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TWO-POINT BOUNDARY VALUE PROBLEM OF INTEGRO-DIFFERENTIAL EQUATIONS OF MIXED TYPE IN BANACH SPACES

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With the differential and integral inequalities theory and monotone iterative technique, the existence and uniqueness of the solution for the two-point boundary value problem of integro-differential equations of mixed type in Banach spaces are studied, and the error estimations for the convergent iterative sequence are also given. It should be point out that do not use any conditions of compactness in this paper. Some previous results are improved and generalized.

Key Words: Integro-Differential Equations; Two-Point Boundary Value Problem; Monotone Iterative Technique; Cone and Order; Banach Space

1. INTRODUCTION

Let E be a real Banach space and P a cone in E. The order "$\leq$" is introduced by cone P, i.e., $x, y \in E$, $x \leq y$ if and only if $y - x \in P$. Let $D \subseteq E$, operator $A : D \times D \to E$ is said to be mixed monotone if $A(x, y)$ is nondecreasing in x and nonincreasing in y. Element $x^* \in D$ is called a fixed point of A if $A(x^*, x^*) = x^*$. Recall that cone P is said to be normal if there exist a positive constant N_p such that $\theta \leq x \leq y$ implies $\| x \| \leq N_p \| y \|$; N_p is called the normal constant of P (see [4, 5]). Let E^* be the dual space of E, $P^* = \{\varphi \in E^* \mid \varphi(x) \geq 0, \ \forall x \in P\}$ is called the dual cone. Obviously. $x \in P$ iff $\varphi(x) \geq 0$ for all $\varphi \in P^*$ (see[5]).

Throughout this paper we always assume that E is a real Banach space, P is normal cone in E. Without loss of generality we may assume that normal constant the $N_p = 1$ (see[4]).

The following symbols are applied in this paper:

$$C[I, E] = \{u(t) : I \to E \mid u(t) \text{ is continuous}\};$$

$$C^2[I, E] = \{u(t) : I \to E \mid u(t) \text{ possesses a continuous second order derivative}\}.$$

For any $u = u(t) \in C[I, E]$, let $\| u \|_c = \max_{t \in I} \| u(t) \|$, and obviously $C[I, E]$ is a Banach space in norm $\| \, \|_c$. For any $u_0 = u_0(t)$, $v_0 = v_0(t) \in C[I, E]$, $u_0 \leq v_0$ if and only if $u_0(t) \leq v_0(t)$ $(\forall t \in I)$, and ordered interval $[u_0, v_0] = \{u \in C[I, E] \mid u_0 \leq u \leq v_0\}$.

The purpose of this paper is to study the following two-point boundary value problem for a second order nonlinear integro-differential equation of mixed type in Banach space E :

$$\begin{cases} -u'' = f(t, u, T_1 u, T_2 u), t \in I \\ u(0) = x_0, u(1) = x_1, \end{cases} \quad \text{... (1)}$$

where $I = [0, 1]$, $T_1 u(t) = g_1(t) + \int_0^t K_1(t, s)\, u(s) ds$, $T_2 u(t) = g_2(t) + \int_0^1 K_2(t, s)\, u(s)\, ds$, $g_i \in C[I, E]$ ($i = 1, 2$), $x_0, x_1 \in E$, $K_1 \in C[D, R^+]$, $K_2 \in C[I \times I, R^+]$, $D = \{(t, s) \in R^2 \mid 0 \le s \le t \le 1\}$, $R^+ = [0, +\infty)$, $R = (-\infty, +\infty)$.

Let $$\Omega = \left\{ (t, u, v, w) \mid t \in I, u \in [u_0, v_0], v \in [T_1 u_0, T_1 v_0], w \in [T_2 u_0, T_2 v_0] \right\}.$$

This paper uses the following assumptions:

(H_1) There exist $M > 0, N_i \ge 0$ ($i = 1, 2$) such that

(*a*) $f(t, u_2, T_1 v_2, T_2 v_2) - f(t, u_1, T_1 v_1, T_2 v_1)$

$\ge -M(u_2 - u_1) - N_1(T_1 v_1 - T_1 v_2) - N_2(T_2 v_1 - T_2 v_2)$,

for $u_i, v_i \in [u_0, v_0]$ ($i = 1, 2$), and $u_2 \ge u_1, v_1 \ge v_2, t \in I$.

