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MULTILINEAR MAPPINGS IN BANACH MODULES OVER A C^*-ALGEBRA

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We prove the stability of multilinear functional equations in Banach modules over a unital C^*-algebra.

Key Words: Banach Module over C^*-algebra; Stability; Unitary Group; Multilinear Functional Equation

1. INTRODUCTION

Let E_1 and E_2 be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f: E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\varepsilon \geq 0$ and $p \in [0, 1)$ such that

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon (\| x \|^p + \| y \|^p)$$

for all $x, y \in E_1$. Rassias [7] shows that there exists a unique $\mathbb{R}$-linear mapping $T: E_1 \to E_2$ such that

$$\| f(x) - T(x) \| \leq \frac{2\varepsilon}{2 - 2^p} \| x \|^p$$

for all $x \in E_1$.

In [3, 6], the authors proved the generalized Hyers-Ulam-Rassias stability of functional equations in Banach modules over a C^*-algebra. In [4], the author proved the stability of multi-quadratic mappings in Banach spaces, and in [5], the author proved the stability of multilinear Trif d-mappings in Banach modules over a C^*-algebra.

Throughout this paper, let A be a unital C^*-algebra with norm $|\cdot|$, $\mathcal{U}(A)$ the unitary group of $A, A_1 = \{a \in A \mid |a| = 1\}$, and A_1^+ the set of positive elements in A_1. Let ${}_A\mathcal{B}_l$ be a left A-module for each $l = 1, \ldots, d$. Let ${}_A\mathcal{D}$ be a left Banach A-module with norm $\|\cdot\|$ (see [8]).

The main purpose of this paper is to prove the stability of multilinear functional equations in Banach modules over a unital C^*-algebra.

2. Stability of Multilinear Functional Equations in Banach Modules over a C^*-Algebra

For a given mapping $f: \prod_{s=1}^{d} {}_A\mathcal{D}$ and given $a_1, ..., a_d \in A$, we set

$$D_{a_1, ..., a_d} f(x_1, y_1, ..., x_d, y_d) :$$

$$= \sum_{l=1}^{d} f(x_1, ..., x_{l-1}, a_l x_l + a_l y_l, x_{l+1}, ..., x_d)$$

$$- \sum_{l=1}^{d} a_l f(x_1, ..., x_l, ..., x_d) - \sum_{l=1}^{d} a_l f(x_1, ..., x_{l-1}, y_l, x_{l+1}, ..., x_d)$$

for all $(x_1, ..., x_d), (y_1, ..., y_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$.

Theorem 1 — *Let $f: \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ be a mapping for which there exists a functional $\varphi : \prod_{s=1}^{d} {}_A\mathcal{B}_s^2 \to [0, \infty)$ such that*

$$\tilde{\varphi}(x_1, y_1, ..., x_d, y_d) := \sum_{j=0}^{\infty} \sum_{l=1}^{d} \frac{1}{2^{l+jd}} \varphi(2^{j+1} x_1, 0, ..., 2^{j+1} x_{l-1}, 0,$$

$$2^j x_l, 2^j y_l, 2^j x_{l+1}, 0, ..., 2^j x_d, 0) < \infty \quad ... (2.i)$$

$$\| D_{u_1, ..., u_d} f(x_1, y_1, ..., x_d, y_d) \| \le \varphi(x_1, y_1, ..., x_d, y_d) \quad ... (2.ii)$$

for all $u_1, ..., u_d \in \mathcal{U}(A)$ and all $(x_1, ..., x_d), (y_1, ..., y_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. Assume that $f(x_1, ..., x_d) = 0$ if $x_l = 0$ for any $l = 1, ..., d$. Then there exists a unique A-multilinear mapping $M : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ such that

$$\| f(x_1, ..., x_d) - M(x_\cdot, ..., x_d) \| \le \tilde{\varphi}(x_1, x_1, ..., x_d, x_d) \quad ... (2.iii)$$

for all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$.

