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ON 3-DIMENSIONAL KENMOTSU MANIFOLDS

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The object of the present paper is to study a 3-dimensional Kenmotsu manifold satisfying certain curvature conditions. Among other it is proved that a 3-dimensional Kenmotsu manifold with η-parallel Ricci tensor is of constant scalar curvature and a 3-dimensional Kenmotsu manifold satisfying cyclic Ricci tensor is a manifold of constant negative curvature-1.

Key Words: Kenmotsu Manifold; Locally ϕ-Symmetric; η-Parallel Ricci Tensor; Manifold of Constant Negative Curvature

1. INTRODUCTION

The object of the present paper is to study the 3-dimensional Kenmotsu manifolds. After preliminaries in section 2, we prove in section 3 that a 3-dimensional Kenmotsu manifold satisfying the condition $R(X, Y). S = 0$ is a manifold of constant negative curvature, where $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. In section 4, we study a 3-dimensional locally ϕ-symmetric Kenmotsu manifold and obtain a necessary and sufficient condition for a 3-dimensional Kenmotsu manifold to be locally ϕ-symmetric. Section 5 of our paper deals with a 3-dimensional Kenmotsu manifold with η-parallel Ricci tensor and obtain some interesting results. In this section, we also obtain another meaningful result.

2. PRELIMINARIES

Let (M, ϕ, ξ, η, g) be an n-dimensional almost contact Riemannian manifold, where ϕ is a (1, 1) tensor field, ξ is the structure vector field, η is a 1-form and g is the Riemannian metric. It is well known that (ϕ, ξ, η, g) satisfy (Blair[1])

$$(a)\ \ \eta(\xi) = 1,$$

$$(b)\ \ g(X, \xi) = \eta(X) \qquad \text{... (1)}$$

$$\phi^2 X = -X + \eta(X)\xi \qquad \text{... (2)}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y) \qquad \text{... (3)}$$

$$(a)\ \ \phi(\xi) = 0,$$

$$(b)\ \ \eta(\phi X) = 0 \qquad \text{... (4)}$$

for any vector fields, X, Y on M.

If, moreover,

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\,\phi(X) \qquad \text{... (5)}$$

$$\nabla_X \xi = X - \eta(X)\xi \qquad ... (6)$$

where ∇ denotes the Riemannian connection of g, then (M, ϕ, ξ, η, g) is called an almost Kenmotsu manifold[2].

Kenmotsu manifold have been studied by Prasad and Bagewadi[4] and also by Tripathi[7].

Also in a Kenmotsu[2] manifold, the following relations hold

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y) \qquad ... (7)$$

$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z) \qquad ... (8)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \qquad ... (9)$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi \qquad ... (10)$$

$$S(X, \xi) = -(n-1)\eta(X) \qquad ... (11)$$

for any vector fields, X, Y, Z where $R(X, Y)Z$ is the Riemannian curvature tensor and S is the Ricci tensor.

In a 3-dimensional Riemannian manifold we have,

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y$$

$$-\frac{r}{2}[g(Y, Z)X - g(X, Z)Y], \qquad ... (12)$$

where Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ and r is the scalar curvature of the manifold.

Putting $Z = \xi$ in (12) and using (9) and (11), we have

$$\eta(Y)QX - \eta(X)QY = \left(\frac{r}{2} + 1\right)[\eta(Y)X - \eta(X)Y]. \qquad ... (13)$$

Putting $Y = \xi$ in (13) and then using (1) (a) and (11) (for $n = 3$), we get

$$QX = \frac{1}{2}[(r+2)X - (r+6)\eta(X)\xi]$$

i.e.,

$$S(X, Y) = \frac{1}{2}[(r+2)g(X, Y) - (r+6)\eta(X)\eta(Y)]. \qquad ... (14)$$

A Kenmotsu manifold M is said to be η-Einstein if its Ricci tensor S is of the form $S(X, Y) = a\,g(X, Y) + b\eta(X)\eta(Y)$ for any vector fields X, Y where a, b are functions on M.

It is known that if a Kenmotsu[2] manifold M of dimension n is an η-Einstein manifold, then $a + b = -(n-1)$.

Hence from (14) we can state that a 3 dimensional Riemannian manifold is an η-Einstein manifold.

Lemma 2.1 — A 3-dimensional Riemannian manifold is a manifold of constant negative curvature if and only if the scalar curvature $r = -6$.

PROOF : Using (14) in (12), we get

$$R(X, Y)Z = \left(\frac{r+4}{2}\right)[g(Y, Z)\,X - g(X, Z)Y]$$

$$-\left(\frac{r+6}{2}\right)[g(Y, Z)\,\eta(X)\,\xi - g(X, Z)\,\eta(Y)\,\xi$$

$$+\,\eta(Y)\,\eta(Z)\,X - \eta(X)\,\eta(Z)Y]. \quad \text{... (15)}$$

From (15) the Lemma is obvious.

