Indian J. pure appl. Math., **35**(1) : 85-100, January 2004

MODELS OF IRREDUCIBLE q-REPRESENTATIONS OF sl (2, $\mathbb{C}$) AND GENERALIZED q-LAURICELLA FUNCTIONS

VIVEK SAHAI AND SHALINI SRIVASTAVA

Department of Mathematics and Astronomy, Lucknow University, Lucknow 226 007, *India*
e-mail:vsahai@hotmail.com

(*Received* 16 *April* 2002; *after revision* 1 *November* 2002; *accepted* 23 *September* 2003)

We construct new (m + 1)-variable models of irreducible q-representations of Lie algebra sl (2, $\mathbb{C}$) by using techniques of fractional q-calculus in terms of q-derivative operators. Some interesting identities are obtained.

Key Words : **Irreducible q-Representations; Lie Algebra sl (2, $\mathbb{C}$), q-Lauricella Functions; Fractional q-Calculus**

1. INTRODUCTION

The theory of q-representations of Lie algebra sl (2, $\mathbb{C}$) has been immensely successful in obtaining results in q-special function theory. The first work in this direction have come from Manocha[4]. Models of the special complex Lie algebra sl (2, $\mathbb{C}$) were constructed acting on q-hypergeometric functions ${}_2\Phi_1$ of single variable. In Sahai[11], these models were generalized and acted on generalized q-hypergeometric functions ${}_{k^2+1}\Phi_{k^2}$ of single variable. Both these models were obtained using the fractional q-calculus techniques. Apart from this, the method of q-Euler integral transformation has also been used to obtain models of sl (2, $\mathbb{C}$) and identities and recurrence relations involving q-hypergeometric functions, see Sahai[9, 10]. It has also been established in Manocha[4] that these models of q-representations of the Lie algebra sl (2, $\mathbb{C}$) are equivalent to the ones introduced by Jimbo[3] in his work on $U_q(g)$. We remark that the corresponding work on the classical case (that is, for q = 1) has been carried out much earlier by Al-Bassam and Manocha[1] and Sahai[7], using ordinary fractional calculus.

In this paper, we apply the technique of fractional q-calculus to obtain an entirely new set of models of Lie algebra sl (2, $\mathbb{C}$) acting on multivariable q-Lauricella functions. As we will see, these (m +1)-variable models are obtained with the help of an operator $\mathcal{D}_q$ which consists of partial q-fractional derivative operators and its inverse $\mathcal{D}_q^{-1}$. The identities resulting from the study involve generalized multivariable q-Lauricella series.

Section-wise treatment is as follows. In Section 2, we list few definitions and results needed for our discussion. In Section 3, we briefly describe Lie algebra sl (2, $\mathbb{C}$) along with its q-deformed version. A theorem is reproduced which classifies irreducible q-representations of sl (2, $\mathbb{C}$) and gives

$(m+1)$-variable models of q-representations $D_q(\alpha, u)$ and $\uparrow_q(u)$ in the particular case $u=0$ in terms of q-derivative operators. These models are then transformed using the operators $\mathcal{D}_q$ and $\mathcal{D}_q^{-1}$ constructed in Section 2. In Section 4, these transformed models are used in obtaining identities involving $\Phi_{1:1;\ldots;1}^{1:2;\ldots;2}$ series.

2. PRELIMINARIES

The generalized basic or q-hypergeometric series is defined as

$$ {}_r\Phi_s\left(\begin{matrix} a_1, >.., a_r \\ \\ b_1, \ldots, b_s \end{matrix} ; q, x\right) \quad \ldots (1)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s, q; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} x^n,$$

where $(a_1, \ldots, a_r; q)_n = \prod_{i=1}^{r} (a_i; q)_n$, $(a_i; q)_n$ being the q-shifted factorial[2, 12]. We now list q-analogues of some important functions which will be needed in the discussion. The q-analogue of the binomial function is

$$ {}_1\Phi_0\left(\begin{matrix} a \\ -; q, x \end{matrix}\right) = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |q|<1, |x|<1. \quad \ldots (2)$$

The q-analogues of the exponential function are

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_\infty}, \quad |x|<1 \quad \ldots (3)$$

and

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q; q)_n} = (-x; q)_\infty. \quad \ldots (4)$$

$E_q(x)$ converges for all x. Note that $e_q(x)\, E_q(-x) = 1$.

We shall also make use of the function

$$\Gamma_q(\alpha) = \frac{e_q(q^\alpha)}{e_q(q)} (1-q)^{1-\alpha} \quad \ldots (5)$$

defined for $\alpha \neq 0, -1, -2, \ldots$. This is a q-analogue of the gamma function and satisfies the functional equation

$$\Gamma_q(\alpha+1) = \frac{1-q^\alpha}{1-q} \Gamma_q(\alpha). \quad \ldots (6)$$

We need the following q-extension of the generalized Lauricella series in m-variables,

$$\Phi^{A:B_1;\ldots;B_m}_{C:E_1;\ldots;E_m}\left(\begin{matrix}(a):(b^{(1)});\ldots;(b^{(m)}) \\ ;q;x_1,\ldots,x_m \\ (c):(e^{(1)});\ldots;(e^{(m)})\end{matrix}\right) \qquad \ldots (7)$$

$$= \sum_{n_1,\ldots,n_m=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j;q)_{n_1+\ldots+n_m}\prod_{j=1}^{B_1}(b_j^{(1)};q)_{n_1}\cdots\prod_{j=1}^{B_m}(b_j^{(m)};q)_{n_m}}{\prod_{j=1}^{C}(c_j;q)_{n_1+\ldots+n_m}\prod_{j=1}^{E_1}(e_j^{(1)};q)_{n_1}\cdots\prod_{j=1}^{E_m}(e_j^{(m)};q)_{n_m}}$$

$$\times \frac{x_1^{n_1}}{(q;q)_{n_1}}\cdots\frac{x_m^{n_m}}{(q;q)_{n_m}}.$$

Eq. (7) is a particular case of Srivastava and Karlsson[12] (eq. (284), p. 350), when all associated coefficients are equal to 1. For convenience, we denote the series $\Phi^{1:1;\ldots;1}_{1:0;\ldots\,0}$ by $\Phi^{(m)}_D$, as when $q \to 1$, $\Phi^{(m)}_D \to F^{(m)}_D$, the ordinary Lauricella series in m-variables (see also Srivastava and Manocha[13]).

