Indian J. pure appl. Math., 35(2) : 121-132, February 2004

UNIQUENESS THEOREMS FOR MEROMORPHIC FUNCTION*

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(*Received* 2 *July* 2002; *accepted* 16 *October* 2003)

In this paper, we deal with the uniqueness problems on meromorphic functions concerning differential polynomials and improve some result given by Fang and Hong (*Indian J. pure appl. Math.*, 32 (2001), 1343-1348).

Key Words: Shared-value; Uniqueness; Meromorphic Function

1. INTRODUCTION AND RESULTS

In this paper the term "meromorphic" will always mean meromorphic in the complex plane C. Let a be a complex number, we say f and g share the value a CM, if $f-a$ and $g-a$ assume the same zeros with the same multiplicities. It is assumed that the reader is familiar with the notations of the Nevanlinna theory that can be found, for instance, in[1]. We denote by $S(r,f)$ any function satisfying

$$S(r,f) = o\{T(r,f)\},$$

as $r \to \infty$, possibly outside of finite measure.

It is well known that if f and g share four distinct values CM, the f is a Möbius transformation of g. In 1997, corresponding to one famous question of Hayman[3], Yang and Hua[4] showed that similar conclusions hold for certain types of differential polynomials when they share only one value. They proved the following result.

Theorem A — *Let* f *and* g *be two nonconstant meromorphic functions* $n \geq 11$ *an integer and* $a \in C - \{0\}$. *If* $f^n f'$ *and* $g^n g'$ *share the value* a *CM, then either* $f = dg$ *for some* $(n+1)$*th root of unity* d *or* $g(z) = c_1 e^{cz}$ *and* $f(z) = c_2 e^{-cz}$, *where* c, c_1 *and* c_2 *are constants and satisfy* $(c_1 c_2)^{n+1} c^2 = -a^2$.

Recently, by using the same argument as did in[4] Fand and Hong[5] obtained the following result.

Theorem B — *Let* f *and* g *be two transcendental entire functions,* $n \geq 11$ *an integer. If* $f^n (f-1) f'$ *and* $g^n (g-1) g'$ *share the value* 1 *CM, then* $f(z) \equiv g(z)$.

The purpose of this paper is to generalize and improve the above result by deriving the following:

*Supported by the NNSF of China (No. 10371065) and NSF of Fujian Province.

Theorem 1 — *Suppose that the condition "$n \geq 11$" is replaced by "$n \geq 7$" in Theorem B, then the conclusion remains valid.*

Theorem 2 — *Suppose that the condition "f and g are two transcendental entire functions" is replaced by "f and g are two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n+1}$" in Theorem B, then the conclusion remains valid.*

Remark : The following example shows that Theorem 2 is sharp.

Example — Let $f(z) = \frac{(n+1)h(h^{n+1}-1)}{(n+2)(h^{n+2}-1)}$ and $g(z) = \frac{(n+1)(h^{n+1}-1)}{(n+2)(h^{n+2}-1)}$, where $h = \frac{u^2 e^z - u}{e^z - 1}$ and $u = \exp\frac{2\pi i}{n+2}$. It is easy to see that $\Theta(\infty, f) = \frac{2}{n+1}$, moreover, $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM. However, $f \not\equiv g$.

Theorem 3 — *Let* f and g be two distinct nonconstant meromorphic functions, $n \geq 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \qquad f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

where h is a nonconstant meromorphic function.

Theorem 4 — *Let* f and g be two nonconstant meromorphic functions, $n \geq 13$ an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share the value 1 CM, then $f(z) \equiv g(z)$.

On the other hand, we have the following results related to the above uniqueness theorems.

Theorem 5 — *Let f be an entire function, $n \geq 1$ positive integer. If $f^n(f-1)^2 f' \neq 1$, then $f(z)$ is a constant.*

Theorem 6 — *Let f be a meromorphic function, $n \geq 2$ positive integer. If $f^n(f-1)^2 f' \neq 1$, then $f(z)$ is a constant.*

2. Main Lemmas

For the proof of our theorems we need the following lemmas.

