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INTRODUCING OF AIR BUBBLE EFFECTS ON A SIMPLIFIED MODEL FOR WAVE HEIGHT

ASHABUL HOQUE

Department of Mathematics, University of Rajshahi, Rajshahi 6205, *Bangladesh*

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The present paper is concerned with the study of air bubble entrainment by breaking waves on a shelf beach in the surf zone. An energy dissipation model is outlined to evaluate air bubble effets, which provides explicit expressions for the wave height within the surf zone. The wave height decreases significantly with increasing the air bubble into water.

Key Words: Air Entrainment; Wave Height; Energy Flux; Energy Dissipation Rate

1. INTRODUCTION

Wave breaking near the shore region is an important phenomenon of wave hydrodynamics. During the wave breaking, a large amount of air bubble is entrained into water, causing a large-scale disturbance in flow. Horikawa and Kau[3] suggested that this entrained air bubbles are responsible for dissipating the wave energy. Führböter[2] proposed that the sudden reduction of wave height and wave energy inside the surf zone could be explained by the entrained air bubbles into water. The main

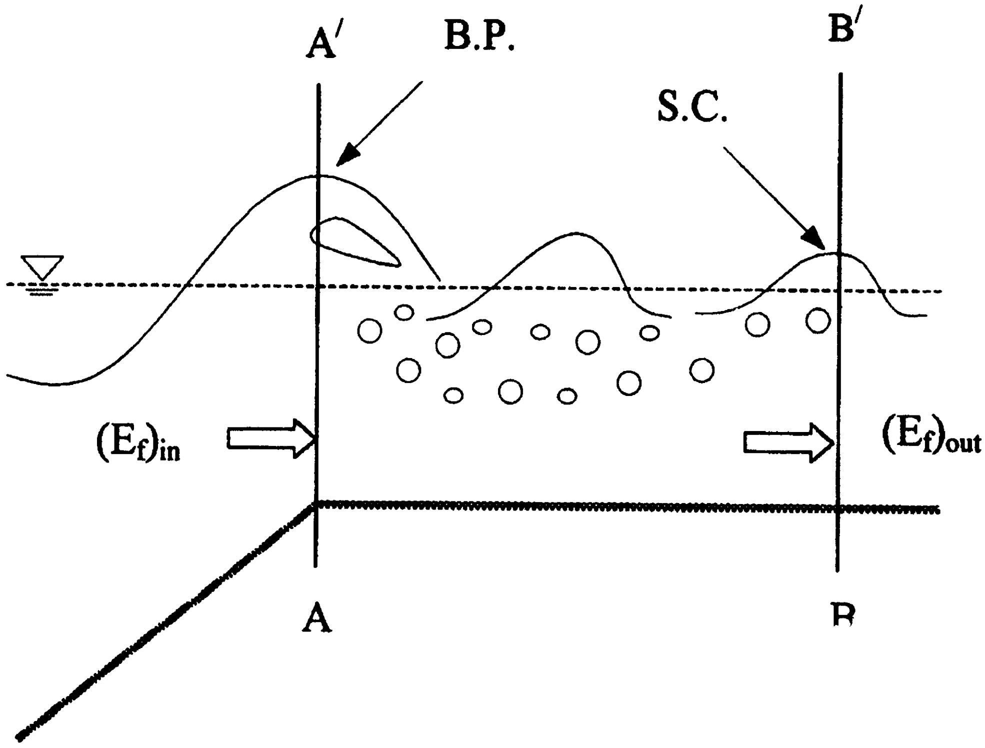


FIG. 1. Shelf beach idealization of the surf zone definition diagram is taken with slight modification from Dally *et al.*[1]. B. P. represents the breaking point and S. C. is the stable condition. Wave breaking starts from *AA′* and continues until stable condition is attained at *BB′* and air entrainment occurs in this region.

issue involves an estimation of wave energy dissipation in the surf zone. Although the energy dissipation has been extensively studied in the past four decades, but still its mechanism remains poorly understood. Two models have received considerable attention from last two decades: the surface roller concept[5] and energy flux difference model[1]. Both a literature review and personal observations indicate that air bubbles have significant effects in the surf zone but up to date, no model considering explicitly the air entrainment process has been reported, except that reported by Führböter[2]. The author has proposed an energy dissipation model considering air bubble effects but the model is not calibrated and verified.