The q-derivative operator is defined as

$$D_{x,q}f(x) = \frac{f(x)-f(qx)}{(1-q)\,x}, \qquad \ldots (8)$$

Indeed, as $q \to 1$, the q-derivative operator $D_{x,q} \to \frac{d}{dx}$. Applying the Taylor formula to the right hand side of (8), we obtain the following expression for the q-derivative operator, see Vilenkin and Klimyk[14],

$$D_{x,q}f(x) = \sum_{n=0}^{\infty} \frac{(q-1)^n}{(n+1)\,!}\,x^n\,\frac{d^{n+1}}{dx^{n+1}}f(x), \qquad \ldots (9)$$

provided that the expression on the right hand side exists.

From (8), it follows that

$$D^n_{x,q}(x^p) = \frac{\Gamma_q(p+1)}{\Gamma_q(p-n+1)}\,x^{p-n}. \qquad \ldots (10)$$

The above derivative formula can be extended to a fractional q-derivative operator of order λ as

$$D^{\lambda}_{x,q}(x^{\mu}) = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu-\lambda+1)}\,x^{\mu-\lambda}, \quad \mu \neq -1,-2,\ldots \qquad \ldots (11)$$

When $q \to 1$, (11) reduces to

$$\frac{d^{\lambda}}{dx^{\lambda}} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda+1)} x^{\mu-\lambda}, \quad \mu \neq -1, -2, \ldots, \qquad \ldots (12)$$

where $\frac{d^{\lambda}}{dx^{\lambda}}$ is fractional derivative of order λ.

The generalized q-Leibniz formula[4] for q-fractional derivative of product of two functions in terms of q-derivatives of each of the functions, is established as

$$D^{\lambda}_{x,q} [f(x)\, g(x)] = \sum_{r=0}^{\infty} \begin{bmatrix} \lambda \\ r \end{bmatrix}_q q^{-r(\lambda-r)} D^{\lambda-r}_{x,q} f(xq^r)\, D^{r}_{x,q}\, g(x), \qquad \ldots (13)$$

where the q-binomial coefficient is defined by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\, \Gamma_q(\alpha-\beta+1)}, \alpha, \beta \in \mathbb{C}, |q| < 1. \qquad (14)$$

We introduce the operators $D_{q,i}$ and $D^{-1}_{q,i}$ defined as

$$D_{q,i} f(x_1, \ldots, x_m) = D^{\beta_i - \gamma_i}_{x_i, q} \left[x_i^{\beta_i - 1} f(x_1, \ldots, x_m) \right], \qquad \ldots (15)$$

$$D^{-1}_{q,i} f(x_1, \ldots, x_m) = x_i^{1-\beta_i} D^{\gamma_i - \beta_i}_{x_i, q} [f(x_1, \ldots, x_m)]. \qquad \ldots (16)$$

Construct the operators $\mathcal{D}_q$ and $\mathcal{D}^{-1}_q$ as

$$\mathcal{D}_q = D_{q,m} \cdots D_{q,1} \qquad \ldots (17)$$

$$\mathcal{D}^{-1}_q = D^{-1}_{q,1} \cdots D^{-1}_{q,m}. \qquad \ldots (18)$$

Indeed, in general,

$$\mathcal{D}_q \mathcal{D}^{-1}_q [f(x_1, \ldots, x_m)] = f(x_1, \ldots, x_m) = \mathcal{D}^{-1}_q \mathcal{D}_q [f(x_1, \ldots, x_m)]. \qquad \ldots (19)$$

It can be easily seen that

$$\mathcal{D}_q \Phi^{(m)}_D = \prod_{i=1}^{m} \frac{\Gamma_q(\beta_i)}{\Gamma_q(\gamma_i)} x_i^{\gamma_i - 1} \Phi^{1:2;\ldots;2}_{1:1;\ldots;1}. \qquad \ldots (20)$$

Using (13), we obtain the following

$$\mathcal{D}_q (x_i D_{x_i, q}) \mathcal{D}^{-1}_q = D_{q,i} (x_i D_{x_i, q}) D^{-1}_{q,i}$$

$$= \frac{1}{1-q} \left(1 - \frac{q}{c_i} T_{x_i} \right), \qquad \ldots (21)$$

$$\mathcal{D}_q (x_i)\, \mathcal{D}_q^{-1} = x_i\, q^{\beta_i - \gamma_i} + \frac{1 - q^{\beta_i - \gamma_i}}{1 - q} D_{x_i, q}^{-1}, \qquad \text{... (22)}$$

$$\mathcal{D}_q \left(\prod_{i=1}^{m} T_{x_i} \right) \mathcal{D}_q^{-1} = \prod_{i=1}^{m} \frac{q^m}{c_i} T_{x_i}, \qquad \text{... (23)}$$

where the q-dilation operator T_x is defined by $T_x f(x) = f(qx)$. Note that $D_{x_i, q}^{-1}$ is a q-integral operator in disguise.