PROOF : Put $u_1 = \ldots = u_d = 1 \in \mathcal{U}(A)$. For each filed l, let $y_1 = \ldots = y_{l-1} = y_{l+1} = \ldots = y_d = 0$ and $y_l = x_i$ in (2.ii). Then we get

$$\| f(x_1, \ldots, x_{l-1}, 2x_l, x_{l+1}, \ldots, x_d) - 2f(x_1, \ldots, x_{l-1}, x_l, x_{l+1}, \ldots, x_d) \|$$

$$\leq \varphi(x_1, 0, \ldots, x_{l-1}, 0, x_l, x_l, x_{l+1}, 0, \ldots, x_d, 0),$$

and hence

$$\| f(x_1, \ldots, x_{l-1}, x_l, x_{l+1}, \ldots, x_d) - \frac{1}{2} f(x_1, \ldots, x_{l-1}, 2x_l, x_{l+1}, \ldots, x_d) \|$$

$$\leq \frac{1}{2} \varphi(x_1, 0, \ldots, x_{l-1}, 0, x_l, x_l, x_{l+1}, 0, \ldots, x_d, 0)$$

for all $(x_1, \ldots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. So one can obtain

$$\left\| \frac{1}{2^{l-1}} f(2x_1, \ldots, 2x_{l-1}, x_l, \ldots, x_d) - \frac{1}{2} f(2x_1, \ldots, 2x_l, x_{l+1}, \ldots, x_d) \right\|$$

$$\leq \frac{1}{2^l} \varphi(2x_1, 0, \ldots, 2x_{l-1}, 0, x_l, x_l, x_{l+1}, 0, \ldots, x_d, 0)$$

for all $(x_1, \ldots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. Thus

$$\left\| f(x_1, \ldots, x_d) - \frac{1}{2^d} f(2x_1, \ldots, 2x_d) \right\|$$

$$\leq \sum_{l=1}^{d} \frac{1}{2^l} \varphi(2x_1, 0, \ldots, 2x_{l-1}, 0, x_l, x_l, x_{l+1}, 0, \ldots, x_d, 0)$$

for all $(x_1, \ldots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. And we get

$$\left\| \frac{1}{2^{jd}} f(2^j x_1, \ldots, 2^j, x_d) - \frac{1}{2^{l+jd}} f(2^{j+1} x_1, \ldots, 2^{j+1} l, x_d) \right\|$$

$$\leq \sum_{l=1}^{d} \frac{1}{2^{l+jd}} \varphi(2^{j+1} x_1, 0, \ldots, 2^{j+1} x_{l-1}, 0, x_l, 2^j x_{l+1}, 0, \ldots, 2^j x_d, 0)$$

for all $(x_1, \ldots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. So

$$\left\| f(x_1, \ldots, x_d) - \frac{1}{2^{nd}} f(2^n x_1, \ldots, 2^n x_d) \right\| \leq \sum_{j=0}^{n-1} \sum_{l=1}^{d} \frac{1}{2^{l+jd}}$$

$$\varphi(2^{j+1} x_1, 0, \ldots, 2^{j+1} x_{l-1}, 0, 2^j x_l 2^j x_l 2^j x_{l+1}, 0, \ldots, 2^j x_d, 0) \quad \ldots (2.1)$$

for all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$.

For each $l = 1, ..., d$, let x_l be an element in ${}_A\mathcal{B}_l$. For positive integers n and m with $n > m$,

$$\left\| \frac{1}{2^{md}} f(2^m x_1, ..., 2^m, x_d) - \frac{1}{2^{nd}} f(2^n x_1, ..., 2^n, x_d) \right\| \le \sum_{j=0}^{n-1} \sum_{l=1}^{d} \frac{1}{2^{l+jd}}$$

$$\varphi(2^{j+1} x_1, 0, ..., 2^{j+1} x_{l-1}, 0, 2^j x_l 2^j x_l 2^j x_{l+1}, 0, ..., 2^j x_d, 0)$$

which tends to zero as $m \to \infty$ by (2.i). So $\left\{ \frac{1}{2^{nd}} f(2^n x_1, ..., 2^n x_d) \right\}$ is a Cauchy sequence for all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. Since ${}_A\mathcal{D}$ is complete, the sequence $\left\{ \frac{1}{2^{nd}} f(2^n x_1, ..., 2^n x_d) \right\}$ converges for all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. We can define a mapping $M : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ by

$$M(x_1, ..., x_d) = \lim_{j \to \infty} \frac{1}{2^{jd}} f(2^j x_1, ..., 2^j x_d) \qquad ... (2.2)$$

for all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$.