(*b*) $-u_0'' + N_1(T_1 v_0 - T_1 u_0) + N_2(T_2 v_0 - T_2 u_0) \le f(t, u_0, T_1 v_0, T_2 v_0)$,

$u_0(0) \le x_0, u_0(1) \le x_1$;

$-v_0'' - N_1(T_1 v_0 - T_1 u_0) N_2(T_2 v_0 - T_2 u_0) \ge f(t, u_0, T_1 v_0, T_2 v_0)$,

$v_0(0) \ge x_0, v_0(1) \ge x_1$.

(H_2) There exist $\beta_j \ge 0$ ($j = 0, 1, 2$) such that

$f(t, u, T_1 v, T_2 v) - f(t, v, T_1 u, T_2 u) \le \beta_0(u - v) + \beta_1(T_1 u - T_1 v) + \beta_2(T_2 u - T_2 v)$,

for $u_0(t) \le v \le u \le v_0(t), t \in I$.

(H_3) (*i*) $N_1 k_1 + N_2 k_2 < M$;

(*ii*) $\alpha \equiv 4M + N_1 k_1 + 6N_2 k_2 < 12$;

(*iii*) $\delta \equiv 4(\beta_0 + M) + (\beta_1 + 2N_1) k_1 + 6(\beta_2 + 2N_2) k_2 < 24$.

where $k_1 = \max_D K_1(t, s), k_2 = \max_{I \times I} K_2(t, s)$.

2. Several Lemmas

We first establish several lemmas.

Lemma 1 — If $u \in C^2[I, E]$ and satisfy

$$\begin{cases} u''(t) \le Mu(t) - N_1 \int_0^t K_1(t,s)u(s)ds - N_2 \int_0^1 K_2(t,s)u(s)ds, \\ u(0) \ge \theta, u(1) \ge \theta, \end{cases}$$

and the assumption $(H)_3$ (i) holds. Then, $u(t) \ge \theta$, $\forall t \in I$.

PROOF : If the conclusion is not true, then there exists a $\varphi \in P^*$ and $t_0 \in (0, 1)$ such that $m(t) = \varphi(u(t)), t \in I$ satisfying

$$\begin{cases} m''(t) \le Mm(t) - N_1 \int_0^t K_1(t,s)m(s)ds - N_2 \int_0^1 K_2(t,s)m(s)ds, \\ m(0) \ge 0, m(1) \ge 0, \end{cases}$$

and $m(t_0) = \min_{t \in I} m(t) < 0$. Hence we have

$$0 \le m''(t) \le Mm(t) - N_1 \int_0^{t_0} K_1(t_0,s)m(s)ds - N_2 \int_0^1 K_2(t_0,s)m(s)ds$$

$$\le m(t_0)\left[M - N_1 \int_0^{t_0} K_1(t_0,s)\,ds - N_2 \int_0^1 K_2(t_0,s)ds \right]$$

$$\le m(t_0)\,[M \quad (N_1 k_1 + N_2 k_2)] < 0,$$

which is a contradiction. This completes the proof.

Lemma 2 — *Let $x(t) \in C[I, E]$ and $x(t)$ be increasing on I, then, for any $\lambda \in I$, we have*

$$\int_0^\lambda x(t)\,dt \le \lambda \int_0^1 x(t)dt.$$

PROOF : For any $\varphi \in P^*$, let $m(t) = \varphi(x(t)), t \in I$. Obviously, $m(t) : I \to R$ is a continuous increasing function on I.

Thus, let $t = \lambda s$, we have that $x(t) = x(\lambda s) \le x(s)$, $(t, s \in I)$ and

$$\int_0^\lambda x(t)dt = \lambda \int_0^1 x(\lambda s)\,ds \le \lambda \int_0^1 x(s)ds.$$

This completes the proof.