3. Irreducible q-Representations of $sl\ (2, \mathbb{C})$

The complex special Lie algebra

$$sl\ (2, \mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c, \in \mathbb{C} \right\} \qquad \text{... (24)}$$

has a basis, (see Miller[5]),

$$\mathcal{J}^+ = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \mathcal{J}^- = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \mathcal{J}^0 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \qquad \text{... (25)}$$

satisfying the commutation relations

$$[\mathcal{J}^0, \mathcal{J}^\pm] = \pm \mathcal{J}^\pm, [\mathcal{J}^+, \mathcal{J}^-] = 2\, \mathcal{J}^0. \qquad \text{... (26)}$$

Let V_q be a complex vector space consisting of q-special functions with a basis $\{\phi_\lambda \mid \lambda \in S\}$ such that the functions $\{f_\lambda = \lim_{q \to 1} \phi_\lambda \mid \lambda \in S\}$ form a basis of a vector space, say V. Let $A\,(V_q)$ be the associative algebra of all linear operators on V_q over the complex field. Then a q-representation of $sl\ (2, \mathbb{C})$ on V_q is a mapping $\rho_q : sl\ (2, \mathbb{C}) \to A\,(V_q)$ satisfying

(*i*) $\rho_q (ax + by) = a\, \rho_q (x) + b\, \rho_q (y)$;

(*ii*) There exists a Lie algebra representation ρ of sl (2) on V such that

$$\lim_{q \to 1} \rho_q (x)\, \phi_\lambda = \rho\, (x) f_\lambda,$$

for all $x, y \in sl\ (2, \mathbb{C})$ and $a, b \in \mathbb{C}$.

The representation ρ_q of $sl\ (2, \mathbb{C})$ is said to be irreducible if there is no proper subspace W_q of V_q which is invariant under ρ_q.

Define

$$J_q^+ = \rho_q (\mathcal{J}^+), J_q^- = \rho_q (\mathcal{J}^-), J_q^0 = \rho_q (\mathcal{J}^0), \qquad \text{... (27)}$$

where $\quad J_q^+, J_q^-, J_q^0 \in A\,(V_q)$.

Manocha[4] has defined the following commutator rules satisfied by J_q-operators :

$$J_q^0 J_q^+ - qJ_q^+ J_q^0 = J_q^+,$$

$$q\, J_q^0 J_q^- - J_q^- J_q^0 = -J_q^-, \qquad \text{... (28)}$$

$$q\, J_q^+ J_q^- - J_q^- J_q^+ = 2q^{2u} J_q^0 - (1-q)\, q^{2u} J_q^0 J_q^0,\ u \in \mathbb{C}.$$

These commutator rules were later generalized by Sahai[8] for the 4-dimensional complex Lie algebra $\mathcal{G}(a, b), a, b \in \mathbb{C}$.

If we define the operator C_q on V_q by

$$C_q = q\, J_q^+ J_q^- + q^{2u} J_q^0 J_q^0 - q^{2u} J_q^0, \qquad \text{... (29)}$$

it is easy to check that

$$q\, J_q^+ C_q = C_q J_q^+, \qquad \text{... (30)}$$

$$J_q^- C_q = q\, C_q J_q^-,$$

$$J_q^0 C_q = C_q J_q^0.$$

As $q \to 1$, the operators J_q^+, J_q^-, J_q^0 reduce to operators J^+, J^-, J^0 and the operator C_q reduces to the Casimir operator C and therefore ρ_q reduces to a Lie algebra representation ρ of $sl\,(2, \mathbb{C}.)$ Following the analysis as in Manocha[4] and Miller[5], we have the following theorem :

***Theorem* 1** — *Every q-representation ρ_q of sl (2, $\mathbb{C}$) is isomorphic to a q-representation in the following list :*

1. The q-representations $D_q(\alpha, u), \alpha \in \mathbb{C} - \{0\}$, $0 \le \text{Re}\, \alpha < 1$, such that $\alpha - 2u$ is not an integer. $S = \{\alpha + n : n = 0, \pm 1, \ldots\}$.

2. The q-representations $\uparrow_q(u)$, $u \in \mathbb{C}$ and 2u is not a non-negative integer. $S = \{0, 1, 2, \ldots.\}$.

3. The q-representations $D_q(2u)$ where 2u is a non-negative integer. $S = \{0, 1, \ldots, 2u\}$.

For each of these representations there is a basis of V_q consisting of vectors $\{f_\lambda\}$, defined for each $\lambda \in S$ such that

$$J_q^0 f_\lambda = \frac{1 - q^{\lambda - u}}{1 - q} f_\lambda, \qquad \text{... (31)}$$

$$J_q^+ f_\lambda = \frac{q^{2u} - q^\lambda}{1 - q} f_{\lambda + 1},$$

$$J_q^- f_\lambda = -\frac{1-q^\lambda}{1-q} f_{\lambda-1}.$$

We have the following $(m + 1)$ - variable models for each of the q-representations $D_q(\alpha, u)$ and $\uparrow_q(u)$ as described in the above theorem in the particular case $u = 0$, in which case they will be denoted by $D_q(\alpha)$ and $\uparrow_q$ respectively.