3. 3-DIMENSIONAL KENMOTSU MANIFOLD SATISFYING THE CONDITION $R(X, Y).\ S = 0$

Let us consider a 3-dimensional Riemannian manifold which satisfies the condition

$$R(X, Y).\,S = 0. \quad \text{... (16)}$$

From (16), we have

$$S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0. \quad \text{... (17)}$$

Putting $X = \xi$ and using (10) we get

$$\eta(U)\,S(Y, V) - g(U, Y)\,S(\xi, V) + \eta(V)\,S(U, Y) - g(V, Y)\,S(U, \xi) = 0. \quad \text{... (18)}$$

Using (11) in (18), we have

$$\eta(U)\,S(Y, V) + 2g(U, Y)\,\eta(V) + \eta(V)\,S(U, Y) + 2g(V, Y)\,\eta(U) = 0. \quad \text{... (19)}$$

Let $\{e_i\}$, $i = 1, 2, 3$ be an orthonormal basis of the tangent space at each point of the manifold. Then putting $Y = U = e_i$ in (19) and taking summation for $1 \le i \le 3$, we get

$$S(\xi, V) + 8\eta(V) + r\eta(V) = 0. \quad (20)$$

Using (11) in (20), we obtain

$$(r+6)\,\eta(V) = 0.$$

This gives $r = -6$ (since $\eta(V) \neq O$), which implies by Lemma 2.1 that the manifold is of constant negative curvature.

Hence we can state the following:

***Theorem* 1** — *A 3-dimensional Riemannian manifold satisfying the condition $R(X, Y).\ S = 0$ is a manifold of constant negative curvature-1.*

4. LOCALLY ϕ-SYMETRIC 3-DIMENSIONAL KENMOTSU MANIFOLD

Definition 4.1 — A Kenmotsu manifold is said to be locally ϕ symmetric if $\phi^2\,(\nabla_W R)\,(X, Y)Z = 0$ for all vector fields W, X, Y, Z orthogonal to ξ.

This notion was introduced by Takahashi[6] for Sasakian manifold.

Now differentiating (15) covariantly with respect to W we get

$$(\nabla_W R)\,(X, Y)Z = \frac{dr\,(W)}{2}\,[g(Y, Z)X - g(X, Z)Y)]$$

$$-\frac{dr(W)}{2}[g(Y,Z)\,\eta(X)\,\xi - g(X,Z)\,\eta(Y)\xi$$

$$+\eta(Y)\,\eta(Z)\,X - \eta(X)\,\eta(Z)Y]$$

$$-\left(\frac{r+6}{2}\right)[g(Y,Z)\,(\nabla_W\eta)\,(X)\xi - g(X,Z)\,(\nabla_W\eta)\,(Y)\xi$$

$$+g(Y,Z)\,\eta(X)\,\nabla_W\xi - g(X,Z)\,\eta(Y)\,\nabla_W\xi$$

$$+(\nabla_W\eta)\,(Y)\,\eta(Z)\,X + \eta(Y)\,(\nabla_W\eta)\,(Z)X$$

$$-(\nabla_W\eta)\,(X)\,\eta(Z)\,Y - \eta(X)\,(\nabla_W\eta\,(Z)Y].$$

Taking X, Y, Z, W orthogonal to ξ and using (6) and (7), we get from the above

$$(\nabla_W R)\,(X,Y)Z = \frac{1}{2}dr(W)\,[g(Y,Z)X - g(X,Z)Y]$$

$$-\left(\frac{r+6}{2}\right)[g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)]\,\xi. \quad \text{... (21)}$$

From (21) it follows that

$$\phi^2\,(\nabla_W R)\,(X,Y)\,Z = \frac{1}{2}dr(W)\,[g(Y,Z)\,\phi^2 X - g(X,Z)\,\phi^2 Y].$$

Now taking X, Y, Z, W orthogonal to ξ and using (1)(b) and (2) we get from the above expression

$$\phi^2\,(\nabla_W R)\,(X,Y)Z = -\frac{1}{2}dr(W)\,[g(Y,Z)\,X - g(X,Z)Y]. \quad \text{... (22)}$$

From (22) we can state the following:

Theorem 2 — *A 3-dimensional Kenmotsu manifold is locally ϕ-symmetric if and only if the scalar curvature* r is constant.

Again if the manifold satisfies the condition $R(X,Y).S = 0$, then we have seen that $r = -6$, i.e. r = constant and hence from (22) we can state the following:

Theorem 3 — *If a 3-dimensional Kenmotsu manifold satisfies the condition $R(X,Y).S = 0$, then the manifold is locally ϕ-symmetric.*

5. 3-Dimensional Kenmotsu Manifold with η-Parallel Ricci Tensor

Definition 5.1 — The Ricci tensor S of a Kenmotsu manifold M is called η-parallel if it satisfies

$$(\nabla_X S)\,(\phi Y, \phi Z) = 0 \quad \text{... (23)}$$

for all vector fields X, Y and Z.

The notation of Ricci-η-parallelity for Sasakian manifolds was introduced by Kon[3].

