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NOTES ON A ROGERS-RAMANUJAN TYPE IDENTITY

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In this paper, a Rogers-Ramanujan type identity is proved by a new method. In the meantime, some Rogers-Ramanujan type identities are obtained.

Key Words : Rogers-Ramanujan Type Identities; Basic Hypergeometric Functions; Partition Identities

1. INTRODUCTION

For convenience, we first give some notations.

For $|q| < 1$, let

$$(a;q)_0 = 1,\ (a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1}),\quad n \geq 1,$$

$$(a_1, a_2, \ldots, a_r; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n,\quad (a:q)_\infty = \prod_{n=1}^{\infty} (1-aq^{n-1}),$$

$$(a_1, a_2, \ldots, a_r; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_r;q)_\infty.$$

In literature[1], we note that Agarwal and Singh established a transformation which contained three parameters

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(dq;q)_{n+r} (q^{1-n-r}/b;q)_r\, q^{n^2+5r^2/2+3nr-pn-pr-r/2}\, d^{2n+3r} (-b)^r}{(d^2q;q)_{2n+2r} (q;q)_n (q,b;q)_r}$$

$$= \frac{1}{(d^2q;q)_\infty} \sum_{j=0}^{p} \frac{(q^{-p};q)_j (-d^2)^j q^{j(j+1)/2}}{(q;q)_j}$$

$$\sum_{n=0}^{\infty} \frac{(d, dq/b;q)_n (1-dq^{2n})\, d^{5n} b^n q^{6n^2-(2p+1)n+4nj}}{(q,b;q)_n (1-d)}. \qquad \ldots (1)$$

For $d = -b = q$ and $p = 1$ in (1), by means of Jacobi's triple-product identity[2]

$$\sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n^2} = (zq;q^2)_\infty (z^{-1}q;q^2)_\infty (q^2;q^2)_\infty,\quad z \neq 0, \qquad \ldots (2)$$

they got

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(q^2;q)_{n+r}(-q^{-n-r};q)_r\,q^{n^2+5r^2/2+3nr+n+5r/2}}{(q^2;q)_{2n+2r}(q;q)_n(q,-q;q)_r}$$

$$=\frac{(q^3,q^9,q^{12};q^{12})_\infty}{(q;q)_\infty}-\frac{q(q,q^{11},q^{12};q^{12})_\infty}{(q;q)_\infty}. \qquad \text{... (3)}$$

''sing the simple transformation

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}A(n,r)=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}A(n-r,r)$$

to tranform the double series on the left in (3) into repeated series and summing the resulting inner series by the q-binomianl theorem, they derived a new Rogers-Ramanujan type identity

$$\sum_{n=0}^{\infty}\frac{(1-q^{n+1})(-q^2;q^2)_n\,q^{n^2+n}}{(q;q)_{2n+2}}=\frac{(q^3,q^9,q^{12};q^{12})_\infty}{(q;q)_\infty}$$

$$-\frac{q(q,q^{11},q^{12};q^{12})_\infty}{(q;q)_\infty}. \qquad \text{... (4)}$$

In fact, identity (4) can be proved by a simpler method. The aim of this note is to give (4) a new and simple proof. In the meantime, we will generalize (4) and obtain some identities similar to (4).

As first, we prove (4).

It is clear that

$$\sum_{n=0}^{\infty}\frac{(1-q^{n+1}(-q^2;q^2)_n\,q^{n^2+n}}{(q;q)_{2n+2}}=\sum_{n=1}^{\infty}\frac{(-q^2;q^2)_{n-1}\,q^{n(n-1)}}{(q;q)_{2n}}$$

$$-\sum_{n=1}^{\infty}\frac{(-q^2;q^2)_{n-1}\,q^{n^2}}{(q;q)_{2n}}.$$

Due to the following results (see (122) and (28) of Slater[3], respectively).

$$\sum_{n=1}^{\infty}\frac{-q^2;q^2)_{n-1}\,q^{n^2+n-2}}{(q;q)_{2n}}$$

$$=\frac{(-q^{10},-q^{38},q^{48};q^{48})_\infty-q^3(-q^2,-q^{46},q^{48};q^{48})_\infty}{(q;q)_\infty} \qquad \text{... (5)}$$

and

$$\sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1}\, q^{n(n-1)}}{(q;q)_{2n-1}} = \sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1}\, q^{n(n-1)}}{(q;q)_{2n}}$$

$$-\sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1}\, q^{n(n+1)}}{(q;q)_{2n}} = \frac{(-q^2;q^2)_\infty\,(-q,-q^5,q^6;q^6)_\infty}{(q^2;q^2)_\infty},$$

we get

$$\sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1}\, q^{n(n-1)}}{(q;q)_{2n}} = \frac{(-q^2;q^2)_\infty\,(-q,-q^5,q^6;q^6)_\infty}{(q^2;q^2)_\infty}$$

$$+\frac{q^2(-q^{10},-q^{38},q^{48};q^{48})_\infty - q^5(-q^2,-q^{46},q^{48};q^{48})_\infty}{(q;q)_\infty}. \qquad \ldots (6)$$

