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A NOTE ON LOCAL CONVEXITY OF THE *p*-POWER LAGRANGIAN FUNCTION IN NONCONVEX OPTIMIZATION*

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In this note, we give a new proof on a main result by Li and Sun[1].

Key Words : Nonconvex optimization, Lagrangian function, local convexification, *p*-power formulation

1. INTRODUCTION

We consider a constrained nonconvex optimization problem of the following form :

Problem MP

$$\min f_0(x)$$

$$\text{s.t. } f_j(x) \le b_j, j = i, 2, \ldots, m,$$

$$x \in X,$$

where $f_j : I\!R^n \to I\!R$, $j = 1, 2, \ldots, m$, are twice continuously differentiable functions and X is a nonempty closed set in $I\!R^n$, f_0 is positive over X, $f_j(x)$, $j = 1, 2, \ldots, m$ are nonnegative over X, and b_j, $j = 1, 2, \ldots, m$ are positive.

Let x^* be a local optimal solution of problem (MP). The Lagrangian function associated with problem (MP) is given by

$$L(x, \lambda) = f_0(x) + \sum_{j=1}^{m} \lambda_j [f_j(x) - b_j], \quad \lambda \ge 0.$$

Suppose that x^* is a regular point of the constraints in (MP) and x^* satisfies the second order sufficiency condition. Then, there exists a Lagrangian multiplier vector $\lambda^* \ge 0$ such that

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$$\nabla f_0(x^*) + \sum_{j=1}^{m} \lambda_j^* \nabla f_j(x^*) = 0, \qquad \text{... (1)}$$

$$\lambda_j^* [f_j(x^*) - b_j] = 0, \quad j = 1, 2, \ldots, m, \qquad \text{... (2)}$$

and the Hessian matrix

$$\nabla^2 L(x^*, \lambda^*) = \nabla^2 f_0(x^*) + \sum_{j=1}^{m} \lambda_j^* \nabla^2 f_j(x^*)$$

is positive definite on the tangent subspace

$$M(x^*) = \{d \in \mathbb{R}^n \mid d^T \nabla f_j(x^*) = 0 \ \ \forall j \in J(x^*)\},$$

where $J(x^*) = \{j \mid \lambda_j^* > 0, j = 1, 2, \ldots, m\}$.

Thus, we have

$$d^T \nabla^2 L(x^*, \lambda^*) d > 0, \ \ \forall d \in M(x^*), \ d \neq 0. \qquad \text{... (3)}$$

Applying the power p, with $p \geq 1$, to both the objective function and the constraints results in the following p-power formulation (see Refs. 2-3), which is equivalent to problem (MP) :

Problem (MP_1)

$$\min \ [f_0(x)]^p,$$

$$\text{s.t.} \ [f_j(x)]^p \leq b_j^p, \ j = i, 2, \ldots, m.$$

$$x \in X.$$

Now, we consider the p-power Lagrangian function associated with (MP_1),

$$L_p(x, \lambda) = [f_0(x)]^p + \sum_{j=1}^{m} \mu_j \ \{[f_j(x)]^p - [b_j]^p\}, \qquad \text{... (4)}$$

for $p > 0$ and $\mu \geq 0$. From (1) and (2), the optimal multipliers associated with x^* in the p-power Lagrangian (4) are given by

$$(\mu_p^*)_j = \begin{cases} [f_0(x^*)]^{p-1/b_j^{p-1}} \lambda_j^*, & j \in J(x^*) \\ 0, & \text{otherwise} \end{cases} \qquad \text{... (5)}$$

In Ref. 1, the authors prove following an important result :

***Theorem* 1** — *Let x^* be a local optimal solution of (MP). Assume that $J(x^*) \neq 0, x^*$ is a regular point, and x^* satisfies the second-order sufficiency condition. Then, there exists a $q_1 > 0$ such that Hessian matrix $\nabla^2 L_p(x^*, \mu_p^*)$ of the p-power Lagrangian is positive definite when $p > q_1$.*

2. The Proof of Theorem

Let $\lambda_0^* = 1$, $(\mu_p^*) = 1$, $\bar{J}(x^*) = J(x^*) \cup \{0\}$.

From (2), (4) and (5), we have

$$\nabla^2 L_p (x^*, \mu_p^*)$$

$$= \sum_{j=0}^{m} (\mu_p^*)_j \{p [f_j (x^*)]^{p-1} \nabla^2 f_j (x^*) + p (p-1) [f_j (x^*)^{p-2} \nabla f_j (x^*) \nabla^T f_j (x^*)\}$$

$$= p [f_0 (x^*)]^{p-1} \left\{ \nabla^2 L (x^*, \lambda^*) + (p-1) \sum_{j \in \bar{J}(x^*)} [(\lambda_j^* / f_j (x^*)) \nabla f_j (x^*) \nabla^T f_j (x^*)] \right\}$$

$$= p [f_0 (x^*)]^{p-1} \left\{ \nabla^2 L (x^*, \lambda^*) + (p-1) \sum_{j \in \bar{J}(x^*)} [\mu_j \nabla f_j (x^*) \nabla^T f_j (x^*)] \right\} \qquad \dots (6)$$

where $u_0 = 1/f_0 (x^*)$, and $u_j = \lambda_j^* / f_j (x^*) = \lambda_j^* / b_j, j \in j (x^*)$.

