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ON SUPPLEMENTS OF MAXIMAL SUBGROUPS OF A FINITE GROUP*

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This paper is devoted to discussing the supplements of maximal subgroups of a finite group. A result of Kegel is strengthened and a result which proved a conjecture raised by Deskins is improved. Moreover, we obtained some theorems which characterize the supersolvability of a finite group.

Key Words : Maximal Subgroups; Supplement; Completion; θ-completion; Solvability; Supersolvability

1. INTRODUCTION

In 1965 Kegel proved the following result:

***Theorem* A** — A finite group G is solvable if every maximal subgroup M of G admits a supplement which is cyclic and of prime-power order.

In the same paper Kegel conjectured that any finite group whose maximal subgroups admit an abelian supplement is solvable. Theorem A was generalized by Guralnick[5]. He showed that if G, a finite group, is such that every maximal subgroup has prime-power index, then $G/O_\infty(G) \cong 1$ or $L_2(7)$, where $O_\infty(G)$ is the maximal normal solvable subgroup of G. In 1997, Beidleman and Robinson studied the group satisfying the permutizer condition that for every subgroup H of a group G, there exists a cyclic subgroup X such that $XH = HX$. They showed that every group enjoying this condition is solvable (see [2]). Kegel's conjecture was proved by Baumeister. She showed[1] the following:

***Theorem* B** — A finite group G whose maximal subgroups admit an abelian supplement is solvable.

In the same paper, the author gave a characterization of solvability ([1, Theorem 1.3)].

***Theorem* C** — A finite group G is solvable if and only if every maximal subgroup M of G admits a supplement whose commutator subgroup is contained in M.

On the other hand, Deskins introduced[3,4] the concept of maximal completion associated to a maximal subgroup. By answering a conjecture raised by Deskins[4], Zhao Yaoqing proved[8,10] the following results:

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***Theorem* D** — Given a group G, suppose that for each maximal subgroup M of G there exists a maximal completion (θ-completion) C such that $CM = G$ and $C/k(C)$ (resp. C/M_G) is cyclic. Then G is supersolvable or else it has a homomorphic image isomorphic to the symmetric group S_4.

In the above theorem, $k(C)$ is denoted the product of all proper G-normal subgroups of C and M_G is the core of M in G. (Please read Section 2 for details about the definitions of completion and θ-completion).

In this paper we will strengthen Theorem A and improve Theorem D and establish some results to characterize solvability and supersolvability by using Theorems B and C.

Notice that in Theorem D as well as relative results the condition of the maximality of a completion or θ-completion is crucial. In this paper we will dispense with the condition to prove the main results.

2. Definitions and Main Results

All groups treated are of finite orders.

Definition 1[3,4] — A subgroup C of a group G is said to be a completion for a maximal subgroup M of G if C is not contained in M while every G-normal proper subgroup of C is contained in M. By $k(C)$ we denote the product of all G-normal proper subgroups of C.

Definition 2[9] — A subgroup C of a group G is called a θ-completion for a maximal subgroup M of G if C is not contained in M while M_G, the core of M in G, is contained in C and C/M_G has no proper normal subgroup of G/M_G.

We establish first a characterization of solvability.

***Theorem* 1** — A group G is solvable if and only if every maximal subgroup M of G admits a supplement C such that C is a θ-completion (a completion) with C/M_G (resp. $C/k(C)$) abelian.

By using Theorem C, we can strengthen the conclusion of Theorem A as follows:

***Theorem* 2** — Suppose that every maximal subgroup M of a group G admits a supplement C such that C is cyclic, then G is supersolvable or else G is solvable and has a homomorphic image isomorphic to S_4.

In order to prove Theorem 2, we will prove the following theorem, whose condition, by Lemma 1 in Section 3, is slightly weaker than that of Theorem 2.

***Theorem* 3** — Suppose every maximal subgroup M of a group G admits a supplement C which is a θ-completion of M with C/M_G cyclic, then G is supersolvable or else G is solvable and has a homomorphic image isomorphic to S_4.

As an improvement of Theorem D we prove the following:

***Theorem* 4** — Suppose every maximal subgroup M of a group G admits a supplement C which is a completion of M with $C/k(C)$ cyclic, then G is supersolvable or else G is solvable and has a homomorphic image isomorphic to S_4.

The following two theorems give characterizations of supersolvability:

***Theorem* 5** — A group G is supersolvable if and only if every maximal subgroup M of G admits a supplement C which is a θ-completion (resp. completion) and C/M_G (resp. $C/k(C)$) is cyclic of prime order.

***Theorem* 6** — A group G is supersolvable if and only if every maximal subgroup M of G admits a supplement C whose largest normal subgroup C_C is contained in M.