Models of $D_q(\alpha)$,

$$\text{(I)}\ J_q^+ = \frac{1}{1-q} t \left(1 - T_t \prod_{i=1}^{m} T_{x_i} \right), \qquad \text{... (32)}$$

$$J_q^- = \frac{1}{1-q} t^{-1} \prod_{i=1}^{m} T_{x_i}^{-1} \left(1 - x_1 T_t + (x_1 T_t - 1) \prod_{i=1}^{m} T_{x_i} \right),$$

$$J_q^0 = \frac{1}{1-q}(1 - T_t),$$

$$f_\lambda(x_1, \ldots, x_m, t) = \Phi_D^{(m)} \left(\begin{matrix} q^\lambda;\ q, \ldots, q \\ \\ q \end{matrix} ; q; x_1, \ldots, x_m \right) t^\lambda.$$

$$\text{(II)}\ J_q^+ = \frac{1}{1-q} t \left(1 - x_1 T_t + (x_1 - 1) T_t \prod_{i=1}^{m} T_{x_i} \right), \qquad \text{... (33)}$$

$$J_q^- = \frac{1}{1-q} t^{-1} \prod_{i=1}^{m} T_{x_i}^{-1} \left(T_t - \prod_{i=1}^{m} T_{x_i} \right),$$

$$J_q^0 = \frac{1}{1-q}(1 - T_t),$$

$$f_\lambda(x_1, \ldots, x_m, t) = \Phi_D^{(m)} \left(\begin{matrix} q^{-\lambda};\ q, \ldots, q \\ \\ q \end{matrix} ; q; q^\lambda x_1, \ldots, q^\lambda x_m \right) t^\lambda,$$

where $\lambda \in S = \{\alpha + n : \alpha \in \mathbb{C} - \{0\},\ 0 \le \operatorname{Re} \alpha < 1, n \in \mathbb{Z}\}$ both for model (32) and (33).

Models of $\uparrow_q$: A model of representation $\uparrow_q$ is same as (33) with $\lambda \in S = \{0, 1, 2, \ldots\}$.

Model (32) and (33) obey the following :

$$J_q^+ f_\lambda = \frac{1-q^\lambda}{1-q} f_{\lambda+1}, \qquad \text{... (34)}$$

$$J_q^- f_\lambda = -\frac{1-q^\lambda}{1-q} f_{\lambda-1},$$

$$J_q^0 f_\lambda = \frac{1-q^\lambda}{1-q} f_\lambda,$$

$$C_q f_\lambda = 0, \; C_q = q J_q^+ J_q^- + J_q^0 J_q^0 - J_q^0;$$

and

$$J_q^0 J_q^+ - q J_q^+ J_q^0 = J_q^+, \qquad \text{... (35)}$$

$$q J_q^0 J_q^- - q J_q^- J_q^0 = - J_q^-,$$

$$q J_q^+ J_q^- - J_q^- J_q^+ = 2J_q^0 - (1-q) J_q^0 J_q^0.$$

as well as (30).

To obtain new models of q-irreducible representations of sl (2, $\mathbb{C}$) in terms of fractional q-differ integral operators with basis functions in terms of multivariable q-Kampe de Feriet functions $\Phi^{1:2;\ldots;2}_{1:1;\ldots;1}$ we need the following theorem :

***Theorem* 2** — *Let* ρ_q *be a* q*-representation of* sl *(2,* $\mathbb{C}$*) in terms of* $\left\{ J_q^+, J_q^-, J_q^0 \right\}$ *with basis functions* $\{f_\lambda ; \lambda \in S\}$. *Then* ρ_q *is also a* q*-representation of* sl *(2,* $\mathbb{C}$*) in terms of* $\{K_q^+, K_q^-, K_q^0\}$ *with basis functions* $\{h_\lambda : \lambda \in S\}$, *where* $K_q^+ = \mathcal{D}_q J_q^+ \mathcal{D}_q^{-1}$, $K_q^- = \mathcal{D}_q J_q^- \mathcal{D}_q^{-1}$, $K_q^0 = \mathcal{D}_q J_q^0 \mathcal{D}_q^{-1}$ *and* $h_\lambda = \mathcal{D}_q f_\lambda, \lambda \in S$,

Further,

$$K_q^+ h_\lambda = \frac{1-q^\lambda}{1-q} h_{\lambda+1}, \qquad \text{... (36)}$$

$$K_q^- h_\lambda = -\frac{1-q^\lambda}{1-q} h_{\lambda-1},$$

$$K_q^0 h_\lambda = \frac{1-q^\lambda}{1-q} h_\lambda,$$

$$C'_q h_\lambda = 0, \; C'_q = qK_q^+ K_q^- + K_q^0 K_q^0 - K_q^0.$$

PROOF : The theorem follows from the fact that

$$J_q^+ f_\lambda (x_1, \ldots, x_m, t) = \frac{1-q^\lambda}{1-q} f_{\lambda+1} (x_1, \ldots, x_m, t)$$

can be rewritten as

$$(J_q^+ \mathcal{D}_q^{-1}) (\mathcal{D}_q f_\lambda (x_1, \ldots, x_m, t)) = \frac{1-q^\lambda}{1-q} \mathcal{D}_q^{-1} (\mathcal{D}_q f_{\lambda+1} (x_1, \ldots, x_m, t))$$

$$(\mathcal{D}_q J_q^+ \mathcal{D}_q^{-1})\,(\mathcal{D}_q f_\lambda(x_1, \ldots, x_m, t)) = \frac{1-q^\lambda}{1-q}\,\mathcal{D}_q\,\mathcal{D}_q^{-1}\,(\mathcal{D}_q f_{\lambda+1}(x_1, \ldots, x_m, t)).$$

This gives

$$K_q^+ h_\lambda(x_1, \ldots, x_m, t) = \frac{1-q^\lambda}{1-q} h_{\lambda+1}(x_1, \ldots, x_m, t).$$

Other relations can be proved similarly. ■

We apply Theorem 2 to models (32) and (33) to induce new models of $D_q(\alpha)$ and $\uparrow_q$ as under :