Analytical solutions were derived by Dally *et al.*[1] for wave decay due to wave breaking on a horizontal shelf and a uniform slope as shows in Fig. 1. According to Fig. 1, wave breaking starts from the AA' and would continue until some stable wave heights are reached at line BB'.

The decay in wave height due to dissipation is found from the continuity equation for the wave energy flux

$$\frac{dE_f}{dx} = -D \qquad \text{... (1.1)}$$

where, $E_f = Ec_g$, the wave energy flux, c_g the group velocity, E the wave energy, and D is the energy dissipation rate per unit area. Dally *et al.*[1] assumed D in the surf zone to be proportional to the difference between the local energy flux and the stable energy flux,

i.e., $$D = \frac{K}{h}[E_f - (E_f)_s] \qquad \text{... (1.2)}$$

where h is the still water depth, K the dimensionless decay coefficient, and $(E_f)_s$ is the energy flux associated with the stable wave when no more energy dissipation due to wave breaking occurs (Fig. 1).

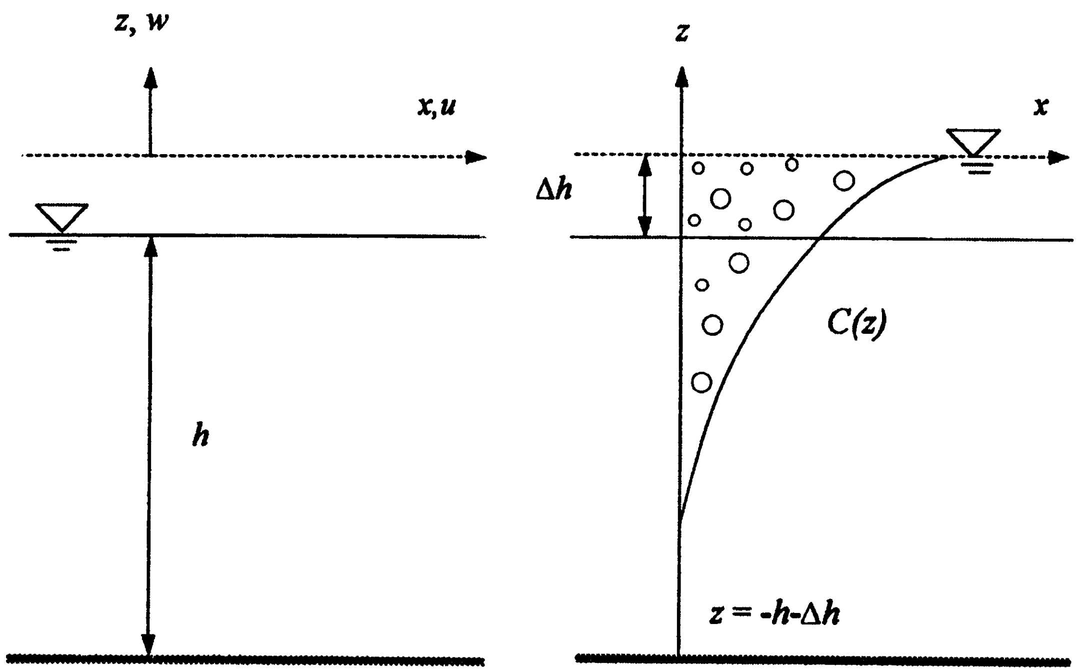


FIG. 2. Schematic outline of water level rise due to the air bubble entrainment. Here h, Δh, and $C(z)$ represent still water depth, volume of entrained air, and void fraction distribution respectively.

The present paper aims to show the effects of air bubble entrainment on the most important parameter, wave evolution. The results provide useful information regarding the surf zone hydrodynamics, such as, wave height, wave set-up, run-up, and long waves.

