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ARBITRARY SUPERSUBDIVISIONS OF STARS ARE GRACEFUL

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In this paper we prove that arbitrary supersubdivisions of star graphs are graceful and hence we verify a conjecture and answer an open problem raised by Sethuraman and Selvaraju[6, 7].

Key Words : Graceful Graphs; Supersubdivision of Graphs; Arbitrary Supersubdivision of Graphs

1. INTRODUCTION

A graph $G = (V, E)$ is numbered if each vertex v is assigned a non-negative integer $\phi(v)$ and each edge uv is assigned the value $|\phi(u) - \phi(v)|$. A numbering is called *graceful* if, further, the vertices are labeled with distinct integers from $\{0, 1, 2, ..., q\}$ and the edges with integers from $\{1, 2, ..., q\}$ where q is the number of edges of G. A graph which admits a graceful numbering is said to be *graceful.*

For a survey of results on graceful graphs see[2] and the relevant references given in them. Many mathematicians have constructed larger graceful graphs from standard graphs by using various operations. Recently Sethuraman and Selvaraju[7] have introduced a new method of construction called supersubdivision of a graph.

Let G be a (p, q) graph. A graph H is said to be a *supersubdivision* of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2, m_i} for some m_i, $1 \leq i \leq q$ in such a way that the ends of e_i are merged with the two vertices of the 2-vertices part of K_{2, m_i} after removing the edge e_i from G.

A supersubdivision H of G is said to be an arbitrary supersubdivision of G if every edge of G is replaced by an arbitrary $K_{2, m}$ (m may vary for each edge arbitrarily).

Sethuraman and Selvaraju proved that arbitrary supersubdivisions of any path are graceful[7]. They also proved that there exists a super subdivision of C_n which is graceful[7]. They raised the following open problem.

Problem : Are there any graphs different from paths whose arbitrary supersubdivisions are graceful?

Koh *et al.*[5] have proved that the one vertex union of t copies of the complete bipartite graph $K_{p,q}$ is graceful. The general problem is whether the one vertex union of complete bipartite graphs K_{p_i, q_i} for any p_i and q_i, $1 \le i \le n$ is graceful? Sethuraman and Selvaraju[6] settled this problem for the cases $p_i = 2$ for $i = 1, 2, 3$ and q_i any positive integers for $i = 1, 2, 3$ and $p_i = 2, 1 \le i \le n$ and at least $(n - 2)$ of the q_i's are not equal. Sethuraman and Selvaraju[6] raised the following conjecture.

Conjecture : "The one vertex union of the complete bipartite graphs K_{2, m_i}, $1 \le i \le n$, [where the union is taken at one of the vertices of the partite sets with 2 vertices of each K_{2, m_i}, $1 \le i \le n$], is graceful for all possible choices of the m_i's".

In this paper we prove that arbitrary supersubdivisions of star graphs are graceful. This result answers the above problem and the conjecture.

2. Arbitrary Supersubdivisions of Stars are Graceful

In this section we prove that arbitrary supersubdivision of stars are graceful.

Theorem 2.1 *Arbitrary supersubdivisions of any star are graceful.*

PROOF : Let S_n be a star with vertices $v_0, u_1, u_2, \ldots, u_n$ and let e_i denote the edge $v_0 u_i$ of S_n for $1 \le i \le n$.

Let H be an arbitrary supersubdivision of S_n. That is for $1 \le i \le n$, each edge e_i of S_n is replaced by a complete bipartite graph K_{2, m_i}, where m_i is any positive integer. Observe that H has $2\ (m_1 + m_2 + \ldots + m_n)$ edges.

The vertex set and edge set of H are given below :

$$V(H) = \{v_0, v_1, v_2, \ldots, v_{m_1}, v_{m_1+1}, v_{m_1+2}, \ldots, v_{m_1+2}, v_{m_1+m_2}$$

$$v_{m_1+m_2+1}, \ldots, v_{m_1+m_2+\ldots m_n}, u_1, u_2, \ldots, u_n\}$$

and

$$E(H) = \left\{ v_0 v_i / i = 1, 2, \ldots, \sum_{i=1}^{n} m_i \right\} \cup \{u_1 v_j / j = 1, 2, \ldots, m_1\} \cup$$

$$\{u_2 v_j / j = m_1 + 1, \ldots, m_1 + m_2\} \cup \ldots \cup \left\{ u_n v_j / j = \sum_{i=1}^{n-1} m_i + 1, \ldots, \sum_{i=1}^{n} m_i \right\}$$

Note that $$p = |V(G)| = \left(\sum_{i=1}^{n} m_i \right) + n + 1 \text{ and } q = |E(G)| = 2 \sum_{i=1}^{n} m_i .$$

An arbitrary supersubdivision of S_5 is given in Fig. 1.