Now let us consider 3-dimensional Kenmotsu manifold with η-parallel Ricci tensor. Then from (14) we get by virtue of (3) and (4)(b),

$$S(\phi X, \phi Y) = \left(\frac{r+2}{2}\right)[g(X, Y) - \eta(X)\,\eta(Y)]. \qquad \text{... (24)}$$

Differentiating (24) covariantly along Z we get

$$(\nabla_Z S)(\phi X, \phi Y) = \frac{1}{2}[dr(Z)\ \{g(X, Y) - \eta(X)\,\eta(Y)\}$$

$$-(r+2)\ \{\eta(Y)(\nabla_Z \eta)(X) + \eta(X)(\nabla_Z \eta)(Y)\}]. \qquad \text{... (25)}$$

By using (23) and (25), we get

$$dr(Z)\ \{g(X, Y) - \eta(X)\,\eta(Y)\}$$

$$-\ (r+2)\ \{\eta(Y)(\nabla_Z \eta)(X) + \eta(X)(\nabla_Z \eta)(Y)\} = 0. \qquad \text{... (26)}$$

Putting $X = Y = e_i$, in (26) and then taking summation over i, $1 \le i \le 3$, we get $dr(Z) = 0$, for all Z.

Proposition 5.1 — If a 3-dimensional Kenmotsu manifold has η-parallel Ricci tensor, then the scalar curvature r is constant.

By virture of proposition 5.1. and Theorem 2, we have the following:

***Theorem* 4** — *A 3-dimensional Kenmotsu manifold with η-parallel Ricci tensor is locally ϕ-symmetric.*

Let us suppose that a 3-dimensional Kenmotsu manifold satisfies the cyclic Ricci tensor.

Then we have

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0. \qquad \text{... (27)}$$

Now putting $Y = Z = e_i$, in (27) and taking summation over i, $1 \le i \le 3$, we get

$$(\nabla_X S)(e_i, e_i) + 2(\nabla e_i S)(e_i, X) = 0. \qquad \text{... (28)}$$

Now,

$$(\nabla_X S)(e_i, e_i) = \nabla_X S(e_i, e_i) - 2S(\nabla_X e_i, e_i). \qquad \text{... (29)}$$

We know that the scalar curvature $r = \sum_i S(e_i, e_i)$. Also in local coordinates $\nabla_X e_i = X^j \Gamma^h_{ji} e_h$, where Γ^h_{ji} are the Christoffel symbols. Since $\{e_i\}$ are orthonormal basis, the metric tensor $g_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker delta and hence the Christoffel symbols are zero. Therefore, $\nabla_X e_i = 0$. Hence from (29) it follows that

$$(\nabla_X S)(e_i, e_i) = \nabla_X r = dr(X). \qquad \text{... (30)}$$

Let Q be the Ricci operator defined by $g(QX, Y) = {}^{\sim}(X, Y)$, that is, Q is the (1, 1) Ricci tensor. Then $(\nabla_Z S)(X, Y) = g((\nabla_Z Q)(X), Y))$.

Taking $Y = Z = e_i$ and taking summation over i, $1 \leq i \leq 3$, we get from the above $(\nabla e_i S)(X, e_i) = g((\nabla e_i Q)(X), e_i)$.

We know

$$(\text{div } Q)(X) = tr(Z \to (\nabla_Z Q)(X))$$

$$= \sum_i g((\nabla e_i Q)(X), e_i).$$

But it is known (Peterson[5]) that $(\text{div } Q)(X) = \frac{1}{2} dr(X)$.

Hence

$$(\nabla e_i S)(X, e_i) = \frac{1}{2} dr(X). \qquad \text{... (31)}$$

Now using (30) and (31) in (28) we obtain

$$dr(X) = 0, \text{ for all } X, \qquad \text{... (32)}$$

which implies r is constant.

From (14), we have

$$(\nabla_Z S)(X, Y) = \frac{1}{2} [dr(Z) \ \{g(X, Y) - \eta(X)\,\eta(Y)\}$$

$$- (r+6) \ \{\eta(Y)\,(\nabla_Z \eta)(X) + \eta(X)\,(\nabla_Z \eta)(Y)\}]. \qquad \text{... (33)}$$

Now using (32) in (33), we have

$$(\nabla_Z S)(X, Y) = -\left(\frac{r+6}{2}\right)[\eta(Y)\,(\nabla_Z \eta)(X) + \eta(X)\,(\nabla_Z \eta)(Y)]. \qquad \text{... (34)}$$

By virtue of (34), we get from (27) that

$$(r+6)\,[\eta(Z)\,(\nabla_X \eta)(Y) + \eta(Y)\,(\nabla_X \eta)(Z) + \eta(Y)\,(\nabla_Z \eta)(X)$$

$$+ \eta(X)\,(\nabla_Z \eta)(Y) + \eta(Z)\,(\nabla_Y \eta)(X) + \eta(X)\,(\nabla_Y \eta)(Z)] = 0. \qquad \text{... (35)}$$

Using (7) and putting $Y = Z = e_i$ in (35), we get

$$(r+6)\,\eta(X) = 0$$

which implies that $r = -6$.

This leads by virtue of Lemma 2.1 to the following theorem:

***Theorem* 5** — *If a 3-dimensional Kenmotsu manifold satisfies the condition (27), then the manifold is a manifold of constant negative curvature-1.*

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