2. Theoretical Analysis

2.1. Basic assumption

The distribution of air bubbles in the vertical direction is given by the following exponential form follows (Wu[6]):

$$C(z) = C_0 \exp(k_1 z) \qquad \text{... (2.1)}$$

where $C(z)$ is the part of the volume locally occupied by bubbles per unit width (time-averaged concentration), k_1 is a decay parameter characterizing vertical distribution of air bubbles and C_0 denotes the reference concentration at the mean water surface $z = 0$.

The following boundary conditions are automatically satisfied:

$$C(z) = C_0 \text{ at the surface } z = 0$$

and $$C(z) \to 0, \text{ for } z \to -\infty$$

The rise of the free-surface level Δh is a function of the amount of entrained air and water depth (Fig. 1). The total volume of entrained air into water per unit width is defined as

$$\Delta h = \int_{-h-\Delta h}^{0} C(z)\, dz \qquad \text{... (2.2)}$$

where z is taken upward from the raised water surface. This yields,

$$\frac{\Delta h}{h} = \frac{C_0}{k_1 h} \frac{(1 - e^{-k_1 h})}{(1 - C_0 e^{-k_1 h})} \qquad \text{... (2.3)}$$

Based on the above assumption, Hoque[4] derived the velocity components, densitv and pressure field in the following manner:

$$\begin{pmatrix} u = u_w \\ w = w_w + C(z)\, w_r \\ p = p_w \\ \rho = (1 - C(z))\, \rho_w \end{pmatrix} \qquad \text{... (2.4)}$$

where, subscripts 'a', 'w' denote air and water, respectively and w_r denotes the rise velocity of bubbles.

According to linear theory, the velocity components and pressure are defined by

$$\left\{\begin{aligned} u_w &= \frac{\pi H}{T}\frac{\cosh k(h+z)}{\sinh kh}\cos(\omega t - kx) \\ W_w &= \frac{\pi H}{T}\frac{\cosh k(h+z)}{\sinh kh}\sin(\omega t - kx) \\ p_w &= -\rho_w g z + \rho_w g\frac{H}{2}\frac{\cosh k(h+z)}{\sinh kh}\cos(\omega t - kx) \end{aligned}\right. \qquad \text{... (2.5)}$$

where H, T, ω, k, and t are the local wave height, the wave period, the wave angle, wave number, and the time, respectively.

2.2. Analytical solution

An analytical solution for wave decay due to breaking of wave on a flat shelf, and a plane slope are derived by Dally *et al.*[1]. Eq. (1.1) can be rewritten with the help of eq. (1.2).

$$\frac{dE_f}{dx} = -\frac{K}{h}[E_f - (E_f)_s] \qquad \text{... (2.6)}$$

When the energy flux is calculated, the contributions from the water in the wave motion and from air bubble entrainment can be found together.

The total energy flux is defined as

$$E_f = \int_{-h-\Delta h}^{\eta}\left\{P_D + \frac{1}{2}\rho(u^2 + w^2)u\right\}dz \qquad \text{... (2.7)}$$

where, η represents the variation of water surface and P_D is the dynamic pressure defined by

$$P_D = p_w + \rho_w g z \qquad \text{... (2.8)}$$

Substituting eqs. (2.4), (2.5) and (2.8) in eq. (2.7), it can be obtained the following relationship

$$E_f = \frac{1}{T}\int_0^T\int_{-h-\Delta h}^{\eta}\left[\rho_w g\frac{H}{2}\frac{\cosh k(h+z)}{\cosh kh}\cos(kx - \omega t)\right]^*$$

$$\left[\frac{\pi H}{T}\frac{\cosh k(h+z)}{\sinh kh} * \cos(kx - \omega t)\right]dz\,dt$$

$$+\frac{1}{T}\int_0^T\int_{-h-\Delta h}^{\eta}\frac{1}{2}\rho_w(1 - C_0 e^{k_1 z})\left[\left\{\frac{\pi H}{T}\frac{\cosh k(h+z)}{\cosh kh} * \cos(kx - \omega t)\right\}\right]^2$$

$$+\left\{\left(-\frac{\pi H}{T\sinh kh}\right)(\sinh k(h+z) + \frac{C_0}{(1 - C_0 e^{k_1 z})}\left(e^{k_1 z}\left(\frac{kk_1}{k^2 - k_1^2}\right)\right)^*\right.$$

$$\cosh k(h+z) - \frac{k_1^2}{k^2 - k_1^2} \sinh k(h+z)) - \frac{kk_1}{k^2 - k_1^2} e^{-k_1 h} \Bigg)\Bigg)$$

$$\sin (kx - \omega t)^2]^* \left[\pi \frac{H]}{T} \frac{\cosh k(h+z)}{\sinh kh} cps\,(kx - \omega t) \right] dz\, dt$$