By (2.i) and (2.2), we get

$$\| D_{1, ..., 1} M(x_1, 0, ..., x_{l-1}, 0, x_l, y_l, x_{l+1}, 0, ..., x_d, 0) \|$$

$$= \lim_{j \to \infty} \frac{1}{2^{jd}} \| D_{1, ..., 1} f(2^j x_1, 0, ..., 2^j x_{l-1}, 0, 2^j x_l, 2^j y_l, 2^j x_{l+1}, 0, ..., 2^j x_d, 0) \|$$

$$\le \lim_{j \to \infty} \frac{1}{2^{jd}} \varphi(2^j x_1, 0, ..., 2^j x_{i-1}, 0, 2^j x_l, 2^j y_l, 2^j x_{l+1}, 0, ..., 2^j x_d, 0)$$

$$= 0,$$

and hence

$$D_{1, ..., 1} M(x_1, 0, ..., x_{l-1}, 0, x_l, y_l, x_{l+1}, 0, ..., x_d, 0) = 0$$

for all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$ and all $y_l \in {}_A\mathcal{B}_l$, which implies that M is additive in the l-th variable for each $l = 1, ..., d$. Moreover, by passing to the limit in (2.1) as $n \to \infty$, we get the inequality (2.*iii*).

$$\|f(x_1, ..., x_d) - L(x_1, ..., x_d)\| \le \tilde{\varphi}(x_1, x_1, ..., x_d, x_d)$$

for all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. Then

$$\| M(x_1, ..., x_d) - L(x_1, ..., x_d) \|$$

$$= \frac{1}{2^{jd}} \| M(2^j x_1, ..., 2^j x_d) - L(2^j x_1, ..., 2^j x_d)$$

$$\le \frac{1}{2^{jd}} \| M(2^j x_1, ..., 2^j x_d) - f(2^j x_1, ..., 2^j x_d)$$

$$+ \frac{1}{2^{jd}} \| f(2^j x_1, ..., 2^j x_d) - L(2^j x_1, ..., 2^j x_d)$$

$$\le \frac{2}{2^{jd}} \tilde{\varphi} (2^j x_1, 2^j x_1 ..., 2^j x_d, 2^j x_d),$$

which tends to zero as $j \to \infty$ by (2.i). Thus $M(x_1, ..., x_d) = L(x_1, ..., x_d)$ for all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. This proves the uniqueness of M.

By the assumption, for each $u_l \in \mathcal{U}(A)$ and all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$,

$$\frac{1}{2^{jd}} \| D_{1, ..., 1, u_l, 1, ..., 1} f(2^j x_1, 0, ..., 2^j x_{l-1}, 0, 2^j x_l, 0, 2^j x_{l+1}, 0, ..., 2^j x_d, 0) \|$$

$$\le \frac{1}{2^{jd}} \varphi (2^j x_1, 0, ..., 2^j x_{l-1}, 0, 2^j x_l, 0, 2^j x_{l+1}, 0, ..., 2^j x_d, 0),$$

which tends to zero as $j \to \infty$ by (2.i). So

$$D_{1, ..., 1, u_l, 1, ..., 1} M(x_1, 0, ..., x_l, 0, ..., x_d, 0)$$

$$= \lim_{j \to \infty} \frac{1}{2^{jd}} D_{1, ..., 1, u_l, 1, ..., 1} f(2^j x_1, 0, ..., 2^j x_l, 0, ..., 2^j x_d, 0) = 0$$

for all $u_l \in \mathcal{U}(A)$ and all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. Hence

$$D_{1, ..., 1, u_l, 1, ..., 1} M(x_1, 0, ..., x_l, 0, ..., x_d, 0)$$

$$= M(x_1, ..., x_{l-1}, u_l x_l, x_{l+1}, ..., x_d)$$

$$- u_l M(x_1, ..., x_{l-1}, x_l, x_{l+1}, ..., x_d) = 0$$

for all $u_l \in \mathcal{U}(A)$ and all $(x_1, \dots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. So

$$M(x_1, \dots, x_{l-1}, u_l x_l, x_{l+1}, \dots, x_d)$$

$$= u_l M(x_1, \dots, x_{l-1}, x_l, x_{l+1}, \dots, x_d) = 0$$

for all $u_l \in \mathcal{U}(A)$ and all $(x_1, \dots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$.

