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OSCILLATION AND NONOSCILLATION FOR HALF-LINEAR SECOND ORDER DIFFERENCE EQUATIONS

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New oscillation and nonoscillation theorems are obtained for the half-linear second order difference equation $\Delta(|\Delta x_{n-1}|^{\sigma} \text{sgn}\, \Delta x_{n-1}) + p_n |x_n|^{\sigma} \text{sgn} x_n = 0$ that are different from most known ones in the sense that they are based only on the summation of p_n in set $\{2^m n_0, 2^m n_0 + 1, ..., 2^{m+1} n_0 - 1\}$ for some $n_0 \in \mathbf{N} = \{0, 1, 2, ...\}$ and for every $m \in \mathbf{N}$.

Key Words : Oscillation; Nonoscillation; Half-linear Difference Equations

1. INTRODUCTION

In this paper, we study the oscillatory and nonoscillatory properties of the half-linear second order difference equation

$$\Delta(|\Delta x_{n-1}|^{\sigma} \text{sgn}\, \Delta x_{n-1}) + p_n |x_n|^{\sigma} \text{sgn} x_n = 0, \quad \text{... (1)}$$

which can also be written as

$$\Delta(|\Delta x_{n-1}|^{\sigma-1} \Delta x_{n-1}) + p_n |x_n|^{\sigma-1} x_n = 0,$$

where the forward difference operator Δ is defined as usual, i.e., $\Delta x_n = x_{x+1} - x_n$, $\sigma > 0$ is a constant, and $\{p\}_{n=1}^{\infty}$ is a real sequence with $p_n \geq 0$.

A nontrivial solution $\{x_n\}$ of (1) is said to be oscillatory if for every $n_0 \geq 0$, there exists $n \geq n_0$ such that $x_n x_{n+1} \leq 0$. Otherwise, it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory and nonoscillatory in the opposite case.

Nonoscillation of eq. (1) has been studied by many authors (see, for example, [2, 6-9] and the references quoted therein). Došly and Řehák[4] showed that the Sturmian separation and comparison theory can be extended to (1). In particular, all solutions of (1) are either oscillatory or nonoscillatory.

When $\sigma = 1$, i.e., eq. (1) becomes the second order linear difference equation

$$\Delta^2 x_{n-1} + p_n x_n = 0. \qquad \text{... (2)}$$

Oscillation and nonoscillation of eq. (2) have been investigated intensively (see Agarwal[5]). Recently, Zhang and Zhou[1] presented several new oscillation and nonoscillation theorems for eq. (2), which are discrete analogues of the corresponding results for the continuous version by Huang[3]. The main results are the following.

***Theorem* A** — *Let $\alpha_0 = 3 - 2\sqrt{2}$. If there exists $n_0 \in \mathbf{N}$ such that for every $m \in \mathbf{N}$,*

$$\sum_{i=2^m n_0}^{2^{m+1} n_0 - 1} p_i \le \frac{\alpha_0}{2^{m+1} n_0}, \qquad \text{... (3)}$$

then eq. (2) is nonoscillatory.

***Theorem* B** — *If there exists $n_0 \in \mathbf{N}$ and $\alpha > \alpha_0$ such that for every $m \in \mathbf{N}$,*

$$\sum_{i=2^m n_0}^{2^{m+1} n_0 - 1} p_i \ge \frac{\alpha_0}{2^m n_0}, \qquad \text{... (4)}$$

then eq. (2) is oscillatory.

It is obvious that (3) and (4) are conditions only concerning the summation of p_n in set $\{2^m n_0, 2^m n_0 + 1, \ldots, 2^{m+1} n_0 - 1\}$ for every $m \in \mathbf{N}$. It is natural to ask whether conditions (3) and (4) are valid for eq. (1) with $\sigma \ne 1$, our answer is affirmative. In this paper, we obtain the following results :

***Theorem* 1** — *Let $0 < \sigma < 1$. If there exists $n_0 \in \mathbf{N}$ such that for every $m \in \mathbf{N}$, condition (3) holds. Then eq. (1) is nonoscillatory.*

***Theorem* 2** — *Let $\sigma > 1$. If there exists $n_0 \in \mathbf{N}$ and $\alpha > \alpha_0$ such that for every $m \in \mathbf{N}$, condition (4) holds. Then eq. (1) is oscillatory.*

2. PROOFS OF THE THEOREMS

The following lemma will be used in the proof of the main results of this paper.