Define $$\phi : V(H) \to \left\{ 0, 1, 2, \ldots, 2 \sum_{i=1}^{n} m_i \right\} \text{ by}$$

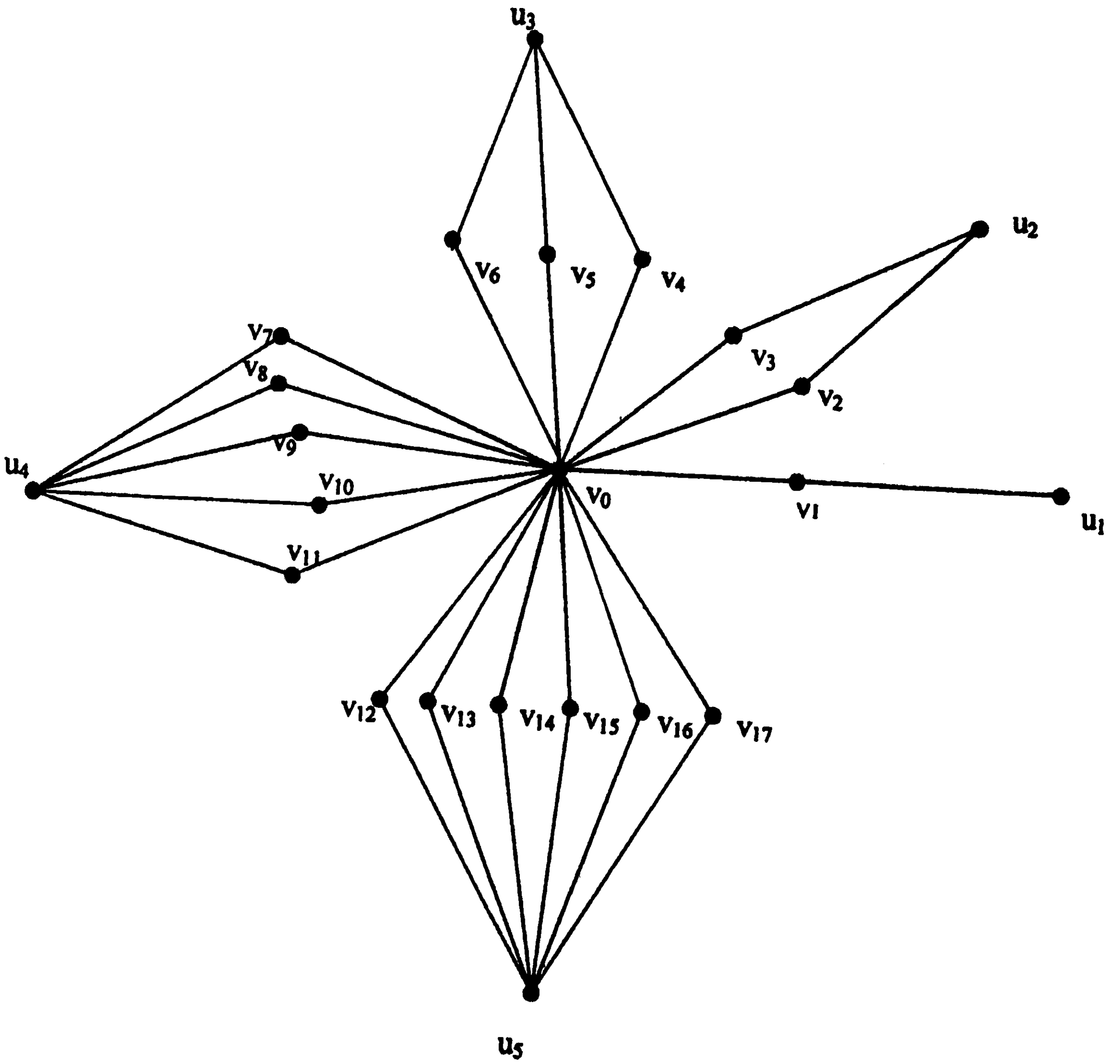

FIG. 1.

$$\phi(v_i) = 2i, \quad 0 \le i \le (m_1 + m_2 + \ldots + m_n)$$

$$\phi(u_k) = \left(2 \sum_{i=1}^{n-k} m_i \right) + 1, \quad 1 \le k \le n-1$$

and $$\phi(u_n) = 1.$$

It is straight forward to verify that this assignment provides a graceful numbering of H. In Fig. 2 we illustrate this graceful numbering.