Lemma 1[6] — Let f be a nonconstant meromorphic function and let

$$R(f) = \sum_{k=0}^{n} a_k f^k \Big/ \sum_{j=0}^{m} b_j f^j$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = \max\{n, m\}$.

In order to state second lemma, we introduce the following notations.

Let f be a meromorphic function. We denote by $n_2(r,f)$ the number of poles of f in $|z|\leq r$, where a simple pole is counted once and a multiple pole is counted two times, $N_2(r,f)$ is defined in terms of $n_2(r,f)$ in the usual way. In the same way, we can define $N_2\left(r,\frac{1}{f}\right)$ (see[7]).

By using the method of[8], we can prove the following result.

Lemma 2 — Let F and G be two nonconstant meromorphic functions such that F and G share 1 CM, and let

$$H=\left(\frac{F''}{F'}-2\frac{F'}{F-1}\right)-\left(\frac{G''}{G'}-2\frac{G'}{G-1}\right)$$

If $H\not\equiv 0$, then

$$T(r)\leq N_2(r,f)+N_2\left(r,\frac{1}{F}\right)+N_2(r,G)+N_2\left(r,\frac{1}{G}\right)+S(r),$$

where $T(r)=\max\{T(r,F),T(r,G)\}$, $S(r)=o(T(r))$ $(r\to\infty, r\notin E)$, E is a set of finite linear measure.

The following result is due to Yi (see[8]), which plays an important role for the proof of our theorems.

Lemma 3[8] — Let H be defined as in Lemma 2. If $H\equiv 0$ and

$$\varlimsup_{\substack{r\to\infty\\ r\in I}}\frac{\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)}{T(r)}<1,$$

where I is a set with infinite linear measure, then $FG\equiv 1$ or $F\equiv G$.

Lemma 4 — Let f and g be two nonconstant meromorphic functions, $n>6$ a positive integer, and let

$$F=f^n(f-1)f',\qquad G=g^n(g-1)g'.$$

If F and G share 1 CM, then $S(r,f)=S(r,g)$.

PROOF : By Lemma 1, we have

$$(n+1)T(r,f)=T(r,f^n(f-1))+S(r,f)\leq T(r,F)+T(r,f')+S(r,f).$$

Therefore,

$$T(r,F)\geq(n-1)T(r,f)+S(r,f).$$

By the second fundamental theorem, we have

$$T(r,F)\leq\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)+S(r,f)$$

$$\leq\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{f'}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,f)$$

$$\leq 5T(r,f) + T(r,G) + S(r,f).$$

Note that $T(r,G) \leq T(r, g^n(g-1)) + T(r,g') \leq (n+3)T(r,g) + S(r,g)$, we deduce that

$$(n-6)\,T(r,f) \leq (n+3)T(r,g) + S(r,g) + S(r,f).$$

It follows that the conclusion of Lemma 4 holds.

Lemma 5[10] — Let

$$Q(\omega) = (n-1)^2\,(\omega^n - 1)\,(\omega^{n-2} - 1) - n(n-2)\,(\omega^{n-1} - 1)^2,$$

then

$$Q(\omega) = (\omega - 1)^4\,(\omega - \beta_1)\,(\omega - \beta_2)\ldots(\omega - \beta_{2n-6}),$$

where $\beta_j \in C \setminus \{0, 1\}$ $(j = 1, 2, \ldots, 2n-6)$, which are distinct respectively.