After integrating and neglecting of some higher-order terms of H, the energy flux becomes

$$E_f = Ec\left(\frac{1}{2} + \frac{kh}{\sinh 2kh}\right) + \frac{2kEc}{\sinh 2kh}\Delta h \qquad \text{... (2.9)}$$

where c and k are the wave celerity and wave number, respectively. The first term on right hand side of eq. (2.9) is the energy flux due to wave and second term represents the air bubble effects.

By applying shallow water linear approximation $c \approx c_g = \sqrt{gh}$ and $\frac{2kh}{\sinh 2kh} \sim 1$, eq. (2.6) can be rewritten with the help of eq. (2.9).

$$\frac{d}{dx}\left[H^2\sqrt{h} + \frac{H^2}{\sqrt{h}}\Delta h \right] = -\frac{K}{h}\left[H^2\sqrt{h} + \frac{H^2}{\sqrt{h}}\Delta h - H_s^2\sqrt{h} \right] \qquad \text{... (2.10)}$$

Horikawa and Kuo[3] measured the stable wave criterion as $H_s = \Gamma h$ where, H_s is the stable wave height. Γ is a dimensionless coefficient called stable wave factor. When a broken wave enters an area with horizontal bed, and moderate slopes the wave height decreases or asymptotically approaches by Γh, where h represents the water depth. On the basis of experiments of Horikawa and Kuo[3], the value of Γ in the range 0.35 to 0.50 was reported by Dally *et al.*[1].

Substituting $H^2\sqrt{h} + \frac{H^2}{\sqrt{h}}\Delta h = y$ in eq. (2.10), it can be written as

$$\frac{dy}{dx} + \frac{K}{h} y = K\Gamma^2 h^{3/2} \qquad \text{... (2.11)}$$

This is the first order linear differential equation for y where h is either a constant or a function of x.

Case 1 — Uniform water depth.

For the idealized beach with a horizontal bottom (i.e., $h(x)$ = const = h), the general solution of eq. (2.11) is

$$y = \left[\exp\left(-K\frac{x}{h}\right)\right]\left[\Gamma^2 h^{5/2} \exp\left(K\frac{x}{h}\right) + C'\right] \qquad \text{... (2.12)}$$

Since the approximation was made for shallow water, the initial condition is applied where the wave breaking occurs

$$y = y_b = H_b^2\left(\sqrt{h} + \frac{\Delta h}{\sqrt{h}}\right)_b \qquad \text{at } x = 0$$

From eq. (2.12), it becomes

$$C' = H_b^2\left(\sqrt{h}+\frac{\Delta h}{h}\right)_b - \Gamma^2 h^{h/2}$$

Finally the decay in dimensionless wave height in water of uniform depth is found as

$$\frac{H}{h} = \left\{\left[\left\{\left(\frac{H}{h}\right)_b^2\left(1+\left(\frac{\Delta h}{h}\right)_b\right)-\Gamma^2\right\}\exp\left(-K\frac{x}{h}\right)+\Gamma^2\right]\Big/\left(1+\frac{\Delta h}{h}\right)\right\}^{1/2} \quad \dots (2.13)$$

where the subscript 'b' represents conditions at the breaking point and x is its origin at the breaker line directing onshore. Note that the wave energy remains constant if $K = 0$ (no breaking).

Case 2 — Uniform bottom slope.