Lemma 3 — Let $u_0, v_0 \in C^2[I, E]$ such that $u_0(t) < v_0(t)$ in I and $f \in C[\Omega, E]$. For $h_i \in [u_0, v_0]$, $(i = 1, 2)$, consider the linear two-point boundary value problem

$$\begin{cases} -u'' = f(t, h_1, T_1h_2, T_2h_2) - M(u - h_1) + N_1(T_1u - T_1h_2) + N_2(T_2u - T_2h_2) \\ u(0) = x_0, u(1) = x_1. \end{cases} \quad \dots (2)$$

If the condition (H_3) (ii) holds, then the (2) has a unique solution in $[u_0, v_0]$.

PROOF : It is easy to observe that $u(t)$ is a solution of the (2) in the order interval $[u_0, v_0]$ if and only if $u(t)$ satisfies the integral equation

$$u(t) = x_0 + t(x_1 - x_0) + t \int_0^1 dr \int_0^r f(s, h_1(s), T_1h_2(s), T_2h_2(s))ds$$

$$- Mt \int_0^1 dr \int_0^r \left[u(s) - h_1(s)\, ds + N_1\, t \int_0^1 dr \int_0^r ds \int_0^s K_1(s, \tau)\, [u(\tau) - h_2(\tau) \right] d\tau$$

$$+ N_2\, t \int_0^1 dr \int_0^r ds \int_0^1 K_2\, (s, \tau) \left[u(\tau) - h_2\, (\tau) \right] d\tau$$

$$- \int_0^1 dr \int_0^r f(s, h_1\, (s), T_1\, h_2(s), T_2\, h_2\, (s))\, ds$$

$$+ M \int_0^1 dr \int_0^r [u(s) - h_1(s))]\, ds - N_1 \int_0^1 dr \int_0^r ds \int_0^s K_1\, (s, \tau)\, [u(\tau) - h_2(\tau)]\, d\tau$$

$$- N_2 \int_0^t dr \int_0^r ds \int_0^1 K_2\, (s, \tau)\, [u(\tau) - h_2\, (\tau)]\, d\tau \equiv Fu(t), \quad (3)$$

Obviously, operator $F : C\,[I, E] \to C\,[I, E]$ and $u(t)$ is a solution of the (2) iff u is a fixed point of the F, i.e., $Fu = u$.

By induction, we verify that

$$\left|\left| F^n u(t) - F^n v(t) \right|\right| \leq \left(M + \frac{1}{3} N_1\, k_1 + N_2\, k_2 \right)$$

$$\left(\frac{\alpha}{12} \right)^{n-1} ||\, u - v\, ||_c \cdot t, t \in I,\ n = 1, 2, \dots, \quad \dots (4)$$

for any $u, v \in C\,[I, E]$.

In fact, when $n = 1$, by (3), we have

$$Fu(t) - Fv(t) = Mt \int_0^1 dr \int_0^r [v(s) - u(s)]\, ds + M \int_0^1 dr \int_0^r [u(s) - v(s)]\, ds$$

$$+N_1 t \int_0^1 dr \int_0^r ds \int_0^s K_1(s, \tau) [u(\tau) - v(\tau)] \, d\tau$$

$$+ N_1 \int_0^t dr \int_0^r ds \int_0^s K_1 (s, \tau) [v(\tau) - u(\tau)] \, d\tau$$

$$+N_2 t \int_0^1 dr \int_0^r ds \int_0^1 K_2(s, \tau) [u(\tau) - v(\tau)] \, d\tau$$

$$+ N_2 \int_0^t dr \int_0^r ds \int_0^1 K_2 (s, \tau) [v(\tau) - u(\tau)] \, d\tau,$$

hence we have

$$\left|\left| F^n u(t) - F^n v(t) \right|\right|$$

$$\leq \left[Mt \int_0^1 dr \int_0^r ds + M \int_0^t dr \int_0^r ds + N_1 k_1 t \int_0^1 dr \int_0^r ds \int_0^s d\tau \right.$$

$$+ N_1 k_1 \int_0^t dr \int_0^r ds \int_0^s d\tau + N_2 k_2 t \int_0^1 dr \int_0^r ds \int_0^1 d\tau$$

$$\left. + N_2 k_2 \int_0^t dr \int_0^r ds \int_0^1 d\tau \right] \cdot \| u - v \|_c$$

$$\leq \left(M + \frac{1}{3} N_1 k_1 + N_2 k_2 \right) \cdot \| u - v \|_c \cdot t, \qquad t \in I.$$