Model I (A)

$$K_q^+ = \frac{1}{1-q}\,t\left(1 - T_t \prod_{i=1}^{m} \frac{q^m}{c_i} T_{x_i}\right), \qquad \ldots (37)$$

$$K_q^- = \frac{1}{1-q}\,t^{-1} \prod_{i=1}^{m} \frac{c_i}{q^m}\, T_{x_i}^{-1}\left(1 - \left\{x_1 q^{\beta_1-\gamma_1} + \frac{1-q^{\beta_1-\gamma_1}}{1-q} D_{x_1,q}^{-1}\right\} T_t\right.$$

$$\left. + \left\{\left(x_1 q^{\beta_1-\gamma_1} + \frac{1-q^{\beta_1-\gamma_1}}{1-q} D_{x_{1,q}}^{-1}\right) T_t - 1\right\} \prod_{i=1}^{m} \frac{q^m}{c_i} T_{x_i}\right),$$

$$K_q^0 = \frac{1}{1-q}(1 - T_t)$$

with basis functions as

$$h_\lambda(x_1, \ldots, x_m, t) = \prod_{i=1}^{m} \frac{\Gamma_q(\beta_i)}{\Gamma_q(\gamma_i)} x_i^{\gamma_i - 1}$$

$$\times \Phi_{1\,;\,1;\ldots;\,1}^{1\,;\,2;\ldots;\,2}\left(\begin{matrix} q^\lambda: & b_1, q; \ldots; b_m, q \\ & \\ q: & c_1\,; \ldots; c_m \end{matrix}\; ; q; x_1, \ldots, x_m\right) t^\lambda \qquad \ldots (38)$$

$\lambda \in S$, where $b_i = q^{\beta_i}$, $c_i = q^{\gamma_i}$.

Model II (A)

$$K_q^+ = \frac{t}{1-q}\left(1 - \prod_{i=1}^{m} \frac{q^m}{c_i} T_t T_{x_i}\right.$$

$$\left. - \left(x_1 q^{\beta_1-\gamma_1} + \frac{1-q^{\beta_1-\gamma_1}}{1-q} D_{x_1,q}^{-1}\right) T_t \left(1 - \prod_{i=1}^{m} \frac{q^m}{c_i} T_{x_i}\right)\right), \qquad \ldots (39)$$

$$K_q^- = \frac{1}{1-q} t^{-1} \prod_{i=1}^{m} T_{x_i}^{-1} \left(\prod_{l=1}^{m} \frac{c_i}{q^m} T_t - \prod_{i=1}^{m} T_{x_i} \right),$$

$$K_q^0 = \frac{1}{1-q} (1 - T_t),$$

with basis functions as

$$h_\lambda (x_1, \ldots, x_m, t) = \prod_{i=1}^{m} \frac{\Gamma_q (\beta_i)}{\Gamma_q (\gamma_i)} x_i^{\gamma_i - 1} \qquad \ldots (40)$$

$$\times \Phi^{1\,;\,2;\,\ldots;\,2}_{1\,;\,1;\,\ldots;\,1} \left(\begin{matrix} q^{-\lambda} : & b_1, q; \ldots; b_m, q \\ & & ; q; q^\lambda x_1, \ldots, q^\lambda x_m \\ q : & c_1 ; \ldots ; c_m \end{matrix} \right) t^\lambda .$$

Both the transformed models satisfy (36) as well as the commutation relations

$$K_q^0 K_q^+ - q K_q^+ K_q^0 = K_q^+, \qquad \ldots (41)$$

$$q K_q^0 K_q^- K_q^- K_q^0 = - K_q^-,$$

$$q K_q^+ K_q^- - K_q^- K_q^+ = 2 K_q^0 - (1 - q) K_q^0 K_q^0 .$$

Moreover,

$$q K_q^+ C_q' = C_q' K_q^+, \qquad \ldots (42)$$

$$K_q^- C_q' = q C_q' K_q^-,$$

$$K_q^0 C_q' = C_q' K_q^0 .$$

4. Identities Based on Transformed Models

In this section, we q-exponentiate the transformed models of $D_q(\alpha)$ and $\uparrow_q$ as given in the previous section and, in the process, obtain identities involving q-series of one and several variables. Our main results are given as eqs. (48), (54) and (58). Eq. (48) originates from Model I(A) whereas (54) and (58) are derived from Model II(A).

Based on model I(A)

We have shown in eq. (36) that

$$h_\lambda (x_1, \ldots, x_m, t) = \prod_{i=1}^{m} \frac{\Gamma_q (\beta_i)}{\Gamma_q (\gamma_i)} x_i^{\gamma_i - 1} \qquad \ldots (43)$$

$$\times \Phi^{1\,;\,2;\,\ldots;\,2}_{1\,;\,1;\,\ldots;\,1}\left(\begin{matrix} q^{\lambda}: & b_1, q; \ldots; b_m, q \\ & \\ q: & c_1\,; \ldots, c_m \end{matrix} ; q; x_1, \ldots, x_m\right) t^{\lambda},$$

is a solution of $C_q^{'} h_{\lambda}(x_1, \ldots, x_m, t) = 0$.