3. Proof of Thorems

3.1. *Proof of Theorem 3*

Let

$$F = f^n(f-1)f', \qquad G = g^n(g-1)g'; \qquad \ldots (1)$$

and

$$F^* = \frac{1}{n+2}f^{n+2} - \frac{1}{n+1}f^{n+1}, \quad G^* = \frac{1}{n+2}g^{n+2} - \frac{1}{n+1}g^{n+1}. \qquad \ldots (2)$$

Thus we obtain that F and G share 1 CM. Moreover, by Lemma 1, we have

$$T(r, F^*) = (n+2)T(r,f) + S(r,f), \qquad \ldots (3)$$

$$T(r, G^*) = (n+2)T(r,g) + S(r,g). \qquad \ldots (4)$$

Since $(F^*)' = F$, we deduce

$$m\left(r, \frac{1}{F^*}\right) \leq m\left(r, \frac{1}{F}\right) + S(r,f),$$

and by the first fundamental theorem

$$T(r, F^*) \leq T(r,F) + N\left(r, \frac{1}{F^*}\right) - N\left(r, \frac{1}{F}\right) + S(r,f). \qquad \ldots (5)$$

Note that $$N\left(r, \frac{1}{F^*}\right) = (n+1)N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f - \frac{n+2}{n+1}}\right), \qquad \ldots (6)$$

$$N\left(r, \frac{1}{F}\right) = nN\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + N\left(r, \frac{1}{f'}\right). \qquad \ldots (7)$$

It follows from (5), (6) and (7) that

$$T(r, F^{*}) \le T(r, F) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f - \frac{n+2}{n+1}}\right)$$

$$- N\left(r, \frac{1}{f-1}\right) - N\left(r, \frac{1}{f'}\right) + S(r, f). \qquad \text{... (8)}$$

Let H be defined as in Lemma 2. Suppose that $H \not\equiv 0$, by Lemma 2, we have

$$T(r) \le N_2(r, F) + N_2\left(r, \frac{1}{F}\right) + N_2(r, G) + N_2\left(r, \frac{1}{G}\right) + S(r), \qquad \text{... (9)}$$

where $T(r) = \max\{T(r, F), T(r, G)\}$, $S(r) = o(T(r))$.

It follows from (1) that

$$N_2(r, F) + N_2\left(r, \frac{1}{F}\right) \le 2\overline{N}(r, f) + 2N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + N\left(r, \frac{1}{f'}\right), \qquad \text{... (10)}$$

$$N_2(r, G) + N_2\left(r, \frac{1}{G}\right) \le 2\overline{N}(r, g) + 2N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g-1}\right) + N\left(r, \frac{1}{g'}\right). \qquad \text{... (11)}$$

By (8), (9), (10) and (11), we obtain

$$T(r, F^{*}) \le 3N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f - \frac{n+2}{n+1}}\right) + 2N(r, f)$$

$$+ 2N(r, g) + 2N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g-1}\right) + N\left(r, \frac{1}{g'}\right) + S(r, f). \qquad \text{... (12)}$$

Note that $N\left(r, \frac{1}{g'}\right) \le N(r, g) + N\left(r, \frac{1}{g}\right) \le 2T(r, g) + S(r, g)$, we have from (3) and (12) that

$$(n-4)\, T(r, f) \le 7T(r, g) + S(r, f) + S(r, g). \qquad \text{... (13)}$$

In the same manner as above, we have

$$(n-4) T(r, g) \le 7T(r, f) + S(r, f) + S(r, g). \qquad \text{... (14)}$$

By (13) and (14), we obtain that $n \le 11$, which contradicts $n \ge 12$. Therefore, $H \equiv 0$. That is

$$\frac{F''}{F'} - 2\frac{F'}{F-1} \equiv \frac{G''}{G'} - 2\frac{G'}{G-1}. \qquad \text{... (15)}$$

By integration, we have from (15)

$$\frac{1}{G-1} = \frac{A}{F-1} + B,$$

where $A(\neq 0)$ and B are constants. Thus,

$$T(r, F) = T(r, G) + S(r,f). \qquad \text{... (16)}$$

By (1) and (16), we have

$$\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + \overline{N}(r, G)$$

$$\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, f) + \overline{N}(r, g)$$

$$+ \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) + \overline{N}\left(r, \frac{1}{f'}\right)$$

$$\leq 6T(r,f) + \overline{N}\left(r, \frac{1}{f'}\right) + \overline{N}\left(r, \frac{1}{g'}\right) + S(r,f). \qquad \text{... (17)}$$