The plane beach can be defined as

$$h(x) = h_n - mx$$

where m is the beach slope. Eq. (2.11) can be solved as

$$y = e^{\frac{K}{m}\ln(h_b - mx)}\left[-\frac{K}{m}\frac{\Gamma^2}{\left(\frac{5}{2}-\frac{K}{m}\right)}(h_b - mx)^{\left(\frac{5}{2}-\frac{K}{m}\right)} + C''\right] \quad \dots (2.14)$$

where C'' is arbitrary constant. Now applying the following initial condition

$$y = y_b = H_b^2\left(\sqrt{h}+\frac{\Delta h}{\sqrt{h}}\right)_b \qquad \text{at } x = 0$$

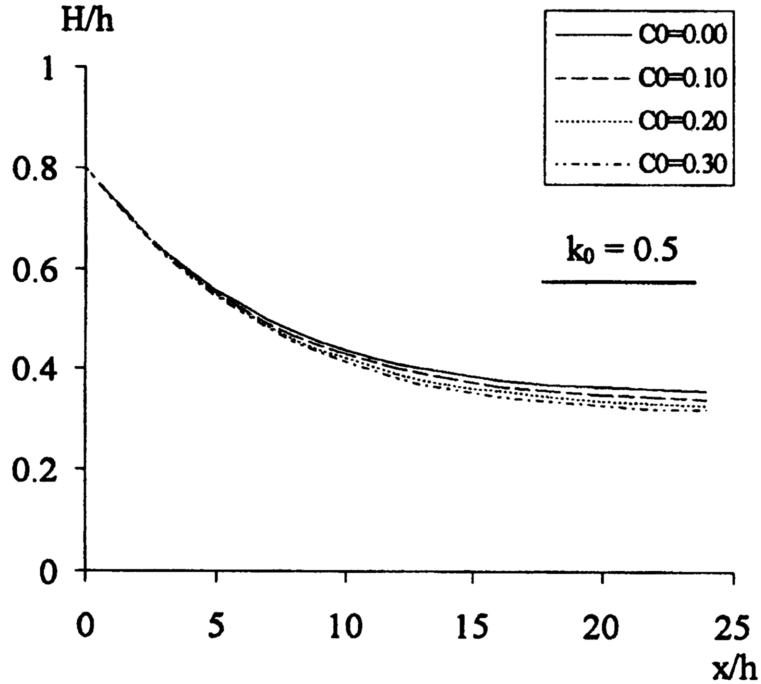
into the eq. (2.14), the arbitrary constant is found as

$$C'' = H_b^2\left(\sqrt{h}+\frac{\Delta h}{\sqrt{h}}\right)_b e^{-\frac{K}{m}\ln h_b} + \frac{K}{m}\frac{\Gamma^2 (h_b)^{\left(\frac{5}{2}-\frac{K}{m}\right)}}{\left(\frac{5}{2}-\frac{K}{m}\right)}$$

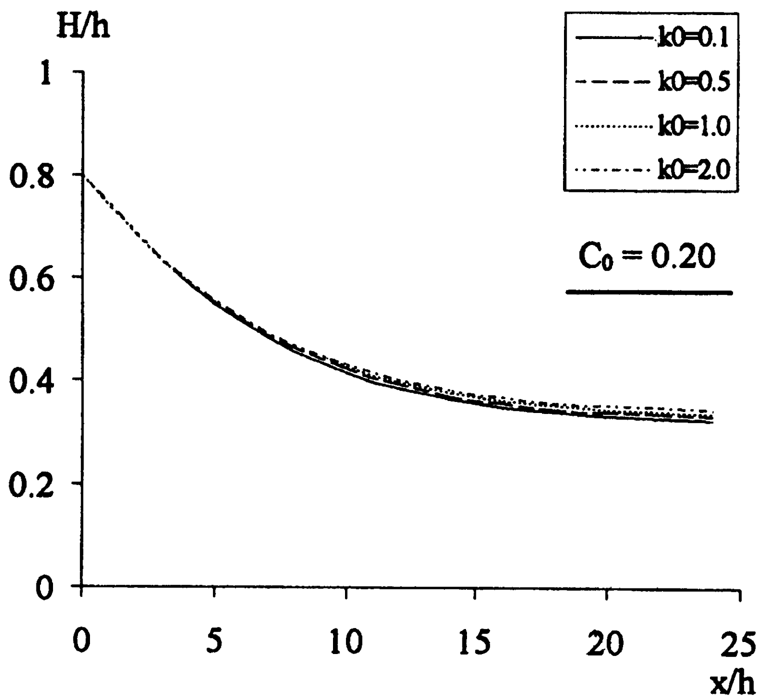
Now substituting the value of C'' into eq. (2.14) and then dividing both sides by $H_b^2\, h_b^{1/2}\, h_b^{-K/m}$, the dimensionless wave height takes the form