Now we assume that (4) is true for $n = k$, i.e.,

$$\left|\left| F^k u(t) - F^k v(t) \right|\right| \leq \left(M + \frac{1}{3} N_1 k_1 + N_2 k_2 \right)$$

$$\left(\frac{\alpha}{12} \right)^{k-1} \left|\left| u - v \right|\right|_c \cdot t, \quad t \in I. \qquad \text{... (5)}$$

Then, when $n = k + 1$, by (3) we have

$$F^{k+1} u(t) - F^{k+1} v(t) = Mt \int_0^1 dr \int_0^r [F^k v(s) - F^k u(s)] \, ds$$

$$+ M \int_0^t dr \int_0^r [F^k u(s) - F^k v(s)] \, ds$$

$$+N_1 t \int_0^1 dr \int_0^r ds \int_0^s K_1(s,\tau)\,[F^k u(\tau) - F^k v(\tau)]\,d\tau$$

$$+N_1 \int_0^t dr \int_0^r ds \int_0^s K_1(s,\tau)\,[F^k v(\tau) - F^k u(\tau)]\,d\tau$$

$$+N_2 \int_0^t dr \int_0^r ds \int_0^1 K_2(s,\tau)\,[F^k u(\tau) - F^k v(\tau)]\,d\tau$$

$$+N_2 \int_0^t dr \int_0^r ds \int_0^s K_2(s,\tau)\,[F^k v(\tau) - F^k u(\tau)]\,d\tau,$$

hence, by (5), we have

$$\left|\left| F^{k+1} u(t) - F^{k+1} v(t) \right|\right| \le \left(M + \frac{1}{3} N_1 k_1 + N_2 k_2 \right)$$

$$\left(\frac{\alpha}{12}\right)^{k-1} || u - v ||_c \cdot \left[Mt \int_0^1 dr \int_0^r s ds \right.$$

$$+ M \int_0^1 dr \int_0^t s ds + N_1 k_1 t \int_0^1 dr \int_0^r ds \int_0^s \tau d\tau + N_1 k_1 \int_0^t dr \int_0^r ds \int_0^s \tau d\tau$$

$$\left. + N_2 k_2 t \int_0^1 dr \int_0^r ds \int_0^1 \tau d\tau + N_2 k_2 \int_0^t dr \int_0^r ds \int_0^1 \tau d\tau \right]$$

$$\le \left(M + \frac{1}{3} N_1 k_1 + N_2 k_2 \right) \left(\frac{\alpha}{12}\right)^{k} || u - v ||_c \cdot t, \quad t \in I.$$

Therefore (4) is proved.

Hence, by (4), we have

$$\left|\left| F^n u - F^n v \right|\right|_c \le \left(M + \frac{1}{3} N_1 k_1 + N_2 k_2 \right)$$

$$\left(\frac{\alpha}{12}\right)^{n-1} || u - v ||_c, \; n = 1, 2, 3, \ldots, \qquad \ldots (6)$$

for any $u, v \in C[I, E]$.

By (6) and the condition $\alpha < 12$, there exist that n_0 is sufficiently large such that

$$\left(M + \frac{1}{3} N_1 k_1 + N_2 k_2 \right) \left(\frac{\alpha}{12}\right)^{n_0 - 1} < 1.$$

Therefore F^{n_0} is a contraction operator on $C[I, E]$. Consequently F^{n_0} has a unique fixed point u, obviously the u is also a unique fixed point of operator F, i.e., the $u(t)$ is a unique solution of (2).

