $k$, $\varphi_k$ vanishes on an interval of length greater than $2\pi(1-D)$. Thus, by Theorem 1.1, $\varphi_k$ is equivalent to zero for sufficiently large $k$. This contradicts the fact that $\{\varphi_k\}$ is a delta sequence. This establishes the theorem. □

Let $\{\xi_n\}$ be a sequence of complex numbers. Suppose there exist an increasing function $\theta(t)$, a constant $M > 0$, and an increasing sequence of positive integers $\{\lambda_n\}$ such that

(i) $\int_1^\infty \frac{\theta(t)}{t^2} dt = \infty,$

(ii) $\lim_{n\to\infty} \frac{n}{\lambda_n} = D > 0,$

(iii) $e^{-\theta(\lambda_n)} |\xi_{\lambda_n}| \geq M, n \in I\!N.$

Then, by Theorem 3.5, $\{\xi_n\}$ is not the Fourier coefficients for any Boehmian.

Thus to show that Theorem 3.5 generalizes Theorem 2.3, we need to show that if $\{\xi_n\}$ is a sequence of complex numbers and $\omega(n)$ is an increasing function for $n = 0, 1, 2, \ldots$ such that $\sum_{n=1}^{\infty} \frac{\omega(n)}{n^2} = \infty$ and $|\xi_n| \geq Ae^{\varepsilon\omega(n)}, n \geq n_0$ (for some $A > 0$, $\varepsilon > 0, n_0 > 0$), then there exist an increasing function $\theta(t)$, a constant $M > 0$, and an increasing sequence of positive integers $\{\lambda_n\}$ such that the conditions (i), (ii) and (iii) are satisfied.

Let the sequence $\{\xi_n\}$ and the function $\omega(n)$ satisfy the conditions above. Let $\tilde{\omega}(t)$ be the linear extension of $\omega(n)$ and $\theta(t) = \varepsilon\tilde{\omega}(t)$. Then $\theta(t)$ is increasing and $\int_1^\infty \frac{\theta(t)}{t^2} dt = \infty$. Let $\lambda_n = n + n_0, n \in I\!N$. Then $|\xi_{\lambda_n}| \geq Ae^{\varepsilon\omega(\lambda_n)} = Ae^{\theta(\lambda_n)}, n \in I\!N$ and $\lim_{n\to\infty} \frac{n}{\lambda_n} = 1.$

*Example* 3.6 — Let $\alpha_n = \dfrac{10n}{\ln 10n}$, for $n = 1, 2, \ldots$ Then by Theorem 3.5, $\sum_{n=1}^{\infty} e^{\alpha_n} e^{-i\,10\,nt}$ is not the Fourier series of any Boehmian. Notice that Theorem 2.3 does not apply. Moreover, notice that $\sum_{n=1}^{\infty} e^{\alpha_n} e^{-i\,10\,nt}$ is the Fourier series for some hyperfunction.

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