Lemma 1[7] — Let $\beta_0 = (2 - \sqrt{2})/2$, $0 < \alpha \le \alpha_0 = 3 - 2\sqrt{2}$, and define

$$f(y) = \frac{1}{2}\left(\frac{y}{1-y} + \alpha\right).$$

Then $\alpha/2 < f(y) < \beta_0$ for all $0 < y < \beta_0$.

Lemma 2 — If $a \ge b \ge 0$, then we have

$$a^\sigma - b^\sigma \le (a-b)^\sigma \text{ for } 0 < \sigma < 1, \text{ and } a^\sigma - b^\sigma \ge (a-b)^\sigma \text{ for } \sigma > 1.$$

PROOF : If $b = 0$, the conclusion is obvious. Let $b > 0$ and consider the function $g(x) = x^\sigma - 1 - (x-1)^\sigma$ defined on $[1, \infty]$. It is easy to show that $g'(x) \le 0$ on $(1, \infty)$ for $0 < \sigma < 1$, and $g'(x) \geq 0$ on $(1, \infty)$ for $\sigma > 1$. Therefore,

$$\left(\frac{a}{b}\right)^{\sigma} - 1 - \left(\frac{a}{b} - 1\right)^{\sigma} \leq 0 \text{ for } 0 < \sigma < 1, \text{ and } \left(\frac{a}{b}\right)^{\sigma} - 1 - \left(\frac{a}{b} - 1\right)^{\sigma} \geq 0 \text{ for } \sigma > 1,$$

i.e.,

$$a^{\sigma} - b^{\sigma} \leq (a-b)^{\sigma} \text{ for } 0 < \sigma < 1, \text{ and } a^{\sigma} - b^{\sigma} \geq (a-b)^{\sigma} \text{ for } \sigma > 1.$$

PROOF OF THEOREM 1 : We are going to show that the solution x_n of (1) subject to the initial conditions $x_{n_0 - 1} = 0$, $x_{n_0} > 0$ remains positive for all $n \geq n_0$. Denote $N_m = 2^m n_0$ for $m = 1, 2, \ldots$, and let

$$n_m = \sup\{n \mid n_0 < n \leq N_m, \Delta x_{s-1} > 0 \text{ for } n_0 \leq s < n\}, \qquad m = 1, 2, \ldots \qquad \ldots (5)$$

Now we define a sequence $\{r_n\}$ as follows:

$$r_0 = \frac{1}{2}\alpha_0, \quad r_{n+1} = \frac{1}{2}\left(\frac{r_n}{1 - r_n} + \alpha_0\right), \qquad n = 0, 1, 2, \ldots$$

Since

$r_0 = \alpha_0/2 = (3 - 2\sqrt{2})/2 < (2 - \sqrt{2})/2 = \beta_0$, from Lemma 1 we see that

$$\frac{1}{2}\alpha_0 < r_1 = f(r_0) < \beta_0.$$

Again using Lemma 1, the increase of $f(y)$ in (0, 1), and by use of an inductive argument we get that

$$0 < r_0 < r_1 < \ldots < r_n < r_{n+1} < \ldots < \beta_0 < 1.$$

Now we prove that the following two formulas are valid for all numbers $m \in \mathbf{N}$:

$$\Delta x_{n-1} > 0, \quad n_0 \leq n \leq N_m, \qquad \ldots (6)$$

$$r_m(\Delta x_{N_m - 1})^{\sigma} \geq \frac{\alpha_0}{2^{m+1}}\left[(\Delta x_{n_0})^{\sigma} + \sum_{i=1}^{m} 2^i (\Delta x_{N_i - 1})^{\sigma}\right]. \qquad \ldots (7)$$

From (5) and the definition of n_m it is obvious that $n_0 < n_1 \leq N_1$. By the definition of n_1, we see that $\Delta x_{n-1} > 0$ for $n_0 \leq n < n_1$, therefore,

$$x_n > 0, \quad \Delta((\Delta x_{n-1})^{\sigma}) \leq 0, \qquad n_0 \leq n \leq n_1 - 2,$$

which means that $(\Delta x_n)^{\sigma}$ is nonincreasing in $n_0 - 1 \leq n \leq n_1 - 1$. By Lemma 2, we get

$$x_n^\sigma = \sum_{i=n_0-1}^{n-1} (x_{i+1}^\sigma - x_i^\sigma) \le \sum_{i=n_0-1}^{n-1} (\Delta x_i)^\sigma \le (\Delta x_{n_0-1})^\sigma (n_1 - n_0). \qquad \dots (8)$$