Now let $a \in A (a \neq 0)$ and K an integer greater than $4 \mid a \mid$. Then

$$\left| \frac{a}{K} \right| = \frac{1}{K} \mid a \mid < \frac{\mid a \mid}{4 \mid a \mid} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By [2, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(A)$ such that $3\frac{a}{K} = u_1 + u_2 + u_3$. And

$$M(x_1, \dots, x_{l-1}, x_l, x_{l+1}, \dots, x_d)$$

$$= M\left(x_1, \dots, x_{l-1}, 3\frac{1}{3}x_l, x_{l+1}, \dots, x_d\right)$$

$$= 3M\left(x_1, \dots, x_{l-1}, \frac{1}{3}x_l, x_{l+1}, \dots, x_d\right)$$

for all $(x_1, \dots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. So

$$M\left(x_1, \dots, x_{l-1}, \frac{1}{3}x_l, x_{l+1}, \dots, x_d\right)$$

$$= \frac{1}{3}M\left(x_1, \dots, x_{l-1}, \frac{1}{3}x_l, x_{l+1}, \dots, x_d\right)$$

for all $(x_1, \dots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. Thus

$$M(x_1, \dots, a x_l, \dots, x_d) = M\left(x_1, \dots, \frac{K}{3} \cdot 3\frac{a}{K}x_l, \dots, x_d\right)$$

$$= \frac{K}{3}M\left(x_1, \dots, 3\frac{a}{K}x_l, \dots, x_d\right)$$

$$= \frac{K}{3}M(x_1, \dots, u_1 x_l + u_2 x_l + u_3 x_l, \dots, x_d)$$

$$= \frac{K}{3}(u_1 + u_2 + u_3) M(x_1, \dots, x_l, \dots, x_d)$$

$$= \frac{K}{3} \cdot 3 \frac{a}{K} M(x_1, \ldots, x_l, \ldots, x_d)$$

$$= aM(x_1, \ldots, x_l, \ldots, x_d)$$

for all $(x_1, \ldots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. Hence

$$M(x_1, \ldots, ax_l + by_l, \ldots, x_d)$$

$$= M(x_1, \ldots, ax_l, \ldots, x_d) + M(x_1, \ldots, x_{l-1}, by_l, x_{l+1}, \ldots, x_d)$$

$$= aM(x_1, \ldots, x_l, \ldots, x_d) + bM(x_1, \ldots, x_{l-1}, y_l, x_{l+1}, \ldots, x_d)$$

for all $a, b \in A$ and all $(x_1, \ldots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$ and all $y_l \in {}_A\mathcal{B}_l$. So the unique multi-additive mapping $M : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ is an A-multilinear mapping, as desired. □

***Theorem* 2** — *Let $f : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ be a mapping for which there exists a function $\varphi : \prod_{s=1}^{d} {}_A\mathcal{B}_s^2 \to [0, \infty)$ satisfying (2.i) such that*

$$\| D_{a_1, \ldots, a_d} f(x_1, y_1, \ldots, x_d, y_d) \| \leq \varphi(x_1, y_1, \ldots, x_d, y_d)$$

for all $a_1, \ldots, a_d \in A_1^+ \cup \{i\}$ and all $(x_1, \ldots, x_d)$, $(y_1, \ldots, y_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. Assume that $f(x_1, \ldots, x_d) = 0$ if $x_l = 0$ for any $l = 1, \ldots, d$, and that for each $l = 1, \ldots, d$, $f(x_1, \ldots, x_{l-1}, \lambda x_l, x_{l+1}, \ldots, x_d)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $(x_1, \ldots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. Then there exists a unique A-multilinear mapping $M : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ satisfying the inequality (2.iii).

PROOF : Put $a_1 = \ldots = a_d = 1 \in A_1^+$. By the same reasoning as in the proof of Theorem 1, there exists a unique multi-additive mapping $M : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ satisfying the inequality (2.iii).

For each fixed $l = 1, \ldots, d$, since $f(x_1, \ldots, \lambda x_l, \ldots, x_d)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $(x_1, \ldots, x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$, by the same reasoning as in the proof of [7, Theorem], the multi-additive mapping $M:\prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ is $\mathbb{R}$-linear in the lth variable. So the multi-additive mapping $M:\prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ is $\mathbb{R}$-multilinear.

By the same reasoning as in the proof of Theorem 1,

$$M(x_1, ..., x_{l-1}, ax_l, x_{l+1}, ..., x_d)$$
$$= aM(x_1, ..., x_{l-1}, x_l, x_{l+1}, ..., x_d) \qquad ... (2.3)$$

for all $a \in A_1^+ \cup \{i\}$ and all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$.