$$\frac{H}{H_b} = \left\{\frac{1}{\left(1+\frac{\Delta h}{h}\right)}\left[\left(\frac{h}{h_b}\right)^{\frac{K}{m}-\frac{1}{2}}\left\{\left(1+\left(\frac{\Delta h}{h}\right)_b\right)+\alpha\right\}-\alpha\left(\frac{h}{h_b}\right)^2\right]\right\}^{1/2} \quad \dots (2.15)$$

in which

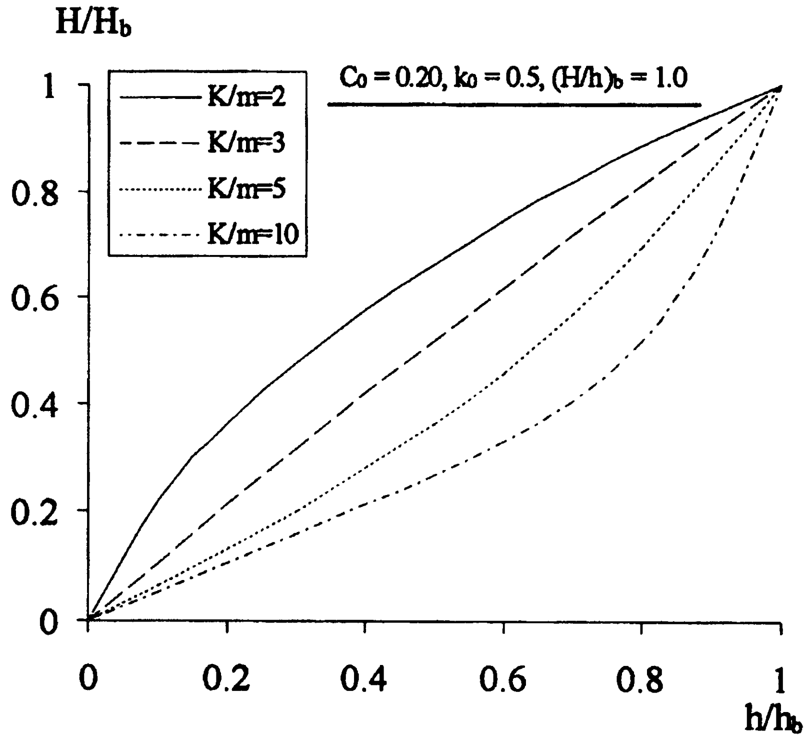


(a)