Summing (1) from n_0 to $n_1 - 1$ and using (4) with $m = 0$ and (8), we obtain

$$(\Delta x_{n_0-1})^\sigma - |\Delta x_{n_1-1}|^\sigma \operatorname{sgn} \Delta x_{n_1-1}$$

$$= \sum_{i=n_0}^{n_1-1} p_i x_i^\sigma \le (\Delta x_{n_0-1})^\sigma (n_1 - n_0) \sum_{i=n_0}^{n_1-1} p_i$$

$$\le (\Delta x_{n_0-1})^\sigma n_0 \sum_{i=n_0}^{2n_0-1} p_i \le \frac{\alpha_0}{2} (\Delta x_{n_0-1})^\sigma,$$

therefore,

$$|\Delta x_{n_1-1}|^\sigma \operatorname{sgn} \Delta x_{n_1-1} \ge \left(1 - \frac{\alpha_0}{2}\right)(\Delta x_{n_0-1})^\sigma > 0,$$

which means that $\Delta x_{n_0-1} > 0$. If $n_1 < N_1$, then it contradicts the definition of n_1. Hence, we get that $n_1 = N_1$ and

$$\Delta x_{n-1} > 0, \quad n_0 \le n \le N_1, \qquad \dots (9)$$

$$(\Delta x_{N_1-1})^\sigma \ge \left(1 - \frac{\alpha_0}{2}\right)(\Delta x_{n_0-1})^\sigma. \qquad \dots (10)$$

Inequality (10) can be rewritten as

$$r_1 (\Delta x_{N_1-1})^\sigma = \frac{1}{2}\left(\frac{r_0}{1 - r_0} + \alpha_0\right)(\Delta x_{n_1-1})^\sigma \ge \frac{\alpha_0}{2} [(\Delta x_{n_1})^\sigma + 2(\Delta x_{N_1-1})^\sigma]. \qquad \dots (11)$$

Relations (9) and (11) are exactly (6) and (7) for $m = 1$, respectively. Now we assume by the induction that (6) and (7) are satisfied for $m = k$, i.e.,

$$\Delta x_{n-1} > 0, \quad n_0 \le n \le N_k, \qquad \dots (12)$$

$$r_k (\Delta x_{N_k-1})^\sigma \ge \frac{\alpha_0}{2^{k+1}}\left[(\Delta x_{n_0-1})^\sigma + \sum_{i=1}^{k} 2^i (\Delta x_{N_i-1})^\sigma\right]. \qquad \dots (13)$$

In view of (12), we obtain that $N_k < n_{k+1} \le N_{k+1}$ and if $N_k \le n \le n_{k+1} - 1$, by using Lemma 2, we get

$$x_n^\sigma = (x_n^\sigma - x_{N_{k-1}}^\sigma) + (x_{N_{k-1}}^\sigma - x_{N_{k-1}-1}^\sigma) + \dots + (x_{N_1-1}^\sigma - x_{n_0-1}^\sigma)$$

$$\le (\Delta x_{N_k-1})^\sigma (N_{k+1}-N_k) + (\Delta x_{N_{k-1}}-1)^\sigma (N_k - N_{k-1}) + \ldots$$

$$+ (\Delta x_{N_1-1})^\sigma (N_2 - N_1) + (\Delta x_{n_0-1})^\sigma (N_1 - N_0)$$

$$= \left[(\Delta x_{n_0-1})^\sigma + \sum_{i=1}^{k} 2^i (\Delta x_{N_i-1})^\sigma \right] N_0, \qquad \ldots (14)$$

where the non-increasing property of $(\Delta x_{n-1})^\sigma$ in $n_0 \le n \le N_k$ is used. Summing (1) from N_k to $n_{k+1}-1$ and using (4) and (14), we have

$$(\Delta x_{N_k-1})^\sigma - |\Delta x_{n_{k+1}-1}|^\sigma \operatorname{sgn} \Delta x_{n_{k+1}-1}$$

$$\le \left[(\Delta x_{n_0-1})^\sigma + \sum_{i=1}^{k} 2^i (\Delta x_{N_i-1})^\sigma \right] N_0 \sum_{i=2^k N_0}^{2^{k+1} N_0} p_i$$

$$\le \frac{\alpha_0}{2^{k+1}} \left[(\Delta x_{n_0-1})^\sigma + \sum_{i=1}^{k} 2^i (\Delta x_{N_i-1})^\sigma \right],$$

which can be rewritten as

$$|\Delta x_{n_{k+1}-1}|^\sigma \operatorname{sgn} \Delta x_{n_{k+1}-1} \ge (\Delta x_{n_k-1})^\sigma$$

$$- \frac{\alpha_0}{2^{k+1}} \left[(\Delta x_{n_0-1})^\sigma + \sum_{i=1}^{k} 2^i (\Delta x_{N_i-1})^\sigma \right]. \qquad \ldots (15)$$