For any element $a \in A$, $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$, and $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = \left(\frac{a+a^*}{2}\right)^+ - \left(\frac{a+a^*}{2}\right)^- + i\left(\frac{a+a^*}{2i}\right)^+ - i\left(\frac{a-a^*}{2i}\right)^-$, where $\left(\frac{a+a^*}{2}\right)^+$, $\left(\frac{a+a^*}{2}\right)^-$, $\left(\frac{a-a^*}{2}\right)^+$ and $\left(\frac{a-a^*}{2i}\right)^-$ are positive elements (see [1, Lemma 38.8]). Using the $\mathbb{R}$-multilinearity and (2.3), one can easily show that

$$M(x_1, ..., x_{l-1}, ax_l, x_{l+1}, ..., x_d)$$
$$= aM(x_1, ..., x_{l-1}, x_l, x_{l+1}, ..., x_d)$$

for all $a \in A$ and all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. Hence

$$M(x_1, ..., x_{l-1}, ax_l, by_l, x_{l+1}, ..., x_d)$$
$$= M(x_1, ..., x_{l-1}, ax_l, x_{l+1}, ..., x_d) + M(x_1, ..., x_{l-1}, by_l, x_{l+1}, ..., x_d)$$
$$= aM(x_1, ..., x_{l-1}, x_l, x_{l+1}, ..., x_d) + bM(x_1, ..., x_{l-1}, y_l, x_{l+1}, ..., x_d)$$

for all $a, b \in A$, all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$ and all $y_l \in {}_A\mathcal{B}_l$. So the unique multi-additive mapping $M:\prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ is an A-multilinear mapping, as desired. □

***Theorem* 3** — *Let* $f:\prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ *be a mapping for which there exists a function* $\varphi:\prod_{s=1}^{d} {}_A\mathcal{B}_s^2 \to [0, \infty)$ *such that*

$$\tilde{\varphi}(x_1, y_1, ..., x_d, y_d) := \sum_{j=0}^{\infty} \sum_{l=0}^{d} 2^{l-1+jd} \varphi\left(\frac{1}{2^{j+1}} x_1, 0, ..., \frac{1}{2^{j+1}} x_{l-1}, 0, \right.$$

$$\left. \frac{1}{2^{j+1}} x_l, \frac{1}{2^{j+1}} y_l, \frac{1}{2^j} x_{l+1}, 0, ..., \frac{1}{2^j} x_x, 0 \right) < \infty \qquad ... (2.iv)$$

$$\| D_{u_1, ..., u_d} f(x_1, y_1, ..., x_d, y_d) \| \le \varphi(x_1, y_1, ..., x_d, y_d)$$

for all $u_1, ..., u_d \in \mathcal{U}(A)$ *and all* $(x_1, ..., x_d), (y_1, ..., y_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. *Assume that* $f(x_1, ..., x_d) = 0$ *if* $x_l = 0$ *for any* $l = 1, ..., d$. *Then there exists a unique A-multilinear mapping* $M : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ *such that*

$$\| f(x_1, ..., x_d) - M(x_1, ..., x_d) \| \le \tilde{\varphi}(x_1, x_1, ..., x_d, x_d) \qquad ... (2.v)$$

for all $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$.

PROOF : The proof is similar to the proof of Theorem 1. □

***Theorem* 4** — *Let* $f : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ *be a mapping for which there exists a function* $\varphi : \prod_{s=1}^{d} {}_A\mathcal{B}_s^2 \to [0, \infty)$ *satisfying (2.iv) such that*

$$\| D_{a_1, ..., a_d} f(x_1, y_1, ..., x_d, y_d) \| \le \varphi(x_1, y_1, ..., x_d, y_d)$$

for all $a_1, ..., a_d \in A_1^+ \cup \{i\}$ *and all* $(x_1, ..., x_d), (y_1, ..., y_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. *Assume that* $f(x_1, ..., x_d) = 0$ *if* $x_l = 0$ *for any* $l = 1, ..., d$, *and that for each* $l = 1, ..., d$, $f(x_1, ..., x_{l-1}, \lambda x_l, x_{l+1}, ..., x_d)$ *is continuous in* $\lambda \in \mathbb{R}$ *for each fixed* $(x_1, ..., x_d) \in \prod_{s=1}^{d} {}_A\mathcal{B}_s$. *Then there exists a unique A-multilinear mapping* $M : \prod_{s=1}^{d} {}_A\mathcal{B}_s \to {}_A\mathcal{D}$ *satisfying the inequality (2.v).*

PROOF : The proof is similar to the proof of Theorem 2. □

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