Note that

$$\overline{N}\left(r, \frac{1}{f'}\right) \leq T(r, f') - m\left(r, \frac{1}{f'}\right) \leq 2T(r,f) - m\left(r, \frac{1}{f'}\right) + S(r,f). \qquad \text{... (18)}$$

and

$$T(r, F) + m\left(r, \frac{1}{f'}\right) = T(r, f^n (f-1) f') + m\left(r, \frac{1}{f'}\right) \geq T(r, f^n (f-1)). \qquad \text{... (19)}$$

By (17), (18) and (19), we apply Lemma 3 and get $F \equiv G$ or $FG \equiv 1$.

We discuss the following two cases.

Case 1 : Suppose that $FG \equiv 1$, that is

$$f^n (f-1) f' g^n (g-1) g' \equiv 1. \qquad \text{... (20)}$$

Let z_0 be a zero of f of order p. From (20) we know that z_0 is a pole of g. Suppose that z_0 is a pole of g of order q. Again by (20) we obtain

$$np + p - 1 = nq + 2q + 1,$$

that is, $(n + 1)(p - q) = q + 2$, which implies that $p \geq q + 1$ and $q + 2 \geq n + 1$. Hence, $p \geq n$.

Let z_1 be a zero of $f - 1$ of order p_1, then we can also deduce that $p_1 \geq \frac{n}{2} + 2$.

Let z_2 be a zero of f' of order p_2 that is not zero of $f(f - 1)$, we similarly have $p_2 \geq n + 3$. Moreover, in the same manner as above, we have the similar results for the zeros of $g(g-1)g'$. On the other hand, suppose that z_3 is a pole of f. From (20) we get that z_3 is the zero of $g(g-1)g'$, thus

$$\overline{N}(r, f) \leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) + \overline{N}\left(r, \frac{1}{g'}\right)$$

$$\leq \frac{1}{12} N\left(r, \frac{1}{g}\right) + \frac{1}{8} N\left(r, \frac{1}{g-1}\right) + \frac{1}{15} N\left(r, \frac{1}{g'}\right)$$

$$< \frac{2}{3} T(r, g) + S(r, g). \quad \text{... (21)}$$

By the second fundamental theorem, we have from (21) that

$$T(r,f) \le \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}(r,f) + S(r,f)$$

$$< \frac{5}{24} T(r,f) + \frac{2}{3} T(r, g) + S(r,f) + S(r, g). \quad \text{... (22)}$$

Similarly, we have

$$T(r, g) < \frac{5}{24} T(r, g) + \frac{2}{3} T(r,f) + S(r,f) + S(r, g). \quad \text{... (23)}$$

From (22) and (23) we deduce a contradiction.

Case 2 : If $F \equiv G$, that is

$$F^* \equiv G^* + c, \quad \text{... (24)}$$

where c is constant.

It follows that

$$T(r,f) = T(r, g) + S(r,f). \quad \text{... (25)}$$

Suppose that $c \neq 0$. By the second fundamental theorem, from (2), (3) and (25) we have

$$(n+2)\, T(r, g) = T(r, G^*) < \overline{N}\left(r, \frac{1}{G^*}\right)$$

$$+ \overline{N}\left(r, \frac{1}{G^* + c}\right) + \overline{N}\left(r, G^*\right) + S(r, g)$$

$$\le \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g - \frac{n+2}{n+1}}\right) + \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{f}\right)$$

$$+ \overline{N}\left(r, \frac{1}{f - \frac{n+2}{n+1}}\right) + S(r,f)$$

$$\le 5T(r,f) + S(r,f),$$

which contradicts the assumption. Therefore $F^* \equiv G^*$, that is

$$f^{n+1}\left(F - \frac{n+2}{n+1}\right) = g^{n+1}\left(g - \frac{n+2}{n+1}\right)$$

Let $h = \frac{f}{g}$. Since $f \not\equiv g$, we have $h \not\equiv 1$, and hence we deduce that

$$g=\frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})},\qquad f=\frac{(n+2)(1-h^{n+1})h}{(n+1)(1-h^{n+2})},$$

which proves Theorem 3.