Now we prove that the $u \in [u_0, v_0]$. In fact, by (2) and the condition (H_1), we have

$$(u-u_0)''(t) = u''(t) - u_0''(t)$$

$$\leq f(t, u_0, T_1 v_0, T_2 v_0) - f(t, h_1, T_1 h_2, T_2 h_2) + M(u-h_1)(t)$$

$$- N_1(T_1 u - T_1 h_2)(t) - N_2(T_2 u - T_2 h_2)(t)$$

$$- N_1(T_1 v_0 - T_1 u_0)(t) - N_2(T_2 v_0 - T_2 u_0)(t)$$

$$\leq M(h_1 - u_0)(t) + N_1(T_1 v_0 - T_1 h_2)(t) + N_2(T_2 v_0 - T_2 h_2)(t)$$

$$+ M(u - h_1(t) - N_1(T_1 u - T_1 h_2)(t) - N_2(T_2 u - T_2 h_2)(t)$$

$$- N_1(T_1 v_0 - T_1 u_0)(t) - N_2(T_2 v_0 - T_2 u_0)(t)$$

$$= M(u - u_1)(t) - N_1(T_1 u - T_1 u_0)(t) - N_2(T_2 u - T_2 u_0)(t)$$

$$= M(u-u_0)(t) - N_1 \int_0^t K_1(t,s)[u(s) - u_0(s)]ds$$

$$- N_2 \int_0^1 K_2(t,s)[u(s) - u_0(s)]ds,$$

$$(u-u_0)(0) = u(0) - u_0(0) \geq x_0 - x_0 = \theta,$$

$$(u-u_0)(1) = u(1) - u_0(1) \geq x_1 - x_1 = \theta.$$

Hence, by Lemma 1, $(u-u_0)(t) \geq \theta$ for $t \in I$, i.e., $u_0 \leq u$. Similarly, we can show $u \leq v_0$. This completes the proof.

We now define a mapping $A : [u_0, v_0] \times [u_0, v_0] \to [u_0, v_0]$ such that $A(h_1, h_2)(t) = u(t)$ is the unique solution of (2), for $h_1, h_2 \in [u_0, v_0]$.

Obviously, the $\bar{u}(t)$ is a solution of (1) iff $\bar{u}$ is a fixed pooint of the A, i.e., $\bar{u} = A(\bar{u}, \bar{u})$.

3. Main Result

Theorem — *Let* $u_0, v_0 \in C^2[I, E]$ *such that* $u_0(t) \leq v_0(t)$ *in* I and $f : \Omega \to E$ be continuous. Assume that the conditions $(H_1), (H_2)$ and (H_3) are satisfied. Then the two-point boundary value problem (1) has a unique solution $\bar{u}$ in $[u_0, v_0]$ and there exist monotone sequences $\{u_n\}$, $\{v_n\}$ such that

$$\lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} v_n(t) = \bar{u}(t),$$

uniformly on [0, 1] and have the following error estimate

$$\left\| \bar{u} - u_n(v_n) \right\|_c \leq \frac{1}{6}\left[3(\beta_0 + M) + (\beta_1 + 2N_1)k_1 + 3(\beta_2 + 2N_2)k_2 \right]$$

$$\left(\frac{\delta}{24}\right)^{n-1} \left\| v_0 - u_0 \right\|_c, \quad n = 1, 2, 3, \ldots . \qquad \ldots (7)$$

Here $u_n = A(u_{n-1}, v_{n-1}), v_n = A(v_{n-1}, u_{n-1}), (n = 1, 2, 3, \ldots)$ are defined by the Lemma 3 respectively.