Remark 2.2 : An arbitrary supersubdivision of S_n is nothing but the one vertex union of the complete bipartite graphs K_{2,m_i} where m_i is arbitrary, $1 \le i \le n$.

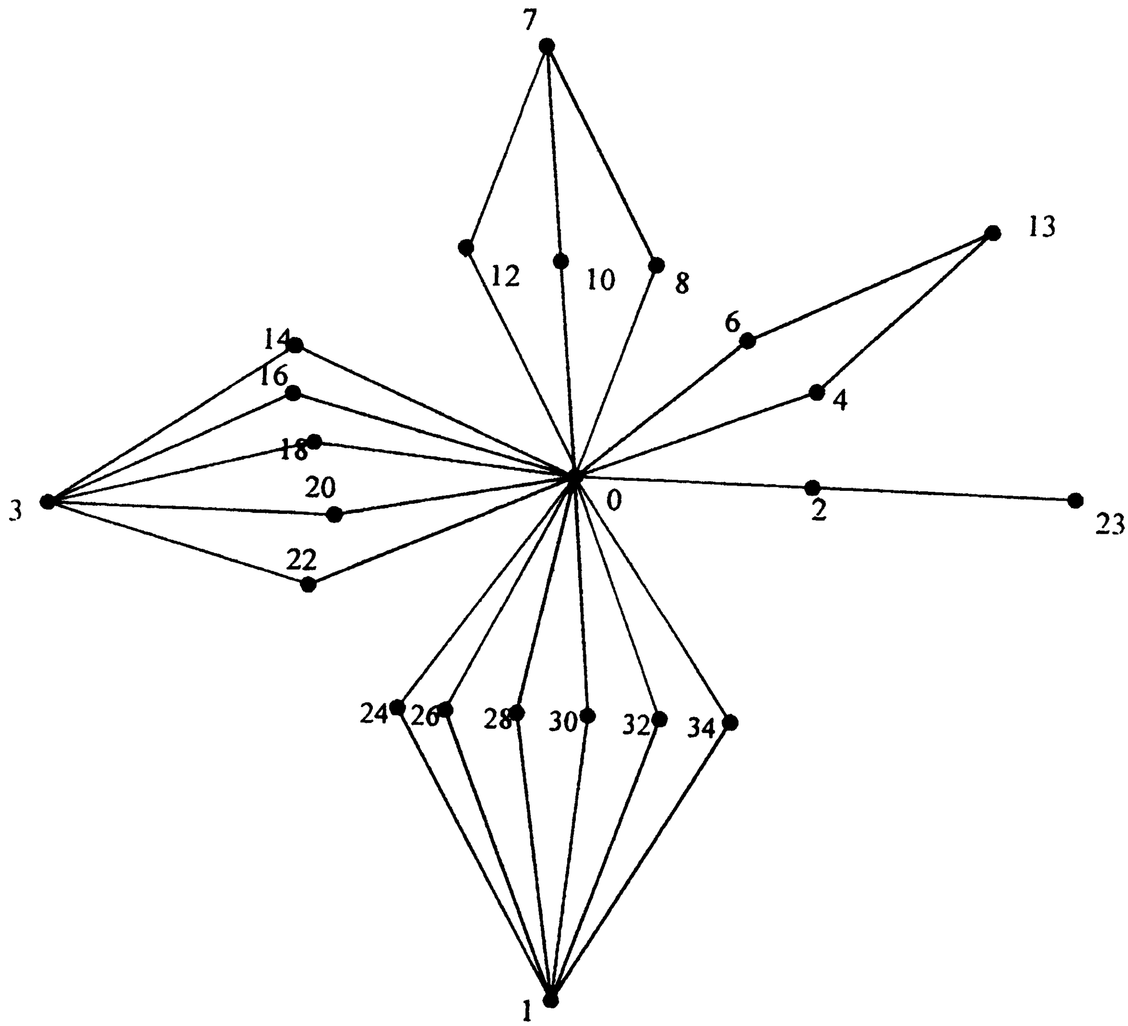

FIG. 2.

Thus we have settled the conjecture of Sethuraman and Selvaraju[6] affirmatively.

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