3.2. Proof of Theorem 1

Let F, G, F^*, G^* and H be given as above. Suppose that $H \not\equiv 0$. Note tha

$$N\left(r,\frac{1}{g'}\right)\le N\left(r,\frac{1}{g}\right)+S(r,g)\le T(r,g)+S(r,g),$$

we have from (12) that

$$T(r,F^*)\le 3N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-\frac{n+2}{n+1}}\right)+2N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-1}\right)$$

$$+N\left(r,\frac{1}{g'}\right)+S(r,f).$$

Thus, we have

$$(n-2)\,T(r,f)\le 4T(r,g)+S(r,f)+S(r,g), \qquad \text{... (26)}$$

and

$$(n-2)\,T(r,g)\le 4T(r,f)+S(r,f)+S(r,g). \qquad \text{... (27)}$$

By (26) and (27), we obtain that $n\le 6$, which contradicts $n\ge 7$. Therefore, $H\equiv 0$. Using this and proceeding as in the proof of Theorem 3, we obtain that $FG\equiv 1$ or $F\equiv G$. Since f and g are entire functions, we easily derive a contradiction when the case $FG\equiv 1$. Therefore, $F\equiv G$. In the same manner as in the proof of Theorem 3, we obtain $f\equiv g$.

This completes the proof of Theorem 1.

3.3. Proof of Theorem 2

Let F, G, F^*, G^* and H be given as above. Suppose that $H\not\equiv 0$.

Note that

$$N\left(r,\frac{1}{g'}\right)\le N\left(r,\frac{1}{g}\right)+N(r,g)+S(r,g)\le 2T(r,g)+S(r,g).$$

Similar to (13), we have for $\varepsilon\,(>0)$ arbitrary

$$(n-4+2\Theta(\infty,f)-\varepsilon)\,T(r,f)\le 7T(r,g)+S(r,f)+S(r,g).$$

From this and (14), by Lemma 4 we obtain that $n<11$, which contradicts $n\ge 11$. Therefore, $H\equiv 0$.

Suppose that $f \not\equiv g$. Proceeding as in the proof of Theorem 3, we have

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \qquad f = \frac{(n+2)(1-h^{n+1})h}{(n+1)(1-h^{n+2})},$$

where h is nonconstant meromorphic function.

It follows that

$$T(r,f) = (n+1)\,T(r,h) + S(r,f).$$

On the other hand, by the second fundamental theorem, we deduce

$$\overline{N}(r,f) = \sum_{j=1}^{n+1} \overline{N}\left(r, \frac{1}{h-\alpha_j}\right) \geq (n-1)\,T(r,h) + S(r,f),$$

where $\alpha_j\,(\neq 1)\,(j = 1, 2, ..., n+1)$ are distinct roots of the algebraic equation $h^{n+2} = 1$.

Therefore, we have $\Theta(\infty, f) \leq \frac{2}{n+1}$, which contradicts the assumption. Thus, $f \equiv g$.

This completes the proof of Theorem 2.

3.4. Proof of Theorem 4

Set

$$F = f^n (f-1)^2 f', \qquad G = g^n (g-1)^2 g'; \qquad \text{... (28)}$$

and

$$F^* = \frac{1}{n+3} f^{n+3} - \frac{2}{n+2} f^{n+2} + \frac{1}{n+1} f^{n+1},$$

$$G^* = \frac{1}{n+3} g^{n+3} - \frac{2}{n+2} g^{n+2} + \frac{1}{n+1} g^{n+1}. \qquad \text{... (29)}$$