PROOF : First, we prove that $A : [u_0, v_0] \times [u_0, v_0] \to [u_0, v_0]$ is the mixed monotone operator. In fact, suppose $u_i, v_i \in [u_0, v_0], (i = 1, 2)$ and $u_1 \leq u_2, v_2 \leq v_1$, let

$$u = A(u_2, v_2), v = A(u_1, v_1),$$

then, by (2) and the condition $(H_1)(a)$, we have

$$(u - v)'' = u'' - v'' = -f(t, u_2, T_1 v_2, T_2 v_2)$$

$$+ M(u - u_2) - N_1(T_1 u - T_1 v_2) - N_2(T_2 u - T_2 v_2)$$

$$+ f(t, u_1, T_1 v_1, T_2 v_1) - M(v - u_1) + N_1(T_1 v - T_1 v_1) + N_2(T_2 v - T_2 v_1)$$

$$\leq M(u - v) - N_1(T_1 u - T_1 v) - N_2(T_2 u - T_2 v)$$

$$= M(u - v)(t) - N_1 \int_0^t K_1(t, s)[u(s) - v(s)]\, ds$$

$$- N_2 \int_0^1 K_2(t, s)[u(s) - v(s)]\, ds, \quad t \in I,$$

$$(u - v)(0) = x_0 - x_0 = \theta, (u - v)(1) = x_1 - x_1 = \theta.$$

By Lemma 1, we know that $u - v \geq \theta$, i.e., $(Au_1, v_1) \leq A(u_2, v_2)$. Therefore, the A is a mixed monotone operator on $[u_0, v_0]$.

By induction and A is a mixed monotone operator, it is easy to verify that

$$u_0 \leq u_1 \leq \ldots \leq \ldots u_n \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0, \qquad \ldots (8)$$

by (8), the condition (H_2) and Lemma 2, we have

$$\theta \leq (v_n - u_n)(t) = A(v_{n-1}, u_{n-1})(t) - A(u_{n-1}, v_{n-1})(t)$$

$$= t \int_0^1 dr \int_0^r \Big[f(s, v_{n-1}(s), T_1 u_{n-1}(s), T_2 u_{n-1}(s))$$

$$- f(s, u_{n-1}(s), T_1 v_{n-1}(s), T_2 v_{n-1}(s)) \Big] ds$$

$$+ \int_0^t dr \int_0^r \Big[f(s, u_{n-1}(s), T_1 v_{n-1}(s), T_2 v_{n-1}(s))$$

$$- f(s, v_{n-1}(s), T_1 u_{n-1}(s), T_2 u_{n-1}(s)) \Big] ds$$

$$- M \int_0^t dr \int_0^r \Big[v_{n-1}(s) - u_{n-1}(s) \Big] ds - N_1 \int_0^t dr \int_0^r \Big[T_1 v_{n-1}(s) - T_1 u_{n-1}(s) \Big] ds$$

$$- N_2 \int_0^t dr \int_0^r \Big[T_2 v_{n-1}(s) - T_2 u_{n-1}(s) \Big] ds$$

$$+ M \int_0^t dr \int_0^r \Big[v_n(s) - u_n(s) \Big] ds - Mt \int_0^1 dr \int_0^r \Big[v_n(s) - u_n(s) \Big] ds$$

$$+ N_1 \int_0^t dr \int_0^r ds \int_0^s K_1(s, \tau) [u_n(\tau) - v_n(\tau)] \, d\tau$$

$$+ N_2 \int_0^t dr \int_0^r ds \int_0^1 K_2(s, \tau) [u_n(\tau) - v_n(\tau)] \, d\tau$$

$$+ Mt \int_0^1 dr \int_0^r \Big[v_{n-1}(s) - u_{n-1}(s) \Big] ds$$

$$+ N_1 t \int_0^1 dr \int_0^r ds \int_0^s K_1(s, \tau) \Big[v_n(\tau) - u_n(\tau) \Big] d\tau$$

$$+ N_1 t \int_0^1 dr \int_0^r ds \int_0^s K_1(s, \tau) \Big[v_{n-1}(\tau) - u_{n-1}(\tau) \Big] d\tau$$

$$+ N_2 t \int_0^1 dr \int_0^r ds \int_0^1 K_2(s, \tau) \Big[v_n(\tau) - u_n(\tau) \Big] d\tau$$

$$+N_2 t \int_0^1 dr \int_0^r ds \int_0^1 K_2(s,\tau)\left[v_{n-1}(\tau)-u_{n-1}(\tau)\right]d\tau$$

$$\leq (\beta_0+M)\, t \int_0^1 dr \int_0^r \left[v_{n-1}(s)-u_{n-1}(s)\right]ds$$

$$+(\beta_1+2N_1)\, k_1\, t \int_0^1 dr \int_0^r ds \int_0^s \left[v_{n-1}(\tau)-u_{n-1}(\tau)\right]d\tau$$

$$+(\beta_2+2N_2)\, k_2\, t \int_0^1 dr \int_0^r ds \int_0^s \left[v_{n-1}(\tau)-u_{n-1}(\tau)\right]d\tau, \quad t\in I, \qquad \dots (9)$$