We combine (15) with (13) to obtain that

$$|\Delta x_{n_{k+1}-1}|^\sigma \operatorname{sgn} \Delta x_{n_{k+1}-1} \ge \frac{1-r_k}{r_k} \cdot \frac{\alpha_0}{2^{k+1}}$$

$$\left[(\Delta x_{n_0-1})^\sigma + \sum_{i=1}^{k} 2^i (\Delta x_{N_i-1})^\sigma \right],$$

which reduces to $\Delta x_{n_{k+1}-1} > 0$. Since $0 < r_m < 1$ and $\Delta x_{n_0-1} > 0, \Delta x_{N_i-1} > 0$ ($i = 1, 2, \ldots, k$). From the definition of n_{k+1} we get $n_{k+1} = N_{k+1}$. Therefore, we have

$$\Delta x_{n-1} > 0, \quad n_0 \le n \le N_{k+1}, \qquad \ldots (16)$$

and

$$\frac{1}{2}\left[\frac{r_k}{1-r_k} + \alpha_0 \right] (\Delta x_{N_{k+1}-1})^\sigma \ge \frac{\alpha_0}{2^{k+2}}$$

$$\left[(\Delta x_{N_0-1})^{\sigma} + \sum_{i=1}^{k+1} 2^i (\Delta x_{N_i-1})^{\sigma} \right],$$

which is exactly the following inequality since $r_{k+1} = [r_k/(1-r_k) + \alpha_0]/2$,

$$r_{k+1} (\Delta x_{N_{k+1}})^{\sigma} \geq \frac{\alpha_0}{2^{k+2}} \left[(\Delta x_{n_0-1})^{\sigma} + \sum_{i=1}^{k+1} 2^i (\Delta x_{N_i-1})^{\sigma} \right] \qquad \dots (17)$$

From (16) and (17) we see that (6) and (7) are true for $m = k + 1$, and therefore, (6) and (7) are valid for all numbers $m \in \mathbf{N}$, hence $\Delta x_{n-1} > 0$ for all $n \geq n_0$ and x_n is nonoscillatory. The proof of Theorem 1 is completed.

PROOF OF THEOREM 2 : Without loss of generality, we can assume that $3 - 2\sqrt{2} < \alpha < 3 + 2\sqrt{2}$. The proof will be accomplished by the contradiction. We suppose that (1) has a nontrivial nonoscillatory solution x_n and $x_n > 0$ for arbitrarily large n. Then we have $\Delta x_{n-1} > 0$ for arbitrarily large n. Suppose to the contrary, there exists a positive integer m such that $\Delta x_{m-1} < 0$. From (1), we obtain $|\Delta x_{n-1}|^{\sigma} \operatorname{sgn} \Delta x_{n-1} \leq -(-\Delta x_{m-1})^{\sigma} < 0$ for $n \geq m$ i.e., $\Delta x_{n-1} \leq \Delta x_{m-1}$, which is a contradiction with $x_n > 0$. Therefore, we can take a number $m_0 \in \mathbf{N}$ such that $x_{2^{m_0} n_0} > 0$ and $\Delta x_{n-1} > 0$ for all $n \geq 2^{m_0} n_0$. Here, we denote $N_0 = 2^{m_0} n_0$, $N_m = 2^m N_0 = 2^{m+m_0} n_0$. Define $\left\{ \overset{*}{r}_n \right\}_{n=1}^{\infty}$ as

$$\overset{*}{r}_1 = \frac{\alpha}{2}, \quad \overset{*}{r}_{n+1} = \frac{1}{2} \left(\frac{\overset{*}{r}_n}{1 - \overset{*}{r}_1} + \alpha \right), \qquad n = 1, 2, \dots$$

Later, we will prove that $0 < \overset{*}{r}_n < 1$ $(n = 1, 2, \dots)$ and therefore, the definition of $\overset{*}{r}_n$ is meaningful. It is obvious that $\Delta((\Delta x_{n-1})^{\sigma}) \leq 0$ for $n \geq 2^{m_0} n_0$ and $(\Delta x_{N_1-1}) \geq ((\Delta x_{N_2-1})^{\sigma} \geq \dots \geq (\Delta x_{N_m-1})^{\sigma} \geq \dots > 0$.