Thus we obtain F and G share 1 CM. Moreover, by Lemma 1, we have

$$T(r, F^*) = (n+3)\,T(r,f) + S(r,f), \qquad \text{... (30)}$$

$$T(r, G^*) = (n+3)\,T(r,g) + S(r,g). \qquad \text{... (31)}$$

Let H be defined as in Lemma 2. Suppose that $H \not\equiv 0$. Proceeding as in the proof of Theorem 3, we have

$$T(r, F^*) \leq T(r,F) + N\left(r, \frac{1}{F^*}\right) - N\left(r, \frac{1}{F}\right) + S(r,f). \qquad \text{... (32)}$$

Note that

$$N\left(r, \frac{1}{F^*}\right) = (n+1)\,N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-a_1}\right) + N\left(r, \frac{1}{f-a_2}\right), \qquad \text{... (33)}$$

where a_1, a_2 are distinct roots of the algebraic equation $\frac{1}{n+3}z^2 - \frac{2}{n+2}z + \frac{1}{n+1} = 0$,

and

$$N\left(r, \frac{1}{F}\right) = nN\left(r, \frac{1}{f}\right) + 2N\left(r, \frac{1}{f-1}\right) + N\left(r, \frac{1}{f'}\right). \qquad \text{... (34)}$$

By Lemma 2, we have

$$T(r) \le N_2(r, F) + N_2\left(r, \frac{1}{F}\right) + N_2(r, G) + N_2\left(r, \frac{1}{G}\right) + S(r), \qquad \text{... (35)}$$

where $T(r) = \max\{T(r, F), T(r, G)\}$, $S(r) = o(T(r))$.

It follows from (28) that

$$N_2(r, F) + N_2\left(r, \frac{1}{F}\right) \le 2N(r, f) + 2N\left(r, \frac{1}{f}\right) + 2N\left(r, \frac{1}{f-1}\right) + N\left(r, \frac{1}{f'}\right), \qquad \text{... (36)}$$

$$N_2(r, G) + N_2\left(r, \frac{1}{G}\right) \le 2N(r, g) + 2N\left(r, \frac{1}{g}\right) + 2N\left(r, \frac{1}{g-1}\right) + N\left(r, \frac{1}{g'}\right). \qquad \text{... (37)}$$

By (32), (33), (34), (35), (36) and (37), we obtain

$$T(r, F^*) \le 3N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-a_1}\right) + N\left(r, \frac{1}{f-a_2}\right) + 2N(r, f)$$

$$+2N(r, g) + 2N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g-1}\right) + N\left(r, \frac{1}{g'}\right) + S(r, f). \qquad \text{... (38)}$$

Note that $N\left(r, \frac{1}{g'}\right) \le N(r, g) + N\left(r, \frac{1}{g}\right) \le 2T(r, g) + S(r, g)$, we have from (30) and (38) that

$$(n-4)\, T(r, f) \le 8T(r, g) + S(r, f) + S(r, g). \qquad \text{... (39)}$$

Similarly, we have

$$(n-4)\, T(r, g) \le 8T(r, f) + S(r, f) + S(r, g). \qquad \text{... (40)}$$

By (39) and (40), we obtain a contradiction. Therefore, $H \equiv 0$.

Note that

$$T(r, F) + m\left(r, \frac{1}{f'}\right) = T(r, f^n (f-1)^2 f') + m\left(r, \frac{1}{f'}\right) \ge T(r, f^n (f-1)^2). \qquad \text{... (41)}$$

By (17), (18) and (41), we apply Lemma 3 and get $F \equiv G$ or $FG \equiv 1$.