3. Lemmas and Proofs of Theorems

PROOF OF THEOREM 1 : By hypothesis, both C/M_G and $C/k(C)$ are abelian. Hence the commutator subgroup of C is contained in M. It follows by Theorem C that G is solvable.

Conversely, if G is solvable, then so also is G/M_G and it contains a minimal normal subgroup A/M_G which is abelian such that A is θ-completion of M. Moreover, the set of all normal subgroups of G which are not contained in M is nonempty. Choose C as a minimal one in the set, then C is a completion of M with $C/k(C)$ abelian.

Lemma 1 — Suppose that a maximal subgroup M of group G admits a supplement C which is abelian (resp. cyclic), then M admits a supplement D which is a θ-completion of M and D/M_G is abelian (resp. cyclic).

PROOF : Suppose C is not a θ-completion. Then either $M_G \not\trianglelefteq C$ or C/M_G contains a nontrivial normal subgroup D/M_G of G/M_G. In the first case, either CM_G is a θ-completion of M or CM_G/M_G contains a nontrivial normal subgroup D/M_G of G/M_G. Choose a minimal normal subgroup of G/M_G contained in D/M_G, say D/M_G itself, then D is a θ-completion of M. While in the second case, choose D/M_G as above, we have D is a θ-completion of M. Clearly D/M_G is abelian (resp. cyclic). On the other hand, if C or CM_G is itself a θ-completion of M, then C/M_G or CM_G/M_G is abelian (resp. cyclic).

PROOF OF THEOREM 2 : By Lemma 1 the conclusion follows from Theorem 3.

Lemma 2 — Let M be a maximal subgroup of a group G. If C is a θ-completion of $M, N \triangleleft G, N \subseteq M$, then C/N is a θ-completion of M/N and $C/N/(M/N)_{G/N} = \overline{C}/\overline{M}_{\overline{G}} \cong C/M_G$; conversely if C/N is a θ-completion of M/N, then C is a θ-completion of M.

PROOF : Suppose C is a θ-completion of M. Set $\overline{G} = G/N, \overline{M} = M/N$ and $\overline{M}_G = M_G/N$. Since $\overline{M}_G = \overline{M}_{\overline{G}}$, it follows by the definition that C/N is a θ-completion of M/N. Since $(M/N)_{G/N} = M_G/N$, we have $\overline{C}/\overline{M}_{\overline{G}} \cong C/M_G$. If C/N is a θ-completion of M/N, clearly C is a θ-completion for M.

Lemma 3 — (7, Lemma 2 or see 8, Lemma 1). Let p be a prime and P a p-group. Suppose H is a subgroup of index p^n and X a normal complement of H in P which is elementary abelian. If P contains an element y such that $P = \langle y \rangle H$, then $n = 1$ if p is odd or $n \leq 2$ if $p = 2$.

PROOF OF THEOREM 3 : (1) Since for every maximal subgroup M of G, the θ-completion C is such that C/M_G is abelian, we see that the commutator subgroup $C' \subseteq M_G$. By Theorem C, G is solvable.

(2) We prove that M has a normal supplement A such that A is a θ-completion of M and A/M_G is either cyclic or an elementary abelian group with order 4. Let N be a normal subgroup of G contained in M. Then C/N is a supplement of M/N and by Lemma 2 C/N is a θ-completion of M/N. Since $C/N/(M/N)_{G/N} = C/N\,M_G/N \cong C/M_G$ is cyclic, G/N satisfies the hypothesis of the theorem. By induction, M/N has a normal supplement A/N such that $\overline{A}/\overline{M_G}$ is either cyclic or an elementary abelian of order 4, where $\overline{A}$ and $\overline{M_G}$ are respectively the homomorphic images of A and M_G in $\overline{G} = G/N$. Therefore A is the desired supplement for M by Lemma 2. Hence we need only to consider the case $M_G = 1$. Let A be a minimal normal subgroup of G, then $G = MA$ and $M \cap A = 1$ and A is an elementary abelian p-group for some prime p. Clearly A is a θ-completion of M. By hypothesis, M has a θ-completion C such that $G = MC$ and C is cyclic. Since $(C \cap M)^G = (C \cap M)^M$ is trivial, we see that C is a p-group. Assume that C is contained in a Sylow p-group P of G. Then $P = P \cap G = P \cap MC = (P \cap M)C$. Hence $H = P \cap M$ is a Sylow p-subgroup of M such that $HA = P = HC$ and $H \cap A = 1$. By Lemma 3, either $|A| = |P:H| = p$ if p is odd or $|A| = 4$, as desired.