This suggests that

$$u(x_1, \ldots, x_m, t) \qquad \ldots (44)$$

$$= \sum_{n=0}^{\infty} \frac{(a, b_1^{'}, \ldots, b_{m+1}^{'}; q)_n}{(q, c_1^{'}, \ldots, c_m^{'}; q)_n} \prod_{i=1}^{m} \frac{\Gamma_q(\beta_i)}{\Gamma_q(\gamma_i)} x_i^{\gamma_i - 1}$$

$$\times \Phi^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} aq^n: & b_1, q; \ldots; b_m, q \\ & \\ q: & c_1\,; \ldots, c_m \end{matrix} ; q; x_1, \ldots, x_m\right) t^{\alpha+n}$$

is a solution of $C_q^{'} u(x_1, \ldots, x_m, t) = 0$. Using the fact that $q K_q^{+} C_q^{'} = C_q^{'} K_q^{+}$, we have

$$C_q^{'} [e_q(sK_q^{+}) u](x_1, \ldots, x_m, t) = 0, \qquad \ldots (45)$$

where

$$[e_q(sK_q^{+}) u](x_1, \ldots, x_m, t) \qquad \ldots (46)$$

$$= \prod_{i=1}^{m} \frac{\Gamma_q(\beta_i)}{\Gamma_q(\gamma_i)} x_i^{\gamma_i - 1} \frac{\left(\frac{ast}{1-q}; q\right)_\infty}{\left(\frac{st}{1-q}; q\right)_\infty} \sum_{n=0}^{\infty} \frac{(a, b_1^{'}, \ldots, b_{m+1}^{'}\,; q)_n}{\left(q, \frac{ast}{1-q}, c_1^{'}, \ldots, c_m^{'}; q\right)_n}$$

$$\times \Phi^{1:2;\ldots;2}_{2:1;\ldots;1}\left(\begin{matrix} aq^n: & b_1, q; \ldots; b_m, q \\ & \\ q, \frac{ast\,q^n}{1-q}: & c_1\,; \ldots; c_m \end{matrix} ; q; x_1, \ldots, x_m\right) t^{\alpha+n}$$

Now we have, from the Weisner's expansion[15]

$$[e_q(sK_q^{+}) u](x_1, \ldots, x_m, t) = \sum_{n=0}^{\infty} A_n h_{\alpha+n}(x_1, \ldots, x_m, t), \qquad \ldots (47)$$

where A_n is found by putting $x_1 = \ldots = x_m = 0$. After suitable rescaling, we arrive at the following identity :

$$\frac{(at; q)_\infty}{(t; q)_\infty} \sum_{k=0}^{\infty} \frac{(a, b_1, \ldots, b_{m+1}^{'}; q)_k}{(q, at, c_1^{'}, \ldots, c_m^{'}; q)_k} \qquad \ldots (48)$$

$$\times \Phi^{1:2;...;2}_{2:1;...;1}\left(\begin{matrix} aq^k: & b_1, q; \ldots; b_m, q \\ & ; q; x_1, ..., x_m \\ q, atq^k: & c_1; ....; c_m \end{matrix}\right)\left(\frac{-t}{w}\right)^k$$

$$= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} {}_{m+2}\Phi_m\left(\begin{matrix} q^{-n}, b_1', ..., b_{m+1}' \\ ; q, -\frac{q^n}{w} \\ c_1', ..., c_m' \end{matrix}\right)$$

$$\times \Phi^{1:2;...;2}_{1:1;...;1}\left(\begin{matrix} aq^n: b_1\, q; ....; b_m, q \\ ; q; x_1, ..., x_m \\ q: \quad c_1; ....; c_m \end{matrix}\right) t^n$$

where $w = -\frac{s}{1-q}$.

Based on Model II(A).

We follow exactly the same approach as given in previous case with the change that we use the other q-exponential E_q instead of e_q.

We have shown that $C_q' h_\lambda(x_1, ..., x_m, t) = 0$, where

$$h_\lambda(x_1, ..., x_m, t)$$

$$= \prod_{i=1}^{m} \frac{\Gamma_q(\beta_i)}{\Gamma_q(\gamma_i)} x_i^{\gamma_i - 1} \sum_{n=0}^{\infty} \frac{(a, b_1', ..., b_m'; q)_n}{(q, c_1', ..., c_{m+1}'; q)_n} \quad \ldots (49)$$

$$\times \Phi^{1:2;...;2}_{1:1;...;1}\left(\begin{matrix} q^\lambda: b_1, q; ...; b_m, q \\ ; q; q^\lambda x_1, ..., q^\lambda x_m \\ q: \quad c_1; ....; c_m \end{matrix}\right) t^\lambda$$

It therefore follows that

$$u(x_1, ..., x_m, t) \quad \ldots (50)$$

$$= \prod_{i=1}^{m} \frac{\Gamma_q(\beta_i)}{\Gamma_q(\gamma_i)} x_i^{\gamma_i - 1} \sum_{n=0}^{\infty} \frac{(a, b_1', ..., b_m'; q)_n}{(q, c_1', ..., c_{m+1}'; q)_n}$$

$$\times \Phi^{1:2;...;2}_{1:1;...;1}\left(\begin{matrix} a^{-1} q^{-n}: b_1, q; ...; b_m, q \\ ; q; aq^n x_1, ..., aq^n x_m \\ q: \quad c_1; ....; c_m \end{matrix}\right) t^{\alpha+n}$$

is a solution of $C_q' u(x_1, \ldots, x_m, t) = 0$.