Suppose that $FG \equiv 1$. In the same manner in the proof of Theorem 3, we again deduce a contradiction. Therefore $F \equiv G$. Thus $F^* \equiv G^*$, that is

$$\frac{1}{n+3}f^{n+3} - \frac{2}{n+2}f^{n+2} + \frac{1}{n+1}f^{n+1}$$

$$= \frac{1}{n+3}g^{n+3} - \frac{2}{n+2}g^{n+2} + \frac{1}{n+1}g^{n+1}. \qquad \text{... (42)}$$

Set $h=\frac{f}{g}$, we substitute $f=hg$ in (42), it follows that

$$(n+2)(n+1)g^2(h^{n+3}-1)-2(n+3)(n+1)g(h^{n+2}-1)$$
$$+(n+2)(n+3)(h^{n+1}-1)=0. \quad \text{... (43)}$$

If h is not constant, using Lemma 5 and (43), we can conclude that

$$\{(n+1)(n+2)(h^{n+3}-1)g-(n+3)(n+1)(h^{n+2}-1)\}^2$$
$$=-(n+3)(n+1)Q(h)$$

where
$$Q(h)=(h-1)^4(h-\beta_1)(h-\beta_2)\dots(h-\beta_{2n}),$$
$$\beta_j\in C\setminus\{0,1\} \ (j=1,2,\dots,2n),$$

which are pairwise distinct.

This implies that every zero of $h-\beta_j$ $(j=1,2,\dots,2n)$ has a multiplicity of at least 2. By the second fundamental theorem we obtain that $n\le 2$, which is again a contradiction. Therefore, h is a constant. We have from (43) that $h^{n+1}-1=0$ and $h^{n+2}-1=0$, which imply $h=1$, and hence $f\equiv g$.

This completes the proof of Theorem 4.

3.5. Proof of Theorem 6

Set

$$F=f^n(f-1)^2f',$$
$$F_1^*=\frac{1}{n+3}f^{n+3}-\frac{2}{n+2}f^{n+2}+\frac{1}{n+1}f^{n+1},$$

and

$$F_2^*=\frac{1}{n+3}f^{n+3}-\frac{2}{n+2}f^{n+2}+\frac{1}{n+1}f^{n+1}+\left(\frac{2}{n+2}-\frac{1}{n+3}-\frac{1}{n+1}\right).$$

Thus $\left(F_1^*\right)'=F,\left(F_2^*\right)'=F$ and $\left(F_2^*\right)=(f-1)^3P_n(f)$, where P_n is a polynomial of degree n.

By Lemma 1, similar to (5), we have

$$(n+3)T(r,f)=T\left(r\,F_1^*\right)+S(r,f)$$
$$\le T(r,F)+N\left(r,\frac{1}{F_1^*}\right)-N\left(r,\frac{1}{F}\right)+S(r,f). \quad \text{... (44)}$$

Moreover, by the second fundamental theorem we have

$$T(r,F) \le \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) + S(r,f)$$

$$\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) + \overline{N}\left(r,\frac{1}{f'}\right) + S(r,f). \quad \text{... (45)}$$

From (33), (34), (44) and (45), we obtain

$$(n+1)\,T(r,f) \le N(r,f) + 2\overline{N}\left(r,\frac{1}{f}\right) - N\left(r,\frac{1}{f-1}\right) + S(r,f). \quad \text{... (46)}$$

Note that

$$N\left(r,\frac{1}{F_2^*}\right) \le 3N\left(r,\frac{1}{f-1}\right) + nT(r,f) + S(r,f). \quad \text{... (47)}$$

In the same manner as above, we have from (34) and (47) that

$$(n+3)\,T(r,f) = T\left(r, F_2^*\right) + S(r,f)$$

$$\le \overline{N}(r,f) + 2N\left(r,\frac{1}{f-1}\right) - (n-1)\,N\left(r,\frac{1}{f}\right) + nT(r,f) + S(r,f). \quad \text{... (48)}$$

By (46) and (48), we obtain

$$(n+4)\,T(r,f) \le 2\overline{N}(r,f) + N\left(r,\frac{1}{f-1}\right) + (3-n)\,N\left(r,\frac{1}{f}\right) + S(r,f),$$

which implies $T(r,f) \le S(r,f)$. Therefore f is constant.

3.6. Proof of Theorem 5

Proceeding as in the proof of Theorem 6, we shall obtain that Theorem 5 holds.

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