(3) Suppose next that for every maximal subgroup M of G the normal supplement A is such that A/M_G is cyclic, we prove that G is supersolvable. Let N be a minimal normal subgroup of G. By Lemma 2 and a routine inductive argument, G/N is supersolvable and hence we can assume that N is the unique minimal normal subgroup of G and $N \not\trianglelefteq \Phi(G)$. Then we have a maximal subgroup L of G such that $G = LN$. By the result of the above paragraph and our assumption, L has a normal supplement A such that A is cyclic since $L_G = 1$. Therefore, $N \subseteq A$ and so $N = A$ since A is a θ-completion and hence also a minimal normal subgroup of G. Hence G is supersolvable.

(4) We prove that should G happen to have a maximal subgroup M which has a normal supplement A such that A/M_G is elementary abelian of order 4, then G will have a homomorphic image isomorphic to S_4. In fact, set $\overline{G} = G/M_G$ and let $\overline{C}, \overline{M}$ and $\overline{A}$ be the respective images of C, M and A in $\overline{G}$. Then $\overline{G} = \overline{MC} = \overline{MA}$ and $\overline{A}$ is a minimal normal subgroup of $\overline{G}$ by the definition of

θ-completion, hence $\overline{M} \cap \overline{A} = 1$. Considering the permutation representation of $\overline{G}$ on t e four cosets of $\overline{M}, \overline{G}$ is isomorphic to a subgroup of S_4. Since $\overline{M}$ acts faithfully by conjugation on $\overline{A}$, it follows that $|\overline{M}|| 6$. On the other hand, since $\overline{C}$ is cyclic and $\overline{A}$ is elementary abelian, so $|\overline{C} \cap \overline{A} \leq 2$. Notice that $(\overline{C} \cap \overline{M})^{\overline{G}} = (\overline{C} \cap \overline{M})^{\overline{M}}$ must be trivial because $\overline{M}$ is core-free. Thus $|\overline{C}| = 2^2$ and $\overline{C} < \overline{CA}$ and hence $2 || \overline{M}|$. Also $|\overline{M}| \neq 2$ for otherwise $\overline{G}$ would be a 2-group with a maximal subgroup of index 4 and this is certainly impossible. Hence $|\overline{M}| = 6$ and $\overline{G}$ is isomorphic to S_4.

Lemma 4 — Let M be a maximal subgroup of a group G. If M has a completion C such that $C/k(C)$ is abelian (resp. cyclic) with $CM = G$, then M has a θ-completion D such that D/M_G is abelian (resp. cyclic) with $DM = G$.

PROOF : If M_G is contained in C, then clearly $k(C) = M_G$ so that C is a θ-completion of M. Suppose $M_G \not\trianglelefteq C$. Assume first that CM_G is a θ-completion. Then CM_G/M_G has no proper normal subgroup of G/M_G. Since $CM_G/M_G \cong C/C \cap M_G$ and $k(C) \leq C \cap M_G$, and so CM_G/M_G is abelian (resp. cyclic). Assume now CM_G is not a θ-completion. Then CM_G/M_G has a proper normal subgroup E/M_G of G/M_G. Choose D/M_G as a minimal normal subgroup of G/M_G contained in E/M_G. Then D is a θ-completion of M with D/M_G abelian (resp. cyclic) and clearly $DM = G$.

PROOF OF THEOREM 4 : By Lemma 4 and Theorem 3 the conclusion follows:

PROOF OF THEOREM 5 : The proof is analogous to that of Theorem 3 and in part (2) of the proof we see that G has a minimal normal subgroup A, a θ-completion of M. Simultaneously M has a θ-completion C with the given property. We need only to treat the case that $M_G = 1$. Since $C \cap M = 1 = A \cap M$ and H is a Sylow p-subgroup of M, we have $HA = P = HC$ and $H \cap A = 1 = H \cap C$. It follows that $|A| = p$. So part (4) of the proof of Theorem 3 can not occur. This shows that G is supersolvable. The conclusion for completion follows from Lemma 4.

Remark : Theorem 5 can also be obtained by using Theorem 6.

PROOF OF THEOREM 6 : Suppose G enjoys the hypothesis, then $G = CM$ and $|G:M| = |C:C \cap M|$. Since the commutator group C' of C is contained in C_C, we have that C/C_C is an abelian simple and hence has a prime order. It follows that $|G:M|$ is a prime. So G is supersolvable. Conversely, if G is supersolvable and M is a maximal subgroup of G, then there exists a chief factor A/B such that $A \not\trianglelefteq M$ but $B \subseteq M$ and A/B is cyclic with prime order. It forces that $A_A \subseteq B \subseteq M$, where A_A is the largest normal subgroup of A, and A is a supplement of M.

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