Considering that $K_q^- C_q' = q C_q' K_q^-$, we have

$$C_q' [E_q(sK_q^-) u](x_1, \ldots, x_m, t) = 0, \qquad \ldots (51)$$

where

$$[E_q(sK_q^-) u](x_1, \ldots, x_m, t) \qquad \ldots (52)$$

$$= \frac{\left(-\frac{w}{t}; q\right)_\infty}{\left(-\frac{aw}{t}; q\right)_\infty} \prod_{i=1}^{m} \frac{\Gamma_q(\beta_i)}{\Gamma_q(\gamma_i)} x_i^{\gamma_i - 1} \sum_{n, n_2, \ldots, n_m = 0}^{\infty} \frac{\left(a, -\frac{aw}{t}, b_1', \ldots, b_m'; q\right)_n}{(q, c_1', \ldots, c_{m+1}'; q)_n}$$

$$\times \frac{(a^{-1} q^{-n}; q)_{n_2 + \ldots + n_m} (b_2; q)_{n_2} \ldots (b_m; q)_{n_m}}{(q; q)_{n_2 + \ldots + n_m} \left(-\frac{q^{1-n} t}{aw}; q\right)_{n_2 + \ldots + n_m} (c_2; q)_{n_2} \ldots (c_m; q)_{n_m}}$$

$$\times {}_3\Phi_3 \left(\begin{matrix} a^{-1} q^{-n + n_2 + \ldots + n_m}, \; b_1, \; q \\ q^{n_2 + \ldots + n_m + 1}, \; -\frac{q^{1 - n + n_2 + \ldots + n_m}}{aw} t, \; c_1 \end{matrix} ; q, -\frac{q^{n_2 + \ldots + n_m + 1}}{w} t x_1 \right)$$

$$\times \left(\frac{t x_2 q^{\frac{1}{2}(n_2 + 1)}}{w} \right)^{n_2} \ldots \left(\frac{t x_m q^{\frac{1}{2}(n_m + 1)}}{w} \right)^{n_m} q^{n_2 (n_3 + \ldots + n_m)} \ldots q^{n_{m-1} n_m} t^{\alpha + n}.$$

By Weisner's expansion, we have

$$[E_q(sK_q^-) u](x_1, \ldots, x_m, t) = \sum_{n=-\infty}^{\infty} B_n h_{\alpha + n}(x_1, \ldots, x_m, t), \qquad \ldots (53)$$

where B_n is obtained by putting $x_1 = \ldots = x_m = 0$. This gives the following identity :

$$\frac{\left(-\frac{w}{t}; q\right)_\infty}{\left(-\frac{aw}{t}; q\right)_\infty} \sum_{n, n_2, \ldots, n_m = 0}^{\infty} \frac{\left(a, -\frac{aw}{t}, b_1', \ldots, b_m'; q\right)_n}{(q, c_1', \ldots, c_{m+1}'; q)_n} \qquad \ldots (54)$$

$$\times \frac{(a^{-1} q^{-n}; q)_{n_2 + \ldots + n_m} (b_2; q)_{n_2} \ldots (b_m; q)_{n_m}}{(q; q)_{n_2 + \ldots + n_m} \left(-\frac{q^{1-n} t}{aw}; q\right)_{n_2 + \ldots + n_m} (c_2; q)_{n_2} \ldots (c_m; q)_{n_m}}$$

$$\times {}_3\Phi_3\left(\begin{matrix} a^{-1}q^{-n+n_2+\ldots+n_m}, b_1, q \\ q^{n_2+\ldots+n_m+1}, -\dfrac{q^{1-n+n_2+\ldots+n_m}}{aw}t, c_1\end{matrix}; q, -\frac{q^{n_2+\ldots+n_m+1}}{w}tx_1\right)$$

$$\times\left(\frac{tx_2\, q^{\frac{1}{2}(n_2+1)}}{w}\right)^{n_2}\cdots\left(\frac{tx_m\, q^{\frac{1}{2}(n_m+1)}}{w}\right)^{n_m}$$

$$\times q^{n_2(n_3+\ldots+n_m)+n_3(n_4+\ldots+n_m)+\ldots+n_{m-1}n_m} t^n$$

$$= \sum_{n=-\infty}^{\infty} \frac{\Gamma_q(\alpha+n)\Gamma_q(\beta_1'+n)\ldots\Gamma_q(\beta_m'+n)\Gamma_q(\gamma_1')\ldots\Gamma_q(\gamma_{m+1}')}{\Gamma_q(\gamma_1'+n)\ldots\Gamma_q(\gamma_{m+1}'+n)(1-q)^n\Gamma_q(\beta_1')\ldots\Gamma_q(\beta_m')}$$

$$\times {}_{m+2}\Phi_{m+2}\left(\begin{matrix} aq^n, aq^{n+1}, b_1'q^n, \ldots, b_m'q^n \\ q^{n+1}, c_1'q^n, c_2'q^n, \ldots, c_{m+1}'q^n\end{matrix}; q, -w\right)$$

$$\times \Phi^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} a^{-n}q^{-n} : b_1, q; \ldots; b_m, q \\ q : \quad c_1; \ldots; c_m\end{matrix}; q; aq^n x_1, \ldots, aq^n x_m\right)t^n,$$

where $\qquad w = -\dfrac{s}{1-q}.$

The identities obtained above are based on models of representations $D_q(\alpha)$. Identities can also be found by considering model II(A) for the representation $\uparrow_q$, that is eqs. (39) and (40) with $\lambda \in S = \{0, 1, 2, \ldots\}$. It can be verified that

$$u(x_1, \ldots, x_m, t) \qquad \ldots (55)$$

$$= \prod_{i=1}^{m} \frac{\Gamma_q(\beta_i)}{\Gamma_q(\gamma_i)} x_i^{\gamma_i-1} \sum_{n=0}^{\infty} \frac{(a, b_1', \ldots, b_m'; q)_n}{(q, c_1', \ldots, c_{m+1}'; q)_n}$$

$$\times \Phi^{1:2;\ldots;2}_{1:1;\ldots;1}\left(\begin{matrix} q^{-n} : b_1, q; \ldots; b_m, q \\ q : \quad c_1; \ldots; c_m\end{matrix}; q; q^n x_1, \ldots, q^n x_m\right)t^n$$

satisfies $C_q' u(x_1, \ldots, x_m, t) = 0$.