Next, by induction, we verify that

$$\left\| v_n(t)-u_n 9t) \right\| \leq \frac{1}{6}\left[3(\beta_0+M)+(\beta_1+2N_1)\, k_1+3(\beta_2+2N_2)\, k_2\right]$$

$$\times\left(\frac{\delta}{24}\right)^{n-1}\cdot \left\| v_0-u_0 \right\|_c \cdot t, \quad t\in I, n = 1, 2, 3, \dots . \qquad \dots (10)$$

In fact, when $n = 1$, by (9), we have

$$\theta \leq v_1(t)-u_1(t) \leq (\beta_0+M)\, t \int_0^1 dr \int_0^r \left[v_0(s)-u_0(s)\right]ds$$

$$+(\beta_1+2N_1)\, k_1\, t \int_0^1 dr \int_0^r ds \int_0^s \left[v_0(\tau)-u_0(\tau)\right]d\tau$$

$$+(\beta_2+2N_2)\, k_2\, t \int_0^1 dr \int_0^r ds \int_0^s \left[v_0(\tau)-u_0(\tau)\right]d\tau$$

and by the normality of cone P, we know that

$$\left\| v_1(t)-u_1(t) \right\| \leq \left[(\beta_0+M)\int_0^1 dr \int_0^r ds + (\beta_1+2N_1)\, k_1 \int_0^1 dr \int_0^r ds \int_0^s d\tau\right.$$

$$\left.(\beta_2+2N_2)\, k_2 \int_0^1 dr \int_0 ds \int_0^1 d\tau\right]\cdot \left\| v_0-u_0 \right\|_c \cdot t$$

$$=\frac{1}{6}\left[3(\beta_0+M)(\beta_1+2N_1)\, k_1+3(\beta_2+2N_2)\, k_2\right]\cdot \left\| v_0-u_0 \right\|_c \cdot t, \quad t\in I.$$

Now we assume that (10) is true for $n = k$, i.e.,

$$\left\| v_k(t) - u_k(t) \right\| \le \frac{1}{6}\left[3(\beta_0 + M) + (\beta_1 + 2N_1) k_1 + 3(\beta_2 + 2N_2) k_2 \right]$$

$$\times \left(\frac{\delta}{24} \right)^{k-1} \cdot \left\| v_0 - u_0 \right\|_c \cdot t, \quad t \in I.$$

Then, when $n = k + 1$, by (9), we have

$$\theta \le v_{k+1}(t) - u_{k+1}(t) \le (\beta_0 + M)\, t \int_0^1 dr \int_0^r \left[v_k(s) - u_k(s) \right] ds$$

$$+ (\beta_1 + 2N_1) k_1 t \int_0^1 dr \int_0^r ds \int_0^s \left[v_k(\tau) - u_k(\tau) \right] d\tau$$

$$+ (\beta_2 + 2N_2) k_2 t \int_0^1 dr \int_0^r ds \int_0^s \left[v_k(\tau) - u_k(\tau) \right] d\tau$$

and by the normality of cone P again, we get

$$\left\| v_{k+1}(t) - u_{k+1}(t) \right\| \le \frac{1}{6}\left[3(\beta_0 + M) + (\beta_1 + 2N_1) k_1 + 3(\beta_2 + 2N_2) k_2 \right]$$

$$\times \left(\frac{\delta}{24} \right)^{k-1} \cdot t \left\| \mathring{v}_0 - u_0 \right\|_c \cdot \le \left[(\beta_0 + M) \int_0^1 dr \int_0^r s ds \right.$$

$$\left. + (\beta_1 + 2N_1) k_1 \int_0^1 dr \int_0^r ds \int_0^s \tau d\tau + (\beta_2 + 2N_2) k_2 \int_0^1 dr \int_0^r ds \int_0^1 \tau d\tau \right]$$

$$= \frac{1}{6}\left[3(\beta_0 + M) + (\beta_1 + 2N_1) k_1 + 3(\beta_2 + 2N_2) k_2 \right]$$

$$\times \left(\frac{\delta}{24} \right)^{k} \left\| v_0 - u_0 \right\|_c \cdot t, \quad t \in I.$$

Thus, the (10) is proved.