Using the nondecreasing of x_n, condition (4) and Lemma 2, we can estimate

$$(\Delta x_{N_m-1})^{\sigma} - (\Delta x_{N_{m+1}-1})^{\sigma} = \sum_{i=N_m}^{N_{m+1}-1} p_i x_i^{\sigma} \geq x_{N_m}^{\sigma} \sum_{i=N_m}^{N_{m+1}-1} p_i$$

$$= \left[x_{N_0}^{\sigma} + \sum_{j=1}^{m} (x_{N_j}^{\sigma} - x_{N_{j-1}}^{\sigma}) \right] \sum_{i=N_m}^{N_{m+1}-1} p_i$$

$$\geq \left[\sum_{j=1}^{m} (\Delta x_{N_j - 1})^{\sigma} (N_j - N_{j-1}) \right] \sum_{i=N_m}^{N_{m+1}-1} p_i$$

$$\geq \left[\sum_{j=1}^{m} (\Delta x_{N_j - 1})^{\sigma} 2^{j-1+m_0} n_0 \right] \sum_{i=2^{m+m_0} n_0}^{2^{m+1+m_0} n_0 - 1} p_i$$

$$\geq \frac{\alpha}{2^m} \sum_{i=1}^{m} 2^{j-1} (\Delta x_{N_j - 1})^{\sigma}. \qquad \text{... (18)}$$

Especially, we have

$$(\Delta x_{N_1 - 1})^{\sigma} - (\Delta x_{N_2 - 1})^{\sigma} \geq \frac{\alpha}{2} (\Delta x_{N_1 - 1})^{\sigma},$$

hence, $0 < r_1^* = \alpha/2 < 1$ and $r_1^* (\Delta x_{N_1 - 1})^{\sigma} < (\Delta x_{N_1 - 1})^{\sigma}$. Now we claim that for all $m \in \mathbf{N}$,

$$r_m^* (\Delta x_{N_m - 1})^{\sigma} \leq \frac{\alpha}{2^m} \sum_{j=1}^{m} 2^{j-1} (\Delta x_{N_j - 1})^{\sigma} < (\Delta x_{N_m - 1})^{\sigma}, \qquad \text{... (19)}$$

$$0 < r_m^* < 1. \qquad \text{.. (20)}$$

In fact, we have proved that (19) and (20) are valid for $m = 1$. Assume that (19) and (20) are true for $m = k$, i.e.,

$$r_m^* (\Delta x_{N_k - 1})^{\sigma} \leq \frac{\alpha}{2^k} \sum_{j=1}^{k} 2^{j-1} (\Delta x_{N_j - 1})^{\sigma} < (\Delta x_{N_k - 1})^{\sigma}, \qquad \text{... (21)}$$

$$0 < r_k^* < 1. \qquad \text{... (22)}$$

In view of (18), we have

$$0 < (\Delta x_{N_{k+1} - 1})^{\sigma} \leq (\Delta x_{N_k - 1})^{\sigma} - \frac{\alpha}{2^k} \sum_{j=1}^{k} 2^{j-1} (\Delta x_{N_j - 1})^{\sigma}. \qquad \text{... (23)}$$

Combining (21) and (23) we get

$$0 < (\Delta x_{N_{k+1} - 1})^{\sigma} \leq \frac{1 - r_k^*}{r_k^*} \cdot \frac{\alpha}{2^k} \sum_{j=1}^{k} 2^{j-1} (\Delta x_{N_j - 1})^{\sigma}.$$

Hence,

$$0<\frac{1}{2}\left(\frac{r_k^*}{1-r_k^*}+\alpha\right)(\Delta x_{N_{k+1}-1})^{\sigma}\leq\frac{\alpha}{2^{k+1}}\sum_{j=1}^{k+1}2^{j-1}(\Delta x_{N_j-1})^{\sigma}. \qquad \text{... (24)}$$