Considering that $\frac{1}{q}K_q^- C_q' = C_q' K_q^-$, we have

$$C_q'[E_q(sK_q^-)u](x_1, \ldots, x_m, t) = 0, \qquad \ldots (56)$$

where

$$[E_q(sK_q)\,u](x_1, ..., x_m, t) \qquad ... (57)$$

$$= \prod_{i=1}^{m} \frac{\Gamma_q(\beta_i)}{\Gamma_q(\gamma_i)} x_i^{\gamma_i - 1} \sum_{n, n_2, ..., n_m = 0}^{\infty} \frac{\left(a, -\frac{w}{t}, b_1', ..., b_m'; q\right)_n}{(q, c_1', ..., c_{m+1}'; q)_n}$$

$$\times \frac{(q^{-n}; q)_{n_2 + \dots + n_m} (b_2; q)_{n_2} \cdots (b_m; q)_{n_m}}{(q; q)_{n_2 + \dots + n_m} \left(-\frac{q^{1-n}t}{w}; q\right)_{n_2 + \dots + n_m} (c_2; q)_{n_2} \cdots (c_m; q)_{n_m}}$$

$$\times {}_3\Phi_3 \left(\begin{matrix} q^{-n+n_2+\dots+n_m}, b_1, q \\ q^{n_2+\dots+n_m+1}, -\frac{q^{1-n+n_2+\dots+n_m}t}{w}, c_1 \end{matrix} ; q, -\frac{q^{n_2+\dots+n_m+1}}{w} tx_1 \right)$$

$$\times \left(\frac{tx_2\, q^{\frac{1}{2}(n_2+1)}}{w} \right)^{n_2} \cdots \left(\frac{tx_m\, q^{\frac{1}{2}(n_m+1)}}{w} \right)^{n_m} q^{n_2(n_3+\dots+n_m)} \cdots q^{n_{m-1}n_m} t^n.$$

Using Weisner's expansions and proceeding as in the above two cases, we get the following identity :

$$\sum_{n, n_2, \dots n_m = 0}^{\infty} \frac{\left(a, -\frac{w}{t}, b_1', ..., b_m'; q\right)_n}{(q, c_1', ..., c_{m+1}'; q)_n} \qquad ... (58)$$

$$\times \frac{(q^{-n}; q)_{n_2 + \dots + n_m} (b_2; q)_{n_2} \cdots (b_m; q)_{n_m}}{(q; q)_{n_2 + \dots + n_m} \left(-\frac{q^{1-n}t}{w}; q\right)_{n_2 + \dots + n_m} (c_2; q)_{n_2} \cdots (c_m; q)_{n_m}}$$

$$\times {}_3\Phi_3 \left(\begin{matrix} q^{-n+n_2+\dots+n_m}, b_1, q \\ q^{n_2+\dots+n_m+1}, -\frac{q^{1-n+n_2+\dots+n_m}t}{w}, c_1 \end{matrix} ; q, -\frac{q^{n_2+\dots+n_m+1}}{w} tx_1 \right)$$

$$\times\left(\frac{tx_2\, q^{\frac{1}{2}(n_2+1)}}{w}\right)^{n_2}\cdots\left(\frac{tx_m\, q^{\frac{1}{2}(n_m+1)}}{w}\right)^{n_m} q^{n_2(n_3+\dots+n_m)}\dots q^{n_{m-1}n_m}\,t^n$$

$$=\sum_{n=0}^{\infty}\frac{(a, b_1', \dots, b_m'; q)_n}{(q, c_1', \dots, c_{m+1}'; q)_n}\;{}_{m+1}\Phi_{m+1}\left(\begin{matrix} aq^n, b_1' q^n, \dots, b_m' q^n \\ c_1' q^n, c_2' q^n, \dots, c_{m+1}' q^n\end{matrix}; q, -w\right)$$

$$\times\,\Phi^{1:2;\dots;2}_{1:1;\dots;1}\left(\begin{matrix} q^{-n} : b_1, q; \dots; b_m, q \\ q : \quad c_1; \dots; c_m\end{matrix}; q; q^n x_1, \dots, q^n x_m\right)t^n.$$

ACKNOWLEDGEMENT

The authors wish to thank the referee for his valuable suggestions which led to a better presentation of the paper.

REFERENCES

1. M. A. Al-Bassam and H. L. Manocha, *SIAM J. Math. Anal.*, **18** (1987), 1087-1100.
2. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge Univ. Press, 1990.
3. M. Jimbo, *Lett. Math. Phys.*, **10** (1985), 63-69.
4. H. L. Manocha, *Appl. Anal.*, **37** (1990), 19-47.
5. W. Miller, *Lie Theory and Special Functions*, Academic Press, 1968.
6. E. D. Rainville, *Special Functions*, Macmillan, 1960.
7. V. Sahai, *Bull. Cal. Math. Soc.*, **85** (1993), 363-80.
8. V. Sahai, *J. Indian Math. Soc.*, **63** (1997), 83-98.
9. V. Sahai, *Indian J. Pure Appl. Math.*, **29** (1998), 819-27.
10. V. Sahai, *Proc. Amer. Math. Soc.*, **127** (1999), 3201-13.
11. V. Sahai, *New Zealand J. Math.*, **28** (1999), 237-49.
12. H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Limited, 1985.
13. H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Limited, 1984.
14. N. Ja. Vilenkin and A. U. Klimyk, *Representation of Lie Groups and Special Functions, Volume 3 : Classical and Quantum Groups and Special Functions*, Kluwer Academic Publishers, 1992.
15. L. Weisner, *Pacific J. Math.*, **5** (1955), 1083-89.