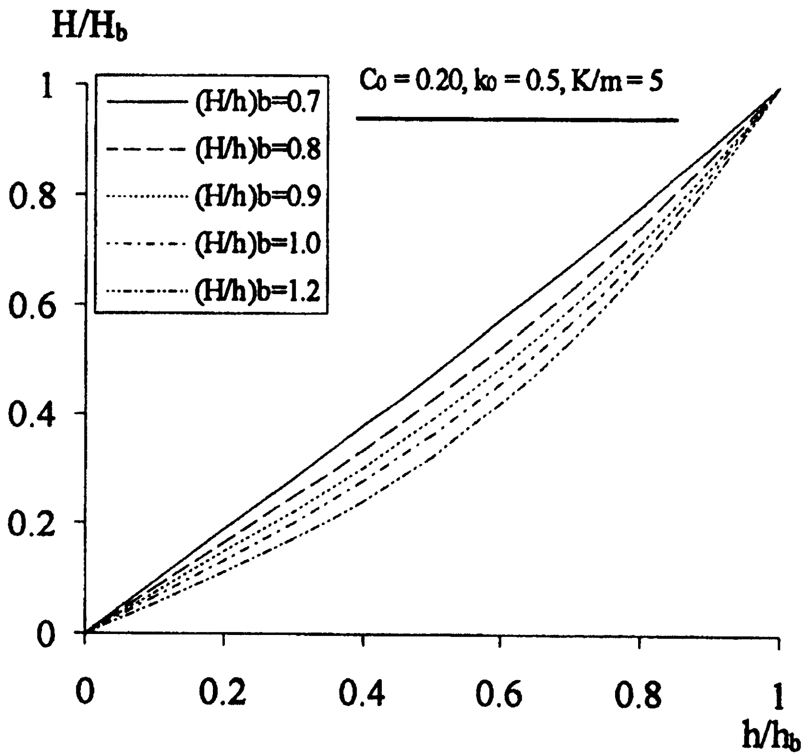


(b)

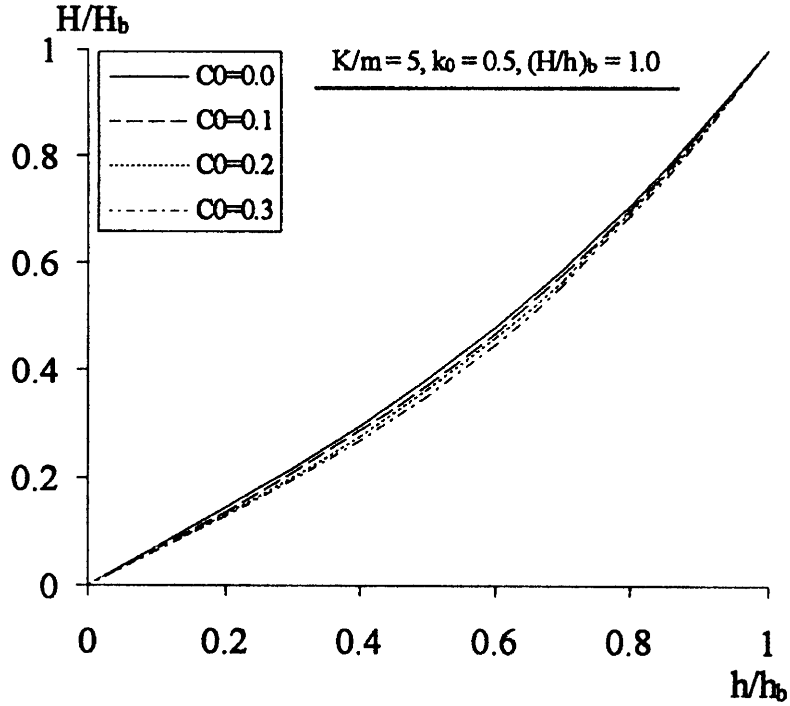
FIG. 3. Dimensionless wave height H/h versus x/h for different (*a*) C_0 and (*b*) k_0 for waves breaking on a flat shelf. The values of 0.2 and 0.35 have been used for K and Γ, respectively.


H/Hb
K/m=2
K/m=3
K/m=5
K/m=10
Co = 0.20, ko = 0.5, (H/h)b = 1.0
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
h/hb

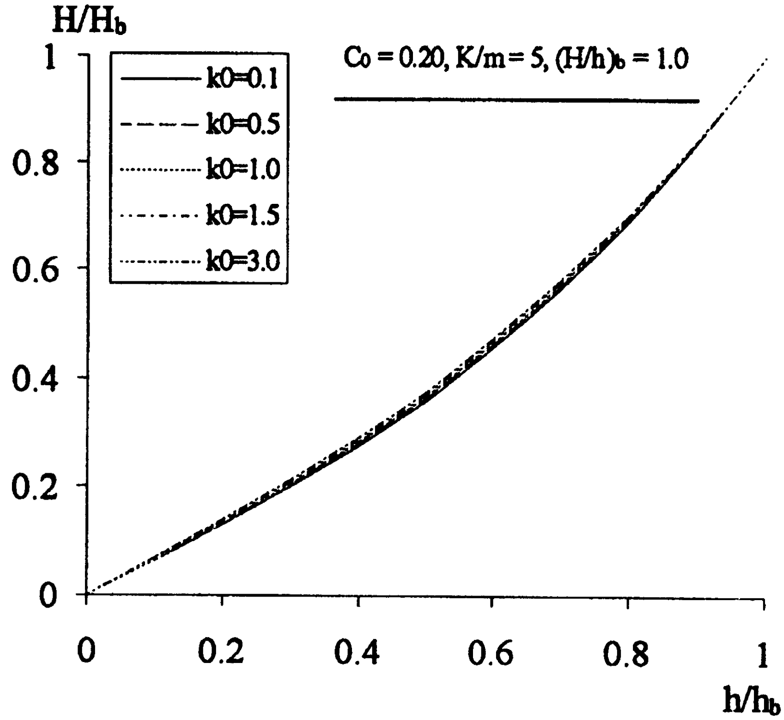
(a)