By (10), we have

$$\left\| v_n - u_n \right\|_c \le \frac{1}{6}\left[3(\beta_0 + M) + (\beta_1 + 2N_1) k_1 + 3(\beta_2 + 2N_2) k_2 \right]$$

$$\times \left(\frac{\delta}{24} \right)^{n-1} \left\| v_0 - u_0 \right\|_c, \quad n = 1, 2, 3, \ldots . \tag{11}$$

It follows from (8) that for any positive integer m

$$\theta \le u_{n+m} - u_n \le v_n - u_n, \qquad \theta \le v_n - v_{n+m} \le v_n - u_n.$$

By the normality of cone P and (11), we have

$$\max\left\{ \left\| u_{n+m} - u_n \right\|_c ,\ \left\| v_n - v_{n+m} \right\|_c \right\} \le \left\| v_n - u_n \right\|_c$$

$$\le \frac{1}{6}\left[3(\beta_0 + M) + (\beta_1 + 2N_1)\, k_1 + 3(\beta_2 + 2N_2)\, k_2 \right] \left(\frac{\delta}{24} \right)^{n-1} \cdot \left\| v_0 - u_0 \right\|_c , \quad \text{... (12)}$$

The results imply that both $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in $C[I, E]$, hence there exist $u^*, v^* \in C[I, E]$ such that

$$u^* = \lim_{n \to \infty} u_n, \qquad v^* = \lim_{n \to \infty} v_n ,$$

and $u_n \le u^* \le v^* \le v_n$, $(n = 1, 2, 3, ...)$. By the normality of cone P and from (11), we have

$$u^* = v^* = \bar{u} \in [u_0, v_0] \text{ and } u_{n-1} \le \bar{u} \le v_{n-1}, \ (n = 1, 2, 3, ...). \quad \text{... (13)}$$

By (13) and the mixed monotocity of A, we have

$$u_n = A(u_{n-1}, v_{n-1}) \le A(\bar{u}, \bar{u}) \le A(v_{n-1}, u_{n-1}) = v_n, \ (n = 1, 2, 3, ...). \quad \text{... (14)}$$

Thus, letting $n \to \infty$ in (14) and by the normality of P, we know that

$$\bar{u} \le A(\bar{u}, \bar{u}) \le \bar{u}$$

i.e., $A(\bar{u}, \bar{u}) = \bar{u}$. This implies $\bar{u}$ is a solution of (1) in $[u_0, v_0]$.

Next we prove that $\bar{u}$ is the unique solution of (1) in $[u_0, v_0]$. In fact, suppose $\bar{v} \in [u_0, v_0]$ is also a solution of (1), then from $u_0 \le \bar{v} = A(\bar{v}, \bar{v}) \le v_0$ we have $u_1 \le \bar{v} \le v_1$. By induction, it is esy to prove that

$$u_n \le \bar{v} \le v_n, \qquad n = 1, 2, 3, \dots . \quad \text{... (15)}$$

Letting $n \to \infty$ in (15) and by the normality of P, we have

$$\bar{u} \le \bar{v} \le \bar{u},$$

i.e., $\bar{v} = \bar{u}$. Therefore, $\bar{u}(t)$ is the unique solution of (1) in $[u_0, v_0]$.

Finally, letting $m \to \infty$ in (12), we can obtain the error estimation (7).

This completes the proof of the Theorem.

Remark — The results presented here unify and extend many recent results. It should be pointed out that we do not use any conditions of compactness in Theorem.

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