On the other hand,

$$\sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1}\, q^{n^2}}{(q;q)_{2n}}$$

$$= \frac{q(-q^{14};-q^{34},q^{48};q^{48})_\infty}{(q;q)_\infty} - \frac{q^5(-q^2,-q^{46},q^{48};q^{48})_\infty}{(q;q)_\infty}. \qquad \ldots (7)$$

For (7), see (123) of Slater[3]. Hence, by (6-7) and (2), we have

$$\sum_{n=0}^{\infty} \frac{(1-q^{n+1})\,(-q^2;q^2)_n\, q^{n^2+n}}{(q;q)_{2n+2}} = \frac{(-q^2;q^2)_\infty\,(-q,-q^5,q^6;q^6)_\infty}{(q^2;q^2)_\infty}$$

$$+\frac{q}{(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty} q^{24n^2-14n+1} - \sum_{n=-\infty}^{\infty} q^{24n^2+10n}\right).$$

It follows from

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{6n^2+5n} = \sum_{n=-\infty}^{\infty} q^{24n^2+10n} - \sum_{n=-\infty}^{\infty} q^{24n^2-14n+1},$$

$$(q^2;q^2)_\infty = (-q;q)_\infty\,(q;q)_\infty = (-q;q^2)_\infty\,(-q^2;q^2)_\infty\,(q;q)_\infty,$$

$$(-q;q^2)_\infty = (-q;q^6)_\infty\,(-q^3;q^6)_\infty\,(-q^5;q^6)_\infty,$$

and

$$(q^6;q^6)_\infty = (q^6;q^{12})_\infty\,(q^{12};q^{12})_\infty = (-q^3;q^6)_\infty\,(q^3;q^6)_\infty\,(q^{12},q^{12})_\infty,$$

$$= (-q^3; q^6)_\infty (q^3; q^{12})_\infty (q^9; q^{12})_\infty (q^{12}; q^{12})_\infty$$

that (4) holds.

Now, we give the generalization of (4).

Theorem 1 — *Assume that m is an integer. Then*

$$\sum_{n=0}^{\infty} \frac{(1-q^{n+m})(-q^2;q^2)_n q^{n^2+n}}{(q;q)_{2n+2}} = \frac{(-q^2;q^2)_\infty (-q,-q^5,q^6;q^6)_\infty}{(q^2;q^2)_\infty}$$
$$+ \frac{q^2[(-q^{10},-q^{38},q^{48};q^{48})_\infty - q^3(-q^2,-q^{46},q^{48};q^{48})_\infty]}{(q;q)_\infty}$$
$$- \frac{q^m[(-q^{14},-q^{34},q^{48};q^{48})_\infty - q^4(-q^2,-q^{46},q^{48};q^{48})_\infty]}{(q;q)_\infty}. \quad \text{... (8)}$$

PROOF : Let

$$X_m = \sum_{n=0}^{\infty} \frac{(1-q^{n+m})(-q^2;q^2)_n q^{n^2+n}}{(q;q)_{2n+2}}.$$

Then

$$X_m = \sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1}\, q^{n(n-1)}}{(q;q)_{2n}} - q^{m-1} \sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1}\, q^{n^2}}{(q;q)_{2n}}.$$

Owing to (6-7), we obtain (8). □

Obviously, (8) is the generalization of (4). From the particular choice of m can obtain some special cases of (8).

In this paper, we can derive some identities similar to (4).