Let $S_n = \{d \in I\!R^n \mid \| d \| = 1\}$, $\eta = \min_{d \in S_n, d \in M(x^*)} d^T \nabla^2 L (x^*, \lambda^*) d$.

Since S_n is compact and $M (x^*)$ is closed, it follows from (3) that $\eta > 0$. We need only to show that

$$\lim_{p \to +\infty} \inf \min_{d \in S_n}$$

$$\left\{ d^T \nabla^2 L (x^* \lambda^*) d + (p-1) \sum_{j \in \bar{J}(x^*)} d^T [\mu_j \nabla f_j (x^*) \nabla^T f_j (x^*)] d \right\} \geq \eta.$$

By contradiction,

$$\lim_{p \to +\infty} \inf \min_{d \in S_n}$$

$$\left\{ d^T \nabla^2 L (x^* \lambda^*) d + (p-1) \sum_{j \in \bar{J}(x^*)} d^T [\mu_j \nabla f_j (x^*) \nabla^T f_j (x^*)] d \right\} < \eta,$$

then, there exist $\varepsilon > 0$, for any n, there exist $p_n > n$ and $d_n \in S_n$, such that

$$d_n^T \nabla^2 L (x^*, \lambda^*) d_n + (p_n - 1) \sum_{j \in \bar{J}(x^*)} d_n^T [\mu_j \nabla f_j (x^*) \nabla^T f_j (x^*)] d_n \leq \eta - \varepsilon.$$

That is,

$$d_n^T \nabla^2 L (x^*, \lambda^*) d_n + (p_n - 1) \sum_{j \in \bar{J}(x^*)} \mu_j [d_n^T \nabla f_j (x^*)]^2 \leq \eta - \varepsilon. \qquad \dots (7)$$

By S_n is compact, there exists a subsequence $\{d_{n_k}\}$ of sequence $\{d_n\}$, such that

$$d_{n_k} \to \bar{d},\ \bar{d} \in S_n.$$

From (7), we have

$$\bar{d}^T \nabla^2 L(x^*, \lambda^*)\bar{d} + \limsup_{k \to +\infty} (p_{n_k} - 1) \sum_{j \in \bar{J}(x^*)} \mu_j [d_{n_k}^T \nabla f_j(x^*)]^2 \le \eta - \varepsilon. \quad \ldots (8)$$

It is obvious that

$$\bar{d}\, \nabla f_j(x^*) = 0,\ \forall j \in \bar{J}(x^*).$$

Otherwise, for some $j \in \bar{J}(x^*), \bar{d}\, \nabla f_j(x^*) \neq 0$, then

$$\limsup_{k \to +\infty} (p_{n_k} - 1) \sum_{j \in \bar{J}(x^*)} \mu_j [d_{n_k}^T \nabla f_j(x^*)]^2 \to +\infty,$$

which contradicts (8).

Now from $\bar{d}\, \nabla f_j(x^*) = 0,\ \forall j \in \bar{J}(x^*)$, we know that $\bar{d} \in M(x^*)$. Thus,

$$\bar{d}^T \nabla^2 L(x^*, \lambda^*)\bar{d} \ge \eta.$$

By $\limsup_{k \to +\infty} (p_n - 1) \sum_{j \in \bar{J}(x^*)} \mu_j [d_{n_k}^T \nabla f_j(x^*)]^2 \ge 0$ again, we obtain

$$\bar{d}^T \nabla^2 L(x^*, \lambda^*)\bar{d} + \limsup_{k \to +\infty} (p_n - 1) \sum_{j \in \bar{J}(x^*)} \mu_j [d_{n_k}^T \nabla f_j(x^*)]^2 \ge \eta,$$

which contradicts (8). Therefore, we have

$$\liminf_{p \to k+\infty} \min_{d \in S_n} \left\{ d^T \nabla^2 L(x^*, \lambda^*)\, d + (p-1) \sum_{j \in \bar{J}(x^*)} d^T [\mu_j (\nabla f_j(x^*)\, \nabla^T f_j(x^*)]\, d \right\} \ge \eta, \quad \ldots (9)$$

By $\eta > 0$, (6) and (9), there exists $q_1 > 0$ such that the Hessian matrix $\nabla^2 L_p(x^*, \mu_p^*)$ of the p-power Lagrangian is positive definite when $p > q_1$. This completes the proof. □

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