H/Hb
(H/h)b=0.7
(H/h)b=0.8
(H/h)b=0.9
(H/h)b=1.0
(H/h)b=1.2
Co = 0.20, ko = 0.5, K/m = 5
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
h/hb

(b)



(c)



(d)

FIG. 4. Dependence of analytical solution (2.15) on (*a*) K/m, (*b*) $(H/h)_b$, (*c*) C_0 and (*d*) k_0 for waves breaking on a plane beach. Dally *et al.*[1] suggested K = 0.17 and Γ = 0.5 for "still water model" on a plane beach.

$$\left(\frac{\Delta h}{h}\right)_b = \frac{C_0}{k_1 h_b} \frac{1 - e^{-k_1 h_b}}{1 - C_0 e^{-k_1 h_b}} = \left(\frac{H}{h}\right)_b \frac{C_0}{k_0} \frac{1 - e^{-k_0\left(\frac{h}{H}\right)_b}}{1 - C_0 e^{-k_0\left(\frac{h}{H}\right)_b}}$$

$$k_1 = \frac{k_0}{H}$$

and

$$\alpha = \frac{K}{m} \frac{\Gamma^2}{\left(\frac{5}{2} - \frac{k}{m}\right)} \left(\frac{h}{H}\right)_b^2$$

where k_0 is the dimensionless parameter. Dally *et al.*[1] recommend $K = 0.17$ and $\Gamma = 0.5$. Note that the solution is invalid if $K/m = 5/2$, since eqs. (2.13) and (2.15) are consistent with the analytical solution reported by Dally *et al.*[1] when $C_0 = 0.0$.

3. Concluding Remarks

In physical concept, most of wave energy is stored at first by the static energy of the air bubbles which are driven into water (Fig. 1). After formation of the air-water mixture, the energy of the air bubbles is transformed by the micro-turbulence of the eddies in the turbulent wakes behind the uprising bubbles (Führböter[2]). Eqs. (2.13) and (2.15) together with Figs. (3) and (4) show that the aeration effect for itself is able to explain the energy loss in breaking waves.

Using shallow water linear wave theory, analytical solutions for wave height transformation due to breaking the wave on a flat shelf, and a plane slope have been derived to evaluate the effect of air bubble entrainment. These analyses have been provided explicit expressions for the wave height within the surf zone and simultaneously some important calculations have been shown graphically (Figs. 3 and 4). Significant deference has been obtained for the variation of H/h for a wide range of C_0 (Figs. 3*a* and 4*c*), whereas the dimensionless wave height has been found insensitive to k_0 in both cases (Figs. 3*b* and 4*d*). A curve consistent with that reported by Dally *et al.*[1] has been found when $C_0 = 0.0$. $(H/h)_b$ showed an increase with increasing the beach slope, whereas the dimensionless wave height H/h decreased remarkably with increasing C_0.

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References

1. W. R. Dally, R. G. Dean and R. A. Dalrymple, *J. Geophys. Res.*, **90** (1985), 11917-928.
2. A. Führböter, *Proc. ICCE,* (1970), 391-98.
3. K. Horikawa and C. T. Kuo, *Proc. ICCE,* (1966), 217-33.
4. A Hoque, *Thesis,* (2002), p. 151.
5. I. A. Svendsen, *Coast, Eng.*, **8** (1984), 303-29.
6. J. Wu, *J. Geophys. Res.*, **93** (1988), 587-90.