Again from (18),

$$(\Delta x_{N_{k+1}-1})^{\sigma}>\frac{\alpha}{2^{k+1}}\sum_{j=1}^{k+1}2^{j-1}(\Delta x_{N_j-1})^{\sigma}. \qquad \text{... (25)}$$

Inequalities (24) and (25) mean that (19) and (20) are also true for $m = k + 1$, therefore, they are true for all $m \in \mathbf{N}$. Since

$$0<r_1^*=\frac{\alpha}{2}<\frac{1}{2}\left(\frac{r_1^*}{1-r_1^*}+\alpha\right)=r_2^*<$$

in view of the increasing of $f(y)$ in $y \in (0, 1)$, by the induction we arrive at

$$0<r_1^*<r_2^*<\ldots<r_n^*<\ldots<1,$$

hence,

$$0<r_{n+1}^*=\frac{1}{2}\left(\frac{r_n^*}{1-r_n^*}+\alpha\right)<1, \qquad \text{... (26)}$$

which asserts that $0<r_n^*<2/3$. Let $r^*=\lim_{n\to\infty} r_n^*$, then $0<r^*\leq 2/3$. Letting n go to infinity in (26) we have that

$$r^*=\frac{1}{2}\left(\frac{r^*}{1-r^*}+\alpha\right)<1,$$

i.e.

$$2(r^*)^2-(1+\alpha)r^*+\alpha=0. \qquad \text{... (27)}$$

But $3-2\sqrt{2}<\alpha<3+2\sqrt{2}$, the discrimination of the quadratic form in the last equation is

$$\alpha^2-6\alpha+1=(\alpha-(3-2\sqrt{2}))\,(\alpha-(3+2\sqrt{2}))<0,$$

which contradicts the existence of $r^* \in (0, 2/3]$ satisfying (27). Therefore, all nontrivial solutions of (1) are oscillatory and the proof is completed.

Corollary 1 — Let $0<\sigma<1$. If

$$\lim_{n\to\infty}\sup n\sum_{i=n}^{2n-1}p_i=\alpha<\frac{\alpha_0}{2}, \qquad \text{... (28)}$$

then eq. (1) is nonoscillatory.

Corollary 2 — Let $\sigma > 1$. If

$$\lim_{n \to \infty} \inf n \sum_{i=n}^{2n-1} p_i = \alpha > \alpha_0, \qquad \text{... (29)}$$

then eq. (1) is oscillatory.

Remark : If conditions (3) and (4) are replaced by

$$\sum_{=c+2^m n_0}^{c+2^{m+1} n_0} p_i \le \frac{\alpha_0}{2^{m+1} n_0}$$

and

$$\sum_{i=c+2^m n_0}^{c+2^{m+1} n_0} p_i \ge \frac{\alpha}{2^m n_0},$$

respectively, where $c \ge 0$ is an integer, then Theorems 1 and 2 are still valid.

In order to illustrate the main results of this paper, let us consider the following difference equation:

$$\Delta(|\Delta x_{n-1}|^{\sigma-1} \Delta x_{n-1}) + \frac{c}{(1+n)^2} |x_n|^{\sigma-1} x_n = 0, \qquad \sigma > 0,\ n \ge 0, \qquad \text{... (30)}$$

where $c > 0$ is a constant. Noting that

$$n \sum_{i=n}^{2n-1} p_i = n \sum_{i=n}^{2n-1} \frac{c}{(1+i)^2} = \frac{c}{n} \sum_{i=n}^{2n-1} \frac{1}{[(1+i)/n]^2}$$

$$= c \left[\frac{1/n}{(1+1/n)^2} + \frac{1/n}{(1+2/n)^2} + \ldots + \frac{1/n}{(1+n/n)^2} \right].$$

Thus, we have

$$\lim_{n \to \infty} \sup n \sum_{i=n}^{2n-1} p_i = \lim_{n \to \infty} \inf n \sum_{i=n}^{2n-1} p_i$$

$$= c \lim_{n \to \infty} \left[\frac{1/n}{(1+1/n)^2} + \frac{1/n}{(1+2/n)^2} + \ldots + \frac{1/n}{(1+n/n)^2} \right]$$

$$= c \int_0^1 \frac{dx}{(1+x)^2} = \frac{c}{2}.$$

By corollaries 1 and 2, we have that eq. (30) with $0 < \sigma < 1$ is nonoscillatory for $c < \alpha_0$ and eq. (30) with $\sigma > 1$ is oscillatory for $c > 2\alpha_0$.

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