Theorem 2 — *Suppose that m is an integer. Then*

$$\sum_{n=1}^{\infty} \frac{(1-q^{n+m})(-q^2;q^2)_n q^{n^2+2n}}{(q;q)_{2n+2}}$$
$$= \frac{(-q^{14},-q^{34},q^{48};q^{48})_\infty}{(q;q)_\infty} - \frac{q^m(-q^{10},-q^{38},q^{48};q^{48})_\infty}{(q;q)_\infty}$$
$$+ \frac{(q^{m+3}-q^4)(-q^2,-q^{46},q^{48};q^{48})_\infty}{(q;q)_\infty}, \quad \text{... (9)}$$
$$+ \sum_{n=1}^{\infty} \frac{(1-q^{n+m})(-q^2;q^2)_n q^{n^2+n}}{(q;q)_{2n+2}}$$

$$= \frac{(-q^2;-q^2)_\infty (-q,-q^5,q^6;q^6)_\infty}{(q^2;q^2)_\infty} + \frac{q^2(-q^{10},-q^{38},q^{48};q^{48})_\infty}{(q;q)_\infty}$$

$$- \frac{q^5(-q^2,-q^{46},q^{48};q^{48})_\infty}{(q;q)_\infty}$$

$$+ \frac{q^m\,[(-q^{14},-q^{34},q^{48};q^{48})_\infty - q^4(-q^2,-q^{46},q^{48};q^{48})_\infty]}{(q;q)_\infty}, \qquad \ldots (10)$$

$$\sum_{n=0}^{\infty} \frac{(1+q^{n+m})\,(-q^2;q^2)_n\, q^{n^2+2n}}{(q;q)_{2n+2}}$$

$$= \frac{(-q^{14},-q^{34},q^{48};q^{48})_\infty - q^4(-q^2,-q^{46},q^{48};q^{48})_\infty]}{(q;q)_\infty}$$

$$+ \frac{q^m\,(-q^{10},-q^{38},q^{48};q^{48})_\infty - q^3(-q^2,-q^{46},q^{48};q^{48})_\infty]}{(q;q)_\infty}, \qquad \ldots (11)$$

$$\sum_{n=0}^{\infty} \frac{(1+q^{2n+m})\,(-q^2;q^2)_n\, q^{n^2+n}}{(q;q)_{2n+2}} = \frac{(-q^2;q^2)_\infty (-q,-q^5,q^6;q^6)_\infty}{(q^2;q^2)_\infty}$$

$$+ \frac{q^2\,[(-q^{10},-q^{38},q^{48};q^{48})_\infty - q^5(-q^2,-q^{46},q^{48};q^{48})_\infty]}{(q;q)_\infty},$$

$$+ \frac{q^m\,[(-q^{10},-q^{38},q^{48};q^{48})_\infty - q^2(-q^2,-q^{46},q^{48};q^{48})_\infty]}{(q;q)_\infty}, \qquad \ldots (12)$$

$$\sum_{n=0}^{\infty} \frac{(1-q^{2n+m})\,(-q^2;q^2)_n\, q^{n^2+n}}{(q;q)_{2n+2}} = \frac{(-q^2;-q^2)_\infty (-q,-q^5,q^6;q^6)_\infty}{(q^2;q^2)_\infty}$$

$$+ \frac{q^2\,[(-q^{10},-q^{38},q^{48};q^{48})_\infty - q^5(-q^2,-q^{46},q^{48};q^{48})_\infty]}{(q;q)_\infty}$$

$$- \frac{q^m\,[(-q^{10},-q^{38},q^{48};q^{48})_\infty - q^3(-q^2,-q^{46},q^{48};q^{48})_\infty]}{(q;q)_\infty}. \qquad \ldots (13)$$

PROOF : We only give the proof of (9). The proofs of (10-13) follow the same pattern and therefore are omitted here.

Evidently,

$$\sum_{n=0}^{\infty} \frac{(1-q^{n+m})(-q^2;q^2)_n q^{n^2+2n}}{(q;q)_{2n+2}} = \sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1} q^{n^2-1}}{(q;q)_{2n}}$$

$$-q^{m-2} \sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1} q^{n^2+n}}{(q;q)_{2n}}.$$

By (5) and (7), we have (9). □

Finally, we give the special case of (13).

When $m = 1$ in (13), we obtain by (2)

$$\sum_{n=0}^{\infty} \frac{(1-q^{2n+3})(-q^2;q^2)_n q^{n^2+n}}{(q;q)_{2n+2}} = \frac{(-q^2;q^2)_\infty (-q,-q^5,q^6;q^6)_\infty}{(q^2;q^2)_\infty}$$

$$+ \frac{q^2(q^5,q^7,q^{12};q^{12})_\infty}{(q;q)_\infty} + \frac{(q^6-q^5)(-q^2,-q^{46},q^{48};q^{48})_\infty}{(q;q